RIEMANN SURFACES

6. Week 7: Differential forms. De Rham complex

6.1. **Introduction.** The notion of differential form is important for us for various reasons. First of all, one can integrate a k-form along a (smooth) k-chain. This, together with the Stokes theorem, allows one to present by differential forms spaces dual to homology. This leads to the notion of de Rham cohomology.

Later on we will find out that de Rham cohomology can be represented by *harmonic differentials*. This allows one to describe the space of holomorphic differentials in the compact case. Thus, we will see that if X is a compact Riemann surface of genus g, the dimension of the space of holomorphic differentials is precisely g. This theory allows to prove the existence of non-constant homolorphic differentials even in non-compact case. AS a result, we will deduce the existence of non-constant meromorphic functions.

6.2. **One-forms.** Informally, a differential form is what can be integrated along a path. In local coordinates, a (smooth) one-form on an open subset U of $\mathbb{R}^n$ is given by an expression

$$\omega=\sum_{i=1}^n f_i dx_i$$

where $f_i\in C^\infty(U)$. If $\gamma:[a,b]\to U$ is a smooth (or piecewise smooth) path $\gamma(t)=(\gamma_1(t),\ldots,\gamma_n(t))$, one defines $\int_\gamma\omega$ by the formula

$$\int_\gamma\omega=\int_a^b(\sum_{i=1}^n f_i(\gamma(t))\gamma_i'(t))dt.$$

Let us find out how an expression for a one-form should behave under a coordinate change. The main (and only) property is, of course, that the value of an intergral $\int_\gamma\omega$ should be invariant under a coordinate change. Thus, if $y_1,\ldots,y_n$ is another coordinate system so that $x=F(y)$ (in vector notation), then

$$\omega=\sum_i f_i(x)dx_i=\sum_{i,j}f_i(F(y))\frac{\partial F_i}{\partial y_j}dy_j.$$

Now we are ready to give a general definition.

6.2.1. **Definition.** A one-form ω on a smooth manifold X is a collection of expressions

$$\omega^\phi=\sum_i f_i(x)dx_i$$

for each local coordinate system $\phi : U \longrightarrow \mathbb{R}^n$ such that the expressions are compatible on the intersections in the sense described above.

In order to define a one-form it suffices to define a collection of one-forms ω_i on a collection of charts $\phi_i : U_i \longrightarrow \mathbb{R}^n$ such that the charts U_i cover the whole manifold and the forms ω_i are compatible on the intersections.

Let us give a typical example.

6.2.2. **Example.** Let $X = \mathbb{R}/\mathbb{Z}$ be the circle. It can be covered by two charts, with the coordinates different by a constant. Thus, any expression of form $f(x)dx$ is a one-form, where f is a smooth function having period 1.

6.2.3. **Example.** Let $X = \mathbb{C}/L$ be an elliptic curve, $L = \mathbb{Z} \oplus \mathbb{Z} \cdot \tau$. The one-forms dx and dy make sense on X.

6.2.4. Differential of a function

If f is a smooth function on X, one defines a one-form df by the formula in local coordinates

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

We will extend later on this formula to all differential forms.

Note that the local formula above gives automatically a compatible family of forms on each local chart:

$$\sum \frac{\partial f}{\partial x_i} dx_i = \sum \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial y_j} dy_j = \sum \frac{\partial f}{\partial y_j} dy_j.$$

Sometimes any one-form is a differential of a function. This is so, for example, for $X = \mathbb{R}$ (existence of integration). Sometimes this is not true.

For instance, one can prove (see Exercise) that $f(x)dx$ is a differential of a function iff $\int_0^1 f(x)dx = 0$.

6.2.5. Restriction

A one-form on a manifold X defines automatically a one-form on any its open subset. All meaningful operations with the differential forms will be compatible with this restriction operation. For instance, if f is a function on X, one can construct a one-form on $U \subset X$ in two seemingly different ways: the one is to restrict the differential form df from X to U, and the second is to restrict the function f to U, and after that to take the differential. It is clear that the two ways give the same result. This means that the operation $f \mapsto df$ *commutes with restrictions.*

Restriction is a special case of a more general operation of *inverse image*, see below.

6.2.6. Inverse image of a one-form

Let $f : X \to Y$ be a smooth map and let ω be a one-form on Y. We will now define a form $f^*(\omega)$ on X called *the inverse image of* ω.

It is enough to write down formulas in local coordinates. Let $f : U \to V$ be a map from an open subset of $\mathbb{R}^m$ with coordinates $x_1, \ldots, x_m$, to an open subset of $\mathbb{R}^n$ with coordinates $y_1, \ldots, y_n$. Let a form ω be given by the local expression

$$\omega = \sum a_i(y) dy_i.$$

Then $f^*(\omega)$ is given by the formula

$$f^*(\omega) = \sum_{i,j} a_i(f(x)) \frac{\partial f_i}{\partial x_j} dx_j. \tag{1}$$

In order to make sure that the formulas above define a one-form on X, one has to check that the local expressions of $f^*(\omega)$ are compatible at the intersections of the charts. This easily follows from the Chain rule.

6.3. **Higher forms.** In order to integrate along higher-dimensional simplices, we need *higher degree forms.*

Locally (that is, in a coordinate chart) a degree k form is just a product of k one-forms. One has, however, take into account that the product of one-form is not commutative, but rather anticommutative!

In local coordinates any k-form is an expression of type

$$\omega = \sum_{i_1 < \ldots < i_k} a_{i_1, \ldots, i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

where $a_{i_1, \ldots, i_k}$ are functions of $x_1, \ldots, x_n$.

The collection of k-forms on an open subset U of $\mathbb{R}^n$ will be denoted $\Lambda^k(U)$. It makes sense to look at functions on U as 0-forms.

Remark. We do not specify at the moment what types of functions are allowed to be coefficients of differential forms. Different choices make sense.

The collection of differential forms on U has a rich structure.

6.3.1. Product

One can multiply a k-form ω to an l-form ω' to get a $k+l$-form $\omega \wedge \omega'$. The formulas can be uniquely reproduced once we know that 0-forms multiply as the functions, 0-forms commute with 1-forms, and two 1-forms anticommute.

One easily deduces from the above that

$$\omega' \wedge \omega = (-1)^{kl} \omega \wedge \omega'.$$

6.3.2. Change of coordinates

In order to be able to define k-forms on a manifold, we have to explain what happens to k-forms under the change of coordinates.

Let $x = f(y)$ define another coordinate system (that is $y = (y_1, \ldots, y_n)$ and f consists of n functions of the variables $y_1, \ldots, y_n$.

We know what happens under the coordinate change to 0-forms (nothing, $\omega(x)$ becomes $\omega(f(y))$), and what happens to 1-forms ($\omega = \sum a_i(x)dx_i$ becomes $\sum_{i,j} a_i(f(y))\frac{\partial f_i}{\partial y_j}dy_j$). Thus, it is enough to add that the product of differential forms is preserved under the base change, and the rest of the formulas can be reconstructed.

Remark. We omit some important details here. The last phrase should be understood as follows. *If we wish that our coordinate change formulas be compatible with the product, we have no choice: any k-form is a product of k one-forms, and so to rewrite it in another coordinate system one has to rewrite the factors and to multiply the results.* However, the existence of coordinate change formulas should be checked.

It is worthwhile to mention separately the coordinate change formula for highest degree forms.

If X is a manifold of dimension n, there are no forms of degree $> n$. The n-forms write in local coordinates as

$$\omega = a(x)dx_1 \wedge \ldots \wedge dx_n.$$

If now $x = f(y)$ is a coordinate change, we have in new coordinates

$$\omega = a(f(y))J(f)dy_1 \wedge \ldots \wedge dy_n,$$

where $J(f)$ is the jacobian (=the determinant ofthe Jacobi matrix) of f.

Note that Riemann surfaces are manifolds of dimension 2, so we need nothing except for 0, 1 and 2-forms.

6.3.3. De Rham differential

The map $d : \Lambda^0(U) \to \Lambda^1(U)$ has been already defined. We claim it extends uniquely to a commection of maps $\Lambda^k(U) \to \Lambda^{k+1}(U)$ if one requires the following properties

- $d(dx_i) = 0$.
- (Leibniz rule) $d(\omega_1 \wedge \omega_2) = d(\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge d(\omega_2)$ for $\omega_1 \in \Lambda^k(U)$.

Uniqueness of the differential satisfying the above properties is clear: if $\omega = a(x)dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ then necessarily

$$d\omega = d(a) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \qquad (2)$$

The existence of the map d satisfying the above properties is less obvious.

Here is the list of properties one has to check.

- That if one defines $d\omega$ by the formula (2), the Leibniz rule is satisfied.

- That the differential of a differential form on U defined as above, is compatible with the coordinate change.

6.4. **De Rham complex. De Rham cohomology.** We have constructed above a collection of vector spaces $\Lambda^k(X)$ and a collection of linear maps

$$d : \Lambda^k(X) \to \Lambda^{k+1}(X).$$

Leu us check that $d \circ d = 0$.

6.4.1. **Lemma.** *For any $k \geq 0$ the composition*

$$d^2 : \Lambda^k \to \Lambda^{k+1} \to \Lambda^{k+2}$$

vanishes.

Proof. It is sufficient to check the claim in local coordinates. Moreover, for $\omega = adx_{i_1} \wedge \ldots \wedge dx_{i_k}$ we have by Leibniz rule and by $ddx_i = 0$ that

$$dd\omega = ddadx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

Thus, we have to check only that $dda = 0$. Here we have

$$dda = d(\sum \frac{\partial a}{\partial x_i} dx_i) = \sum_{i,j} \frac{\partial^2 a}{\partial x_i \partial x_j} dx_i \wedge dx_j.$$

The latter expression vanishes since $dx_i \wedge dx_j = -dx_i \wedge dx_i$.

□

6.4.2. **Definition.** Let X be a manifold. De Rham cohomology of X is a collection of groups $H^k_{\mathrm{DR}}(X) := \mathrm{Ker}(d : \Lambda^k \to \Lambda^{k+1}) / \mathrm{Im}(d : \Lambda^{k-1} \to \Lambda^k)$, $k = 0, \ldots, n$

6.4.3. 0th cohomology

It is easy to calculate the 0th cohomology. This is the vector space of functions f on X satisfying the condition $df = 0$. This means that all partial derivatives of f vanish, that is f is constant (at least, at each connected component).

We have got the following result.

$H^0_{\mathrm{DR}}(X)$ is $\mathbb{R}$ for a connected manifold X.

The following *Poincaré lemma* is very important.

6.4.4. **Proposition.** *Let U be an open disc in $\mathbb{R}^n$. Then $H^i_{\mathrm{DR}}(U) = 0$ for $i > 0$.*

6.5. **Inverse image.** If $f : X \to Y$ is a smooth map and $\omega \in \Lambda^k(Y)$, one can define $f^*(\omega) \in \Lambda^k(X)$ satisfying the following properties.

- f^* is compatible with restrictions on open subsets.
- For $\omega \in \Lambda^0(Y)$ $f^*(\omega) = \omega \circ f$.
- $f^*(d\omega) = d(f^*(\omega))$.
- $f^*(\omega_1 \wedge \omega_2) = f^*(\omega_1) \wedge f^*(\omega_2)$.

As everything in this section, it is much easier to prove uniqueness of f^* satisfying the above property that to check the existence.

We have already f^* on Λ^0. By the third property, $f^*(dy_i) = \sum \frac{\partial f_i}{\partial x_j} dx_j$. This basically yields the formula (1). Then, since f^* preserves the product, we can write down a general formula.

The above reasoning proves the uniqueness. As usual, an effort is needed to prove the existence of f^*.

6.6. **Integration.** Recall that a standard n-simplex is the collection of points $(t_0, \ldots, t_n) \in \mathbb{R}^{n+1}$ satisfying the equation $\sum t_i = 1$ and the inequalities $t_i \geq 0$.

A smooth singular k-simplex in X is a smooth map of a (neighborhood of) the standard simplex Δ^k to X[1]. A k-chain in X is an integral linear combination of singular k-simplices.

Let $\gamma = \sum a_i \gamma_i$ be a (smooth singular) k-chain in X. We define the integral

$$\int_\gamma \omega \in \mathbb{R}$$

as follows. First of all, the integral will be linear in γ, so one can assume that γ is a k-simplex in X. Consider $\gamma : \Delta^k \to X$. The inverse image $\gamma^*(\omega)$ is a k-form on Δ^k (more precisely, on an open neighborhood of Δ^k in $\mathbb{R}^k$). Thus, one can integrate $\gamma^*(\omega)$ along Δ^k. Thus, we define

$$\int_\gamma \omega = \int_{\Delta^k} \gamma^*(\omega).$$

6.6.1. Remarks

1. Even though we defined integration along parametrized simplices $\gamma : \Delta^k \to X$, there is nothing special in the triangular form of Δ^k which forces the definition to work. We could have used any meaningful compact subset of $\mathbb{R}^k$ instead fit for multiple integration; a cube, for instance.
2. Note that the definition of integration is cooked up so that the result does not depend on reparametrization of a singular simplex: if $\gamma' = \gamma \circ \Phi$ for a diffeomorphism $\Phi : \Delta^k \to \Delta^k$ then $\int_{\gamma'} \omega = \int_\gamma \omega$. This is a fact we have already known before (integral along a curve is independent of the parametrization of the curve).
3. Taking into account the above remarks we will allow ourselves to talk about "integrals over a region" without specifying how to parametrize the region and/or how to divide it into simplices.

6.6.2. Stokes We are now ready to formulate Stokes theorem which claims that

[1]A neighborhhod of Δ^n is described by the equation $\sum t_i = 1$ together with the inequalities $t_i > -\epsilon$ for some $\epsilon > 0$

the de Rham differential on forms is compatible with passage to the boundary of a simplex.

Define for a k-simplex $\gamma : \Delta^k \to X$ its boundary $\partial\gamma$ by the formula

$$\partial\gamma = \sum_{i=0}^{k} (-1)^k \gamma \circ \partial^i$$

where $\partial^i : \Delta^{k-1} \to \Delta^k$ is given by the formula

$$\partial^i(t_0, \ldots, t_{k-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}).$$

The map $\gamma \mapsto \partial\gamma$ extends by linearity to a map

$$\partial : C_k(X) \longrightarrow C_{k-1}(X)$$

where in this section we denote by $C_k(X)$ the abelian group of smooth singular k-chains in X.

6.6.3. **Theorem** (Stokes theorem)**.** *For any $\gamma \in C_k(X)$ and $\omega \in \Lambda^{k-1}(X)$ one has*

$$\int_\gamma d\omega = \int_{\partial\gamma} \omega.$$

6.6.4. Complex-valued functions and forms A complex-valued function $f : X \to \mathbb{C}$ can be uniquely expressed as $f = u + iv$ where u, v are real-valued functions. This allows one to extend everything said above to complex-valued functions allowing, of course, differential forms $\sum f_i dx_i$ where f_i are complex-valued functions.

Similarly to the above, one defines complex-valued forms by pairs of real-valued forms. We extend the notion of a differential of a complex-valued function by the formula $d(u + iv) = du + idv$ and the rest of the theory extends immediately.

The space of complex-valued k-forms admits an operation of *complex conjugation*: one defines $\overline{\omega + i\omega'} = \omega - i\omega'$ where $\omega,\ \omega'$ are real k-forms. Complex conjugation commutes with the de Rham differential.

6.7. Riemann surface peculiarities.

6.7.1. dz and $d\bar{z}$.

If X is a Riemann surface, we use the standard notation $\Lambda^k(X)$ for complex-valued k-forms. X is two-dimensional, so we have $\Lambda^i(X) = 0$ for $i > 2$. Note that in local coordinates

$$dz = dx + idy \text{ and } d\bar{z} = dx - idy.$$

This allows one to rewrite any one-form as a linear combination of dz and $d\bar{z}$:

$$fdx + gdy = \frac{1}{2}(f(dz + d\bar{z}) - ig(dz - d\bar{z})) = \frac{f - ig}{2} dz + \frac{f + ig}{2} d\bar{z}.$$

Similarly for two-forms

$$dx \wedge dy = -\frac{i}{4}(dz + d\bar{z})(dz - d\bar{z}) = \frac{i}{2} dz \wedge d\bar{z}.$$

6.7.2. Operators ∂ and $\bar{\partial}$

Define the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ by the formulas

$$f'_z = \frac{1}{2}(f'_x - i f'_y), \quad f'_{\bar{z}} = \frac{1}{2}(f'_x + i f'_y).$$

Then one has

$$df = f'_z dz + f'_{\bar{z}} d\bar{z}.$$

Define the operators $\partial, \quad \bar{\partial} : \Lambda^0(U) \to \Lambda^1(U)$ by the formulas

$$\partial f = f'_z dz, \quad \bar{\partial} f = f'_{\bar{z}} d\bar{z}.$$

We will see in a while that these formulas are compatible with the holomorphic change of coordinates, and so define meaningful operators for functions on a Riemann surface.

Note that $df = \partial f + \bar{\partial} f$ and that

$$\partial^2 = \bar{\partial}^2 = \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0.$$

First of all let us mention that by Cauchy-Riemann equations a function $f = u + iv$ is holomorphic iff

$$f'_{\bar{z}} = \frac{1}{2}(f'_x + i f'_y) = \frac{1}{2}(u'_x - v'_y + i(u'_y + v'_x)) = 0.$$

Let now $z(w)$ be a *holomorphic* coordinate change. Then

$$f'_z dz = f'_z(z'_w dw + z'_{\bar{w}} d\bar{w}) = f'_w dw.$$

This means that the operator ∂ is compatible with the coordinate change; therefore, it is an operator on the functions on the whole Riemann surface.

Note that a function $f \in C^\infty(X)$ is holomorphic iff $\bar{\partial}(f) = 0$.

6.7.3. Two-forms

A two-form is written locally as $\omega = f dz \wedge d\bar{z}$. If not $z = z(w)$ is a holomorphic coordinate change, ω is rewritten in the new coordinates as

$$\omega = f dz \wedge d\bar{z} = f |z'_w|^2 dw \wedge d\bar{w}.$$

6.7.4. Star operation We will now define a new operation on one-forms. Note that in general this operation assigns to a k-form an $n-k$-form where n is the dimensin of the manifold X. In our special case $n = 2$ and $k = 1$ so that we get an operation on smooth 1-forms.

We define an operator
$$*:\Lambda^1(X)\to\Lambda^1(X)$$
by the following formula.

Let $\omega = udz + vd\bar{z}$ in local coordinates, for smooth functions u, v. Then $*\omega = -iudz + ivd\bar{z}$.

One can easily check that the formula is compatible with the holomorphic coordinate change (See Exercise 2), so the operator $*$ is defined on one-forms. Note that $** = -1$.

6.7.5. Closed forms, coclosed forms etc.

A form $\omega \in \Lambda^1(X)$ is *closed* if $d\omega = 0$; it is *exact* if $\omega = df$.

A form ω is coexact or coclosed if $*\omega$ is exact or closed respectively.

Since $d^2 = 0$, any exact form is closed, as well as any coexact form is coclosed.

Let us define a linear map *(the Laplace map)*
$$\Delta : \Lambda^0(X) \longrightarrow \Lambda^2(X)$$
by the formula
$$\Delta(f) = d * d(f).$$
Let us make a calculation in local coordinates.
$$\Delta(f) = d * d(f) = d * (f'zdz + f'_{\bar{z}}d\bar{z}) = d(-if'_z dz + if'_{\bar{z}}d\bar{z}) = 2if''_{z\bar{z}}dz \wedge d\bar{z}.$$

As shows Exercise 3, one has a well-known formula
$$\Delta(f) = (f''_{xx} + f''_{yy})dx \wedge dy.$$

A function f is called *harmonic* if $\Delta(f) = 0$.

6.7.6. Harmonic and holomorphic differentials

A one-form ω is *harmonic* if for any point $x \in X$ there exists a neighborhood U containing x and a function f on U such that $\omega|_U = df$.

We will now give another characterisation of harmonic one-forms.

Proposition. *A one-form ω is harmonic iff it is closed and coclosed simultaneously.*

Proof. If ω is harmonic, it is df locally, so $d\omega = ddf = 0$ locally, and therefore, globally. Thus, harmonic forms are closed. Let us calculate $d * \omega$. Once more, locally one has $d * \omega = d * d(f) = 0$ since f is harmonic. Therefore, harmonic forms are coclosed.

Finally, let $d\omega = 0$. By Poincaré lemma this implies that in any open disc U $\omega = df$. Then $d * df = d * \omega = 0$ if ω is coclosed, so f is harmonic. $\square$

Definition. A one-form is *holomorphic* if it is presented locally as df where f is a holomorphic function.

Lemma. *A form $\omega = udz + vd\bar{z}$ is holomorphic iff $v = 0$ and u is a holomorphic function. If f is a harmonic function, ∂f is a holomorphic differential.*

Proof. These are very elementary claims. If $\omega = df$ with f holomorphic, $\omega = f'_z dz$ and f'_z is as well holomorphic. Moreover, any holomorphic function on a disc is a derivative of another holomorphic function, so the converse is also true. Now, if f is harmonic, f'_z is holomorphic since $f''_{z\bar{z}} = 0$. This proves the last claim. □

And finally

Theorem. *A one-form ω is holomorphic iff $\omega = \alpha + i * \alpha$ for some harmonic one-form α.*

Proof. Let $\omega = fdz + gd\bar{z}$. We have

$$d\omega = (-f'_{\bar{z}} + g'_z)dz \wedge d\bar{z}$$

and

$$d * \omega = (if'_{\bar{z}} + ig'_z)dz \wedge d\bar{z}.$$

Thus, ω is harmonic iff f is holomorphic and g is anti-holomorpihc (that is, $\bar{g}$ is holomorphic).

Now we can easily deduce the claim of the theorem. If ω is holomorphic, it is harmonic and $*\omega = -i\omega$, so that for instance $\alpha = \frac{1}{2}\omega$ furnishes the required presentation for ω. Vice versa, if $\omega = \alpha + i * \alpha$ with harmonic α, we have locally

$$\alpha = fdz + gd\bar{z}, \quad i * \alpha = fdz - gd\bar{z}$$

with holomorphic f, so $\omega = 2fdz$ and we are done.

□

Home assignment.

1. Prove that a one-form $f(x)dx$ on $S^1 = \mathbb{R}/\mathbb{Z}$ is a differential of a function iff $\int_0^1 f(x)dx = 0$. Deduce from this that $H^1(S^1) = \mathbb{R}$.
2. Prove that the $*$-operation on one-forms commutes with holomorphic coordinate change.
3. Check the formula

$$\Delta(f) = (f''_{xx} + f''_{yy})dx \wedge dy.$$

4. Prove that if $*\omega = -i\omega$ and ω is closed then ω is holomorphic.