RIEMANN SURFACES

7. Week 8: Square integrable one-forms.

The conjugation operation $*$ allows one to define square-integrable one-forms, an inner product and L^2 space of one-forms. The latter has a remarkable decomposition.

In a more detail, for $\omega_1, \omega_2 \in \Lambda^1(X)$ we define

$$(\omega_1, \omega_2) = \int_X \omega_1 \wedge *(\bar{\omega_2}),$$

where the right-hand side can be finite or infinite.

A form $\omega \in \Lambda^1(X)$ is called square-integrable if $(\omega, \omega) < \infty$. We have an inner product on the vector space of square-integrable forms; its completion (see details below) is denoted $L^2(X)$.

We define some important subspace of $L^2(X)$:

- E is the closure of the space of differentials df where f is smooth with a compact support.
- $E^* = \{\omega \in L^2(X) | *\omega \in E\}$.

The main result of this section will assert that $L^2(X) = E \oplus E^* \oplus H$ where H is the space of harmonic forms. This, in particular, will imply that any de Rham cohomology class in $H^1_{DR}(X)$ has a unique harmonic representative.

7.1. Basics of Hilbert space theory.

7.1.1. **Definition.** A complex vector space H with a Hermitian inner product is called a Hilbert space if it is complete with respect to the metric $x, y \mapsto ||x - y||$.

Any finite-dimensional vector space admits an inner product. This inner product is unique up to an isomorphism. Completeness is automatic in finite-dimensional case. The most interesting Hilbert spaces are infinite-dimensional.

A typical example is ℓ_2 — the space of infinite sequences $x = (x_1, x_2, \ldots)$ satisfying the condition $\sum |x_i|^2 < \infty$.

The inner product is given by the formula

$$(x, y) = \sum x_i \bar{y_i}.$$

The standard Cauchy-Schwarz inequality proves that the inner product so defined is always finite.

7.1.2. Subspaces The meaningful notion of subspace of H is the following.

Definition. A subspace $V \subset H$ is a vector subspace which is closed in the topology defined by the metric.

Example. The vector subspace of ℓ_2 generated by the "basic" sequences which are zero everywhere except for one place, is **not** closed. On the contrary, it is dense in ℓ_2.

Below some basic properties of Hilbert spaces are listed.

7.1.3. **Lemma.** *(Cauchy inequality)*

$$|(x, y)|^2 \leq (x, x)(y, y).$$

7.1.4. **Lemma.** *A subspace of a Hilbert space is itself Hilbert (that is, complete) with respect to the induced inner product.*

7.1.5. **Lemma.** *Let V be a subspace of H. Denote $V^\perp = \{x \in H | \forall y \in V (x, y) = 0\}$. Then $V^\perp$ is also a subspace and $H = V \oplus V^\perp$.*

7.1.6. **Lemma.** *If V, W are subspaces of H then $V + W$ is also a subspace.*

7.1.7. **Proposition.** *(Riesz theorem) The map $H \longrightarrow H^*$ given by the formula*

$$x \mapsto (x, \quad)$$

is a (norm preserving) isomorphism.

Proof. Injectivity is obvious since $(x, x) = 0$ implies $x = 0$. Let us prove surjectivity. Let $\phi \in H^*$. The kernel K is a subspace of H, therefore it has an orthogonal complement. Obviously, $\dim K^\perp = 1$. Choose $x \in K^\perp \setminus 0$. Both ϕ and $(x, \quad)$ have the same kernel. Their linear combination

$$\phi - \frac{(x, x)}{\phi(x)}(x, \quad)$$

vanishes at x. Therefore, it vanishes everywhere. The preservation of norms follows from Cauchy inequality. □

7.1.8. Completion Our ultimate aim is to construct the space of square-integrable forms L^2. The easiest way to do this is via completion of the space of smooth forms. Here is the general recipe.

Let V be a complex vector space endowed with a Hermitian form

$$v, w \mapsto (v, w) \in \mathbb{C}.$$

We present below an easy construction of a completion $\widehat{V}$ of V with respect to the norm defined by the Hermitian form. The main property of $\widehat{V}$ is the following

7.1.9. **Theorem.** *$\widehat{V}$ is a Hilbert space, that is a vector space endowed with a Hermitian form and complete with respect to the corresponding norm.*

The proof is obvious. We recall below the definitions.

7.1.10. **Definition.** 1. A sequence $\{x_i\}$ in V is a Cauchy sequence if for any $\epsilon > 0$ there exists N such that $\forall m, n > N$ one has $||x_n - x_m|| < \epsilon$.
2. Two Cauchy sequences are equivalent if their mixture is also a Cauchy sequence.

The collection of Cauchy sequences forms a vector space denoted $\widehat{V}$.

Let us check $\widehat{V}$ has a Hilbert space structure. First of all, we have to define a Hermitian form on it.

7.1.11. **Lemma.** *Let $\{x_i\}$ and $\{y_i\}$ be Cauchy sequences. Then the sequence (x_n, y_n) is Cauchy, and, therefore, it has a finite limit.*

□

We define the inner product on $\widehat{V}$ by the formula

$$(\{x_n\}, \{y_n\}) := \lim(x_n, y_n).$$

It is an easy exercise to check that $\widehat{V}$ is complete with respect to the norm defined by the above inner product.

7.2. **Space of L^2 one-forms.** We apply the general completion construction to the space of one-forms on a Riemann surface.

First of all, let us explain in a more detail how to integrate a two-form along X. This is indispensable for the definition of inner product on one-forms.

We explained earlier how to integrate a two-form along a smooth singular simplex $\gamma : \Delta^2 \longrightarrow X$. In order to integrate over the whole X we have to fix an orientation of X and to choose its triangulation.

We have no problems with the orientation: since X is a Riemann surface, it has an atlas with holomorphic transition functions. Such functions, as we have already seen, have a positive Jacobian, which means that the complex-analytic atlas defines an orientation of X.

It is easy to define the integral over X using a triangulation: the integral is just the sum (finite or infinite) of the integrals over all simplices.

One has to take into account that if X is compact then the triangulation has a finite number of simplices and the sum is always finite. In the noncompact case the existence of a triangulation is less obvious, so one may be willing to define the integral in a way which does not rely upon a triangulation.

A standard way of doing this is via *partition of unity.*

Recall a few basic notions.

7.2.1. **Definition.** A cover $X = \cup V_i$ is called *locally finite* if for any $x \in X$ there exists a small neighborhood U of x having nonempty intersection with only finite number of V_i.

7.2.2. **Definition.** An open cover $X = \cup V_j$ is *subordinate* to a cover $\{U_i\}$ if for each j there exist i so that $V_j \subset U_i$.

7.2.3. **Definition.** Let $X = \cup V_j$ be a locally finite covering of a manifold X. A partition of unity corresponding to $\{V_j\}$ is a collection of smooth functions α_i satisfying the following properties

- $\operatorname{Supp} \alpha_j \subset V_j$.
- $\alpha_i(x) \geq 0$.
- For any $x \in X$ one has $\sum_j \alpha_j(x) = 1$.

7.2.4. **Theorem.** *Let X be a manifold (note: it is assumed to be countable at infinity). Then*

1. *Any covering $X = \cup U_i$ admits a locally finite subordinate covering $\{V_j\}$.*
2. *Any locally finite covering $\{V_j\}$ admits a partition of unity.*

7.2.5. Integration via partition of unity

Let now X be a smooth oriented manifold of dimension n (in our case $n = 2$) and let $\omega \in \Lambda^n(X)$.

Choose an oriented locally finite atlas $\{V_j, \phi_j : V_j \to \mathbb{R}^n\}$ for X and a partition of unity $j \mapsto \alpha_j$. We define the integral of ω by the formula

$$\int_X \omega = \sum_j \int_{\mathbb{R}^n} \phi_j^{-1*}(\alpha_j \omega).$$

7.2.6. Some consequences of Stokes theorem

Proposition. *Let D be a compact region in X bounded by a piecewise smooth curve ∂D. Then for a C^1-function f and a C^1-one-form ω one has*

$$\int_{\partial D} f\omega = \int_D d(f) \wedge \omega + \int_D f d\omega.$$

Proof. Apply Stokes theorem to $f\omega$. □

In particular, if ω is closed, one has

$$\int_{\partial D} \omega = 0.$$

Corollary. *Let f or ω have a compact support. Then*

$$\int_X f d\omega = \int_X \omega \wedge df.$$

7.2.7. Construction of $L^2(X)$

Recall that for a Riemann surface X one has two operations on $\Lambda^1(X)$:

1. Star-operation $\omega = udz + vd\bar{z} \quad \mapsto \quad *\omega = -iudz + ivd\bar{z}$.
2. Complex conjugation $\omega = udz + vd\bar{z} \quad \mapsto \quad \bar{\omega} = \bar{u}d\bar{z} + \bar{v}dz$.

One can easily check that the two operations commute. We define

$$(\omega, \omega') = \int_X \omega \wedge *\bar{\omega}'.$$

Let Λ^1_{fin} be the set of all one-forms with finite norm. By the Cauchy inequality, Λ^1_{fin} is a vector subspace of Λ^1.

We define $L^2(X)$ as the completion of Λ^1_{fin} with respect to the norm defined above.

7.3. **Decomposition.** Define $E \subset L^2(X)$ as the closure of the space of df where $f \in \Lambda^0_c(X)$ ($\Lambda^0_c(X)$ is the space of smooth functions with compact support).

Define $E^* = \{\omega | *\omega \in E\}$.

Thus, any element $\omega \in E$ is presented by a sequence $\omega = \lim df_i$ where f_i are smooth functions with compact support.

By the general theory one has

$$L^2(X) = E \oplus E^\perp = E^* \oplus E^{*\perp}.$$

By definition,

$$E^\perp = \{\omega \in L^2 | (\omega, df) = 0 \quad \forall f \in \Lambda^0_c(X)\}$$

and

$$E^{*\perp} = \{\omega \in L^2 | (\omega, *df) = 0 \quad \forall f \in \Lambda^0_c(X)\}.$$

We will deduce now that E and E^* are orthogonal to each other.

7.3.1. **Proposition.** *Let $\alpha \in L^2(X)$ be of class C^1. Then $\alpha \in E^{*\perp}$ iff α is closed and $\alpha \in E^\perp$ iff α is coclosed.*

Proof. Assume α is closed and $f \in \Lambda^0_c$ have support at a compact domain D. Then

$$(\alpha, *df) = -\int_D \alpha \wedge \overline{df} = -\int_D d(\alpha \bar{f}) = -\int_{\partial D} \alpha \bar{f} = 0.$$

Thus, $\alpha \in E^{*\perp}$. Conversely, if $\alpha \in E^{*\perp}$, one has

$$0 = (\alpha, *df) = -\int_D \alpha \wedge \overline{df} = -\int_D (d(\alpha \bar{f}) - d\alpha \bar{f}) = \int_D \bar{f} d\alpha.$$

The latter vanishes for all $f \in \Lambda^0_c(X)$ only if $d\omega = 0$.

The second claim of the proposition follows from the first one, together with Exercise 1 (see below). □

The above proposition immediately implies that the spaces E and E^* are orthogonal. This implies the following

7.3.2. **Lemma.** *There is an orthogonal decomposition*

$$L^2(X) = E \oplus E^* \oplus H$$

where

$$H = E^{\perp} \cap E^{*\perp}.$$

Our next task will be to prove that H is the space of harmonic one-forms. Note that any harmonic form belongs to H and that any form in H of class C^1 is harmonic — by the above proposition. Thus, the only problem is to prove that any element of H is *automatically* of class C^1.

This will be done using the following important Weyl lemma which will be proven later on.

7.3.3. **Theorem** (Weyl's Lemma)**.** *Let D be the unit disc. Let $\omega \in L^2(D)$ be a square-integrable one-form on D. Then ω is harmonic iff*

$$(\omega, df)_D = (\omega, *df) = 0 \qquad (1)$$

for any C^∞-function η with compact support.

We are now ready to prove the following

7.3.4. **Theorem.** *H consists of harmonic one-forms.*

Proof. If ω is harmonic, it is smooth, closed and coclosed. Then by Proposition 7.3.1 $\omega \in H$. Conversely, let $\omega \in H$. We have to prove that ω is smooth. Then it will be smooth, closed and coclosed, therefore, harmonic. The inverse image to D of the restriction $\omega|_U$ satisfies the conditions of Weyl's Lemma, so that $\omega|_D$ is harminic. This implies that ω is smooth which is enough for us. □

7.4. **Weyl's Lemma.** We will now prove Theorem 7.3.3.

First of all, the "only if" part is obvious: a harmonic form is orthogonal to both E and E^*.

The converse is much more difficult. The idea is to "smoothen" and arbitrary L^2-differential so that some important properties are preserved.

7.4.1. Smoothing operators: properties Recall that we live on the disc $D = \{z \in \mathbb{C} |\ |z| < 1\}$.

For each number $\rho \in (0, 1)$ an operator $M_\rho : L^2(D) \to L^2(D)$ will be defined. It will satisfy the following properties

(SM1) For $\omega \in L^2(D)$ one has $M_\rho(\omega) \in C^1(D_\rho)$, where $D_\rho = \{z \in \mathbb{C} |\ |z| < 1-\rho\}$.
(SM2) If ω is harmonic, $M_\rho(\omega) = \omega$ in D_ρ.
(SM3) For $\omega \in L^2(D)$, $\lim_{\rho\to 0} ||\omega - M_\rho(\omega)||_{D_\rho} = 0$.
(SM4) If Supp $\gamma \subset D_\rho$ then Supp $M_\rho(\gamma) \subset D$ and one has

$$(M_\rho(\omega), \gamma)_{D_\rho} = (\omega, M_\rho(\gamma))_D.$$

7.4.2. Construction of smoothing operators

The smoothing operators M_ρ will be first of all defined for L^2-functions, and then extended to L^2-forms by the formula

$$M_\rho(\omega) = M_\rho(f)dx + M_\rho(g)dy$$

for $\omega = fdx + gdy$.

We define a function s_ρ on $\mathbb{C}$ by the formula

$$s_\rho(z) = \begin{cases} k(\rho^2 - |z|^2), & |z| < \rho \\ 0, & |z| \geq \rho \end{cases},$$

where the normalizing constant k is chosen so that

$$\int_{D_{1-\rho}} s_\rho(z) dz \wedge d\bar{z} = 1.$$

and for any complex-valued L^2-function f we define

$$M_\rho(f)(z) = \int_D f(\zeta) s_\rho(z-\zeta) d\zeta \wedge d\bar{\zeta}.$$

The operator M_ρ defined above assigns to each point z (at least for $z \in D_\rho$) the average value of a function f in a disc of radius ρ around z.

This explains especially good properties of $M_\rho(f)$ in D_ρ.

7.4.3. Proof of the properties

See G. Springer, *Introduction to Riemann surfaces*, p. 190–195.

7.4.4. Weyl Lemma: end of the proof

Let ω $L^2(D)$ satisfy the properties $(\omega, df) = (\omega, *df) = 0$ for all $f \in \Lambda^0_c(D)$. Then for all $f \in \Lambda^0_c(D_\rho)$ one has

$$(M_\rho(\omega), df)_{D_\rho} = (\omega, M_\rho(df))_D = (\omega, dM_\rho(f))_D = 0$$

and

$$(M_\rho(\omega), *df)_{D_\rho} = (\omega, M_\rho(*df))_D = (\omega, *dM_\rho(f))_D = 0.$$

since M_ρ commutes with $*$ and with d. Then Proposition 7.3.1 implies that $M_\rho(\omega)$ is harmonic in D_ρ.

We are almost done since $M_\rho(\omega)$ approximate ω; but this is not enough. We will actually prove that $M_\rho(\omega)$ and $M_\sigma(\omega)$ coincide in $D_{\rho+\sigma}$ and his will be enough to deduce equality $\omega = M_\rho(\omega)$ from the estimate of $||\omega - M_\rho(\omega)||$.

By (SM2) one has that $M_\sigma M_\rho(\omega)$ coincides with $M_\rho(\omega)$ in $D_{\sigma+\rho}$ since $M_\rho(\omega)$ is harmonic in D_ρ. Similarly, $M_\rho M_\sigma(\omega)$ coincides with $M_\sigma(\omega)$ in $D_{\sigma+\rho}$. Thus, it is enough to check that $M_\sigma M_\rho = M_\rho M_\sigma$, and we get the required coincindence of $M_\rho(\omega)$ and $M_\sigma(\omega)$ insode $D_{\sigma+\rho}$.

The property $M_\sigma M_\rho = M_\rho M_\sigma$ is a direct result of Fubini theorem.

And now comes the last step. Property (SM3) says that $\lim_{\rho\to 0} ||M_\rho(\omega) - \omega||_{D_\rho} = 0$. One has an obvious inequality

$$||\alpha||_{D_{\sigma+\rho}} \leq ||\alpha||_{D_\rho}$$

just because $D_{\sigma+\rho} \subset D_\rho$. Therefore,

$$\lim_{\rho\to 0} ||M_\rho(\omega) - \omega||_{D_{\sigma+\rho}} = 0.$$

Since $M_\rho(\omega)$ and $M_\sigma(\omega)$ coincide in $D_{\sigma+\rho}$, the latter can be rewriten as

$$\lim_{\rho\to 0} ||M_\sigma(\omega) - \omega||_{D_{\sigma+\rho}} = 0.$$

Note that we are talking about th elimit of a function which increases while $\rho \to 0$. Thus, it is constantly zero.

We have deduced that

$$||M_\rho(\omega) - \omega||_{D_{\sigma+\rho}} = 0$$

for all $\rho > 0$. Thus, $M_\rho(\omega) = \omega$ and the theorem is proven.

7.4.5. **Corollary.** *$E \oplus H$ is the closure in $L^2(X)$ of the space of closed (square integrable) one-forms. Similarly, $E^* \oplus H$ is the closure of the space of coclosed one-forms.*

Proof. Any closed one-form belongs to $E^{*\perp} = E \oplus H$, so the latter contains the closure of the former. Conversely, if $\alpha \in E \oplus H$ then its harmonic component is closed and any element of E is by definition a limit of closed (even exact) forms. The claim for coclosed forms is similar. □

7.4.6. Restriction

Let U be an open subset of X. Restriction of forms defines a map

$$\Lambda^1(X) \longrightarrow \Lambda^1(U)$$

so that $||\omega|| \geq ||\omega|_U||$.

Thus, a map $L^2(X) \to L^2(U)$ is defined. We call it *restriction.*

The restriction of a closed form is closed, so the restriction preserves the spaces $E \oplus H$ and $E^* \oplus H$.

7.5. **One-forms defined by closed curves.** Let γ be a simple closed curve in X. Choose an open neighborhood Ω of γ so that $\Omega - \gamma = \Omega^+ \cup \Omega^-$ is a disjoint union of two annuli. Choose a smaller neighborhood Ω_0 and its two parts $\Omega_o^\pm = \Omega_0 \cap \Omega^\pm$. Choose an orientation of γ so that Ω^- is to the left of γ.

Choose a real-valued function f smooth on $X - \gamma$ such that

$$f|_{\Omega_0^-} = 1, \quad f|_{X-\Omega^-}.$$

We define a one-form η_γ by the formula

$$\eta_\gamma|_{\Omega-\gamma} = df; \eta|_\gamma|_{\Omega_0} = 0.$$

The form η_γ is obviously closed and smooth.

It is called "the one-form associated to γ". Note that it depends on the choice of the function f; however, the difference is an exact form, so the cohomology class is uniquely defined.

7.5.1. **Proposition.** *Let $\alpha \in L^2(X)$ be a closed form of class C^1. Then*

$$\int_\gamma \alpha = (\alpha, *\eta_\gamma).$$

Proof. One has

$$(\alpha, *\eta_\gamma) = -\int_X \alpha \wedge \eta_\gamma = -\int_{\Omega^-} \alpha \wedge df = \int_{\Omega^-} d(f\alpha) - fd\alpha =$$

$$= \int_{\Omega^-} d(f\alpha) = \int_{\partial\Omega^-} f\alpha = \int_\gamma \alpha.$$

□

7.5.2. **Proposition.** *Let $\alpha \in L^2(X)$ be a form of class C^1. Then α is exact (resp., coexact) iff $(\alpha, \beta) = 0$ for any coclosed (resp., closed) $\beta \in \Lambda^1_c(X)$.*

Proof. If α is exact, $\alpha = df$ with $f \in C^2$. Let $d * \beta = 0$, Supp $\beta \subset D$ for a compact domain D. Then

$$(\alpha, \beta) = \int_D df \wedge *\bar{\beta} = \int_D d(f * \bar{\beta} - fd * \bar{\beta}) = \int_{\partial D} f * \bar{\beta} = 0.$$

Conversely, if $(\alpha, \beta) = 0$ for all coclosed compactly supported β, we can apply this to $\beta = \eta_\gamma$ and deduce that

$$\int_\gamma \alpha = 0$$

for all γ. This implies that α is exact (see Exercise 2).

The claim about coexactness of α now follows from the fact that $*$ is an isometry.

□

7.5.3. **Corollary.** *Any form $\omega \in E \cap C^1$ is exact. Any form in $E^* \cap C^1$ is coexact.*

Proof. To prove ω is exact one has to check by 7.5.2 that $(\omega, \beta) = 0$ for any coclosed β with compact support. But we know this since $E^\perp = H \oplus E^*$ contains all coclosed L^2 forms. □

7.5.4. **Corollary.** *Let X be a compact Riemann surface. The first de Rham cohomology of X is isomorphic to the space of harmonic one-forms.*

Proof. Any harmonic form is closed and, therefore, defines a class of $H^1_{DR}(X)$. Since $H \in E^{\perp}$, a nonzero harmonic form on a compact Riemann surface cannot be exact. Therefore, the map $H \to H^1_{DR}(X)$ is injective. The surjectivity is Exercise 3 below. □

7.5.5. Exact forms

The space E is defined as the closure of the space of compactly supported exact forms. Therefore, if X is compact, E contains all exact forms.

This is not true if X is not compact.

Here is a typical example. Let f be a harmonic function in a neighborhood of the closure of the unit disc D. Then (by definition) df is a harmonic differential on D, and therefore cannot belong to E.

Home assignment.

1. Check that the star operation is isometry on $L^2(X)$.
2. Let ω be a closed one-form such that $\int_\gamma \omega = 0$ for any piecewise smooth closed curve γ. Deduce that ω is exact.
3. Prove that the map $H \to H^1_{DR}(X)$ is surjective.