RIEMANN SURFACES

3.4. **Uniformization theorem: Advertisement.** The following very important theorem firstly formulated by Riemann has a tremendous impact on the theory. We will not prove it in this course.

3.4.1. **Theorem.** *Let X be a simply connected Riemann surface. Then either $X = \widehat{\mathbb{C}}$ (the only compact case), or $X = \mathbb{C}$, or $X = \mathbb{H}$ (the upper half-plane).*

The three simply-connected RS listed above are obviously non-isomorphic. In fact, the first one is compact while the two others are not; the third one admits a (global) holomorphic bounded nonconstant function, while $\mathbb{C}$ does not admit.

Taking into account the existence of universal covering and the passage of RS structure to a covering, one immediately gets the following

3.4.2. **Corollary.** *Any Riemann surface is isomorphic to a quotient of X, one of the listed above simply connected RS, modulo a subgroup $G \subset Aut(X)$ of the holomorphic authomorphism group of X acting freely on X.*

Thus, it seems that the task of classification of all Riemann surfaces is not really difficult. One has to describe the authomorphism groups of the simply-connected Riemann surfaces, and then to describe discrete subgroups acting freely on it.

The group $Aut(X)$ can be easily described in all three cases.

3.4.3. $X = \widehat{\mathbb{C}}$ Authomorphisms of the Riemann sphere are always given by Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$. Since the scalar matrices define the identity transformation, the authomorphism group of $\widehat{\mathbb{C}}$ is $PGL(2,\mathbb{C}) = GL(2,\mathbb{C})/\mathbb{C}^*$.

3.4.4. $X = \mathbb{C}$

The authomorphisms are Möbius transformations preserving ∞. These are transformations of tyle $z \mapsto az + b$ with $a \neq 0$.

3.4.5. $X = \mathbb{H}$

The authomorphisms are Möbius transformations with real coefficients. They form the group $PGL(2,\mathbb{R}) = GL(2,\mathbb{R})/\mathbb{R}^*$.

The task is easy for $X = \widehat{\mathbb{C}}$ and $X = \mathbb{C}$.

3.4.6. Subgroups In the first two cases the problem of describing discrete subgroups of $Aut(X)$ acting freely on X is easy.

If $X = \widehat{\mathbb{C}}$ there are no nontrivial subgroups with this property.

If $X = \mathbb{C}$, there are subgroups of shifts $z \mapsto z + b$ with one or two non-collinear generators. In the first case we get cyliner as a quotient, in the second case we get an elliplic curve (as we remember, different elliptic curves correspond to different choices of the lattice).

The case $X = \mathbb{H}$ is much more difficult. Most Riemann surfaces have $\mathbb{H}$ as a universal covering space.

4. Week 4-5. Topological structure of RS. Continuation

We wish to find a simple combinatorial description of compact two-dimensional manifolds.

4.1. **Simplicial complexes.**

4.1.1. An abstract simplicial complex is a purely combinatorial notion. It consists of

- A set V (called the set of vertices).
- A collection S of finite subsets of V (called the set of simplices)

satisfying the following properties

- $\{x\} \in S$ for all $x \in V$.
- $\sigma \in S, \quad \tau \subset \sigma$ implies $\tau \in S$.

Let (V, S) be a simplicial complex. Dimension of a simplex $\sigma \in S$ is $\dim \sigma = |\sigma| - 1$ (so that vertices are 0-simplices). A simplex of dimension n is also called an n-simplex.

4.1.2. Geometric realization and triangulation

Let (V, S) be an abstract simplex. We can construct a topological space called *the geometric realization of* (V, S) as follows.

First of all, we assign to each n-simplex $\sigma = (x_0, \ldots, x_n)$ the geometric simplex

$$|\sigma| = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | t_i \geq 0, \sum t_i = 1\}.$$

Such geometric simplex has $n + 1$ vertices given by the coordinate units of $\mathbb{R}^{n+1}$. We number them by the vertices x_i of σ.

Once more, a geometric realization of an abstract simplex σ is the topological space $|\sigma|$ with the vertices identified with x_i.

Note that if $\tau \subset \sigma$, we can identify $|\tau|$ with $|\sigma|$ taking care that the vertices of $|\tau|$ are identified with the corresponding vertices of $|\sigma|$.

Finally, to define the geometric realization of (V, S) we construct geometric realizations of each simplex from S, and identify them along subsimplices. We get as a result a topological space glued from geometric simplices.

4.1.3. **Example.** Assume $V = \{x, y, z\}$ and S consists of all subsets of cardinality ≤ 2. The realization will have three vertices and three edges. As a topological space this is a circle. If V consists of four tpoints and S of all subsets of cardinality ≤ 3, we will get a sphere.

4.1.4. **Definition.** A triangulation of a topological space X is a homeomorphism between X and a realization of a simplicial complex. X is called trangulated if it admits a triangulation.

"Good" topological spaces are triangulated. We will use in this section that all two-dimensional manifolds are triangulated. This fact is not absolutely obvious. For Riemann surfaces we will be able to deduce it later from the existence of meromorphic functions.

4.2. **Triangulation of oriented two-dimensional manifolds.** We assume without proof that any Riemann surface can be triangulated. Our aim is to find the simplest possible triangulation.

Let X be a simplicial complex homeomorphic to a compact oriented two-dimensional manifold. X is finite. It has simplices of dimensions $0, 1, 2$. We will call them vertices, edges and triangles respectively.

Some obvious properties of X:

- Any edge belongs to two triangles.
- Any two triangles a, b can be connected by a sequence $a = a_0, \ldots, a_n = b$ such that a_{i-1} and a_i have a common edge.

Some less obvious properties will be formulated later.

Let us construct a graph G whose vertices are the triangles of X and an edge between vertices exists if the corresponding triangles have a common edge. G is a degree three graph.

Choose a maximal subtree in G. It contains all vertices of G, that is all triangles. It describes a process of constructing a simplicial complex, each time guluing a new triangle along an edge. As a result we will have a simplicial complex Y satisfying the following properties:

- It contains all the triangles of X.
- Y is a polygon.
- X is obtained from Y by identification of some of its edges.

Since the resulting simplicial complex has no boundary, all edges of the polygon Y are glued pairwise. Thus, Y is a $2n$-gon for some n. All $2n$ edges of Y are divided into pairs and X is obtained by identification of these pairs.

Let us try to find out a way of keeping all necessary information. We know which pairs of edges have to be glued. There are, however, two different ways of gluing a pair of edges, identifying them in different directions.

Let us choose an orientation of each edge so that the edges to be glued have the same orientation. Choose n letters to denote $2n$ edges so that the edges to

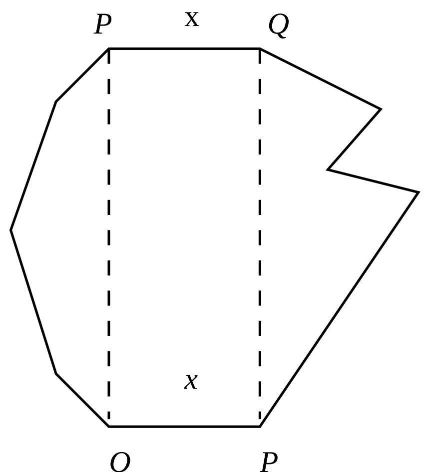

FIGURE 1. The rectangle $PQPQ$ makes up Möbius band after pasting

be glued get the same letter. Let us move along the boundary of the polygon Y; when we pass an edge, write its letter if the orientation of the edge coincides with that of the movement, and its inverse in the opposite case. We get finally a word in the alphabet $x_1, \ldots, x_n$ so that each letter $x_i,\ i = 1, \ldots, n$ (or its inverse) appears twice.

We claim that, since X is oriented, each letter a appears twice in the characteristic word, once as a, and once as a^{-1}. In fact, if two edges marked by a contribute the same sign into the characteristic word, the surface X contains a Möbius band (see Figure 1), and, therefore, cannot be oriented.

The word obtained as above will be called *a characteristic word.* This word defines uniquely the way X is glued out of Y. There are, however, different ways to glue up homeomorphic surfaces. We wish to find a "canonical" presentation.

4.3. **Canonical presentation of a surface.** Even though this will not contain a new information, we will mark the vertices of Y so that the vertices get the same marking if and only if they correspond to the same vertex of X.

4.3.1. Step 1 If two consecutive letters of the characteristic word are x and x^{-1}, and these are not the only letters appearing in the word, one can erase them without changing the topological type of the polygon. Topologically this means that we glue together the edges a and a^{-1} of Y, and obtain a new polygon whose edges do not contain the letter a. Note that the vertex P lying between the edges x and x^{-1} is the only vertex marked by P — otherwise gluing would not produce a manifold, see 2.

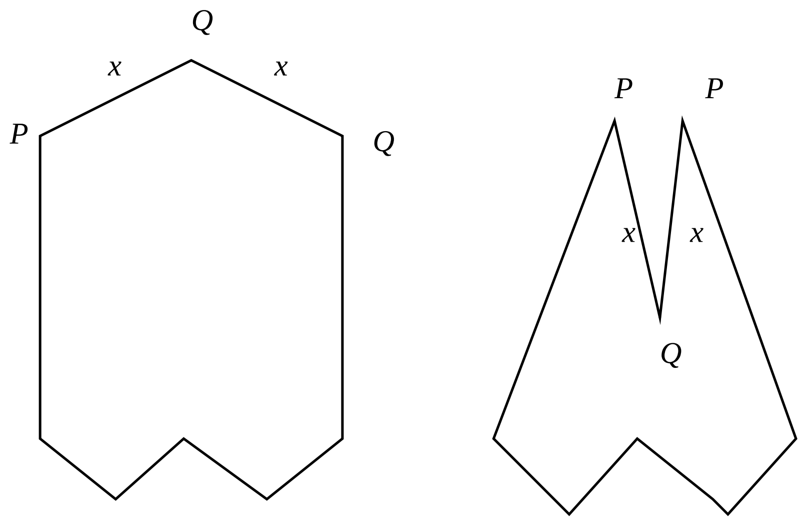

FIGURE 2. Gluing a with a^{-1}

4.3.2. Step 2 The next reduction with leave only one equivalent class (=marking) of vertices.

For if there are two vertex markings, $P \neq Q$ on an edge x, the following procedure diminishes by one the number of appearances of Q. We cut off the triangle RQP on the top of the picture below and paste it to the other appearance of the edge y as in Figure 3. In this way we get another polygon which would give the same X after gluing; the number of appearances of the vertex Y is now less that before.

We assume now that all vertices of Y are covered by the same color (that means that X has only one vertex).

A pair of letters x, y is *linked* if the appear in the order $x, \dots, y, \dots, x^{-1}, \dots, y^{-1}$.

4.3.3. Step 3 This step produces a linked pair of letters going together, so that the corresponding segment of the characteristic word becomes

$$x, y, x^{-1}, y^{-1}.$$

We proceed similarly to Step 2. Each our move will consist of a choice of a pair of vertices, say, P and Q; cutting Y along a diagonal connecting P with Q; pasting two halves along a common edge to get a new polygon. The new polygon so obtained corresponds to a homeomorphic surface. We need two such transformations to put the edges x, y, x^{-1}, y^{-1} together, see Figure 4. Let P and Q be initial vertices of x and x^{-1} (see Picture 4 below). Our first transformation is to cut Y along the diagonal PQ (call the new edges thus obtained z and z^{-1}) and paste along the edges y, y^{-1}. The second transformation make another cut connecting the ends of two edges marked z, and pastes the two halves along x.

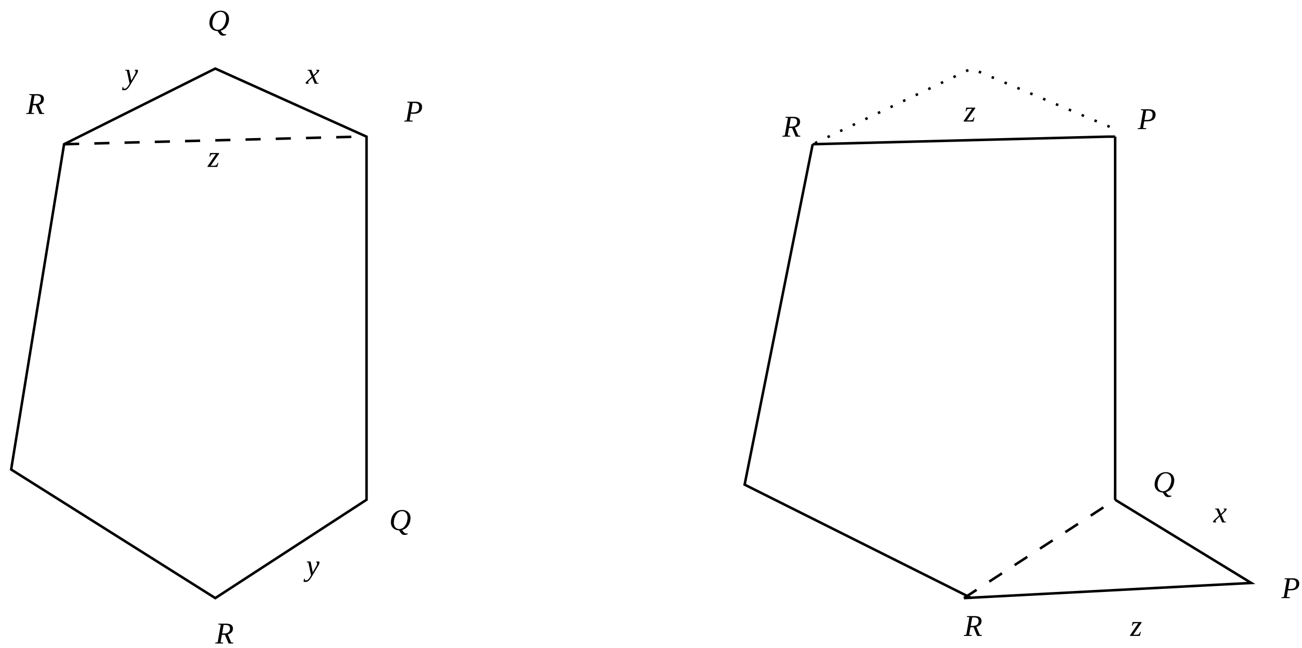

FIGURE 3. Reducing the vertices

We get a new polygon with no edges marked x, y, and with new edges marked z, w linked together forming a segment of characteristic word of form $zwz^{-1}w^{-1}$. We are done.

We have therefore proven the following important theorem.

4.3.4. **Theorem.** *Any compact oriented two-dimensional surface can be presented by pasting pairs of edges in a polygon with a characteristic word which is either*

$$xx^{-1}$$

or

$$x_1y_1x_1^{-1}y_1^{-1}\cdots x_gy_gx_g^{-1}y_g^{-1}.$$

In the first case the surface is called *genus* 0 *surface*, and in the second case it is a *genus* $g > 0$ *surface.*

Of course, genus 0 surface is just a sphere, genus one surface is a torus.

4.4. **Van Kampen theorem.** The following result is extremely efficient if you want to calculate the fundamental group of your topological space. We will first formulate the result, then explain its meaning, and only after that will prove it.

Theorem. *(Van Kampen theorem, imprecise form) Let* $A \xleftarrow{f} C \xrightarrow{g} B$ *be a diagram of connected topological spaces and let* X *be obtained by gluing* A *and* B *along* C*. Then the fundamental group* $\pi_1(X)$ *is isomorphic to the amalgam of the diagram* $\pi_1(A) \longleftarrow \pi_1(C) \longrightarrow \pi_1(B)$.

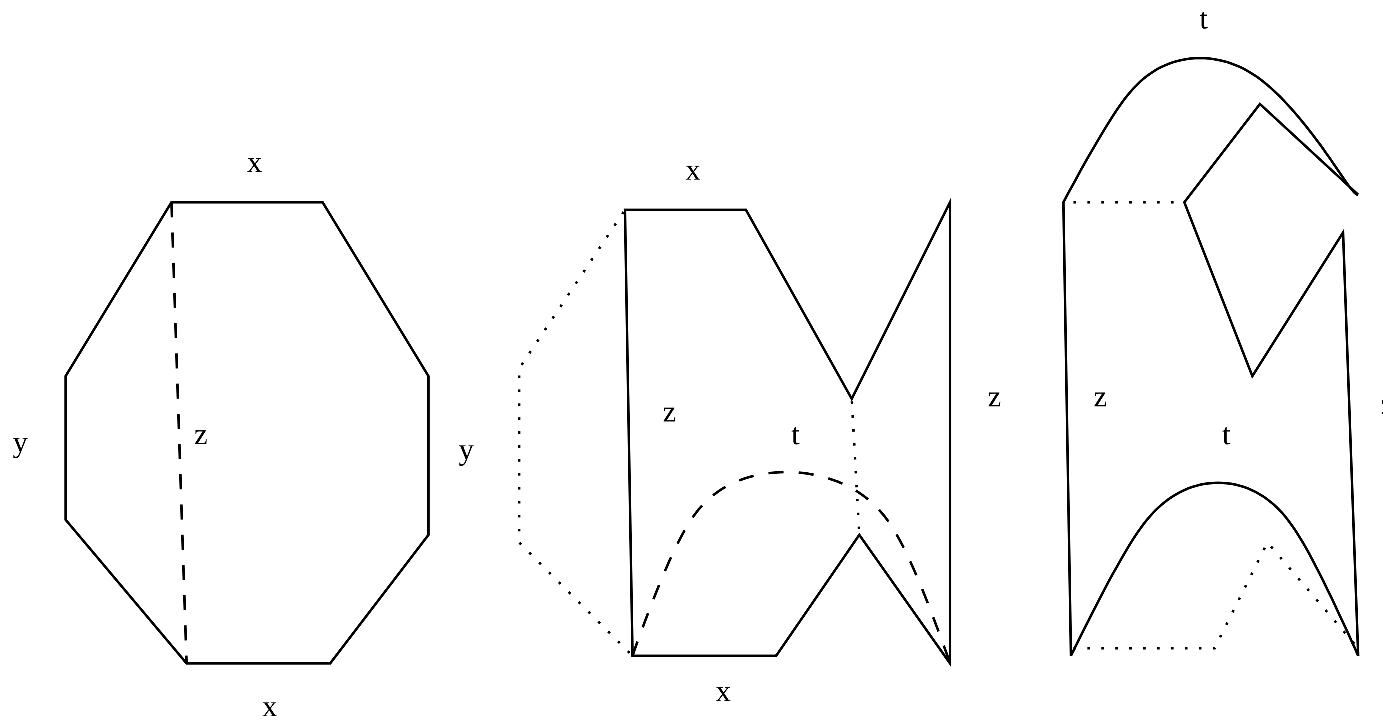

FIGURE 4. Two transformations

First of all, let us explain what is gluing. By definition, X is the quotient of the disjoint union $A \sqcup B$ by the equivalence relation generated by the condition $f(c) \sim g(c)$.

Here are some nice examples.

If C is a point, X is a *wedge* of A and B, denoted $X = A \vee B$. For instance, any one-dimensional simplicial complex is homotopy equivalent to a wedge of circles (prove that!).

If A and B are discs, and C is the circle mapping as the boundary into both A and B, X will be a two-sphere.

In some cases the result of gluing can be a nasty topological space. Note that the theorem formulated above is not completely correct (see below a possible correction).

Let us explain the notion of amalgam (=colimit) of a diagram of groups. First of all, if G and H are groups, one defines their *free product* $G \sqcup H$ as the collection of (finite) words $x_1 \cdots x_n$ such that x_i belong to G or to H, $x_i \neq 1$ and no two consecutive letters belong to the same group. The empty word ($n = 0$) is the unit of the new group; one multiplies words by concatenation followed by, if required, multiplication of the neighboring letters belonging to the same group.

Fir example, free product of n copies of $\mathbb{Z}$ yields the free group on n generators.

Now, given a diagram $G \xleftarrow{f} K \xrightarrow{g} H$ of groups, their amalgam is defined as the quotient of the free product $G \sqcup H$ by the normal subgroup generated by the elements $f(x)g(x)^{-1}$, $x \in K$.

Let us show how the above theorem allows one to calculate fundamental groups.

First of all, recall that $\pi_1(S^1) = \mathbb{Z}$. This can be explained, for example, by the existence of an isomorphism $S^1 = \mathbb{R}/\mathbb{Z}$. Using the theorem we deduce that $\pi_1(S^1 \vee S^1) = \mathbb{Z} \sqcup \mathbb{Z}$, and more generally, $\pi_1(S^1 \vee \ldots \vee S^1)$ (the wedge of n circles) is the free group in n generators.

4.4.1. The fundamental group of a compact oriented surface

Let A be the wedge of $2g$ circles, denoted $x_1, \ldots, x_g, y_1, \ldots, y_g$. Let B be a disc, and let C be a circle. The map $g : C \to B$ is just the embedding of the circle as the boundary of the disc. The map $f : C \to A$ is the path in the wedge of $2g$ circles described by the word $x_1 y_1 x_1^{-1} y_1^{-1} \ldots y_g^{-1}$: it goes first along the circles x_1 and y_1 in the positive direction, then alomg x_1 and y_1 in the opposite direction, then does the same with the other pairs of circles.

Looking carefully at the picture, one deduces that $X = A \sqcup^C B$ is precisely the compact genus g oriented two-dimensional surface.

Applying van Kampen theorem, we get the following result

Theorem. *The fundamental group of a compact genus g oriented surface is isomorphic to the quotient of the free group generated by $x_1, y_1, \ldots, x_g, y_g$ by the normal subgroup generated by the element $\prod_{i=1}^d [x_i, y_i]$.*

4.4.2. A variant of van Kampen theorem

One can chose different ways to make precise the theorem formulated above. Here is one of possible ways.

4.4.3. **Theorem.** *Under the assumptions listed below the conclusion of the van Kampen theorem holds. Here are the assumptions.*

- *The map $g : C \to B$ identifies C with a closed subset of B.*
- *There exists an open neighborhood C' of $g(C)$ in B, a map $p : C' \to C$ such that $pg = \mathrm{id}_C$ and a homotopy between $\mathrm{id}_{C'}$ and $gp : C' \to C'$.*

4.4.4. Van Kampen via coverings

The best way of thinking about van Kampen type theorems is to reformulate the claim about the fundamental groups in terms of coverings of A, B, C and $X = A \sqcup^C B$.

Let as above $X = A \sqcup^C B$ and let $\pi : \widetilde{X} \to X$ be a covering. Using the map $i : A \to X$ we can define a covering $\widetilde{A}$ of A by the formula

$$\widetilde{A} = A \times_X \widetilde{X} = \{(a, \widetilde{x}) \in A \times \widetilde{X} | i(a) = \pi(\widetilde{x})\}.$$

Similarly to the above, we define a covering $\widetilde{B}$ of B and a covering $\widetilde{C}$ of C. We have a diagram of covering spaces

$$
\begin{tikzcd}
 & \widetilde{C} \arrow[dl] \arrow[dr] & \\
\widetilde{A} \arrow[dr] & & \widetilde{B} \arrow[dl] \\
 & \widetilde{X} &
\end{tikzcd}
\tag{1}
$$

and it is an easy exercise to chech that $\widetilde{X} = \widetilde{A} \sqcup^{\widetilde{C}} \widetilde{B}$.

Assume now the conditions of 4.4.3 hold.

4.4.5. **Proposition.** *Let $p_A : \widetilde{A} \to A$ and $p_B : \widetilde{B} \to B$ be coverings. Define $\widetilde{C}_A = \widetilde{A} \times_A C$, $\widetilde{C}_B = \widetilde{B} \times_B C$.*

Assume an isomorphism $\theta : \widetilde{C}_A \to \widetilde{C}_B$ is given. Then the space $\widetilde{X}$ defined by gluing of the diagram

$$\widetilde{A} \xleftarrow{\tilde{f}} \widetilde{C}_A \xrightarrow{\tilde{g}\theta} \widetilde{B},$$

gives a covering of $X = A \sqcup^C B$.

Proof. Let $\widetilde{X} = \widetilde{A} \sqcup^{\widetilde{C}_A} \widetilde{B}$. We have to prove that the natural map $\widetilde{X} \to X$ is a covering.

Thus, for each $x \in X$ we have to find an open neighborhood $\mathcal{O}$ of x in X whose preimage $\widetilde{\mathcal{O}}$ in $\widetilde{X}$ is isomorphic to a product $\mathcal{O} \times F$ where F is discrete.

By definition a subset of X is open iff its preimage in $A \sqcap B$ is open that is is a union of an open subset U of A and an open subset V of B. In other words, open subsets of X correspond to pairs (U, V) of open subsets in A and B respectively satisfying the property $f^{-1}(U) = g^{-1}(V)$.

We will consider two cases for x. Note that $X = A \sqcup (B - C)$ as sets. Assume that $x \in B - C$. Since g is a closed embedding, $B - C$ is open in B and therefore in X; therefore, the preimage of $B - C$ in $\widetilde{X}$ is the same as its preimage in $\widetilde{B}$ and therefore there is a neighborhood in $B - C$ with the required properties.

The case $x \in A$ is more difficult. We choose an open neighborhood U of x in A so that the restriction of $\widetilde{A} \to A$ to U is a trivial covering. Let $W = f^{-1}(U)$. Let $W = \sqcup W_i$ be the decomposition of W into connected components. We finally use the existence of $C' \supset C$, an open subset in B, with a projection $p : C' \to C$. Define $V_i = p^{-1}(W_i)$ and $V = \cup V_i = p^{-1}(W)$.

One has obviously that $W = g^{-1}(V)$ so the pair (U, V) defines an open neighborhood of x in X. We will now prove that the restriction of $\pi : \widetilde{X} \to X$ to (U, V) is a trivial covering.

To simplify the notation, we may assume from the very beginning that $A = U$, $C = W$ and $B = V$. Choose an isomorphism $\widetilde{A} = A \times F$ for a discrete space F. The covering $\widetilde{C}_A$ of C is trivial since it is induced from the trivial covering $\widetilde{A}$.

We claim that the covering $\widetilde{B}$ of $B = \sqcup V_i$ is also trivial. In fact, the maps $W_i \to V_i$ are homotopy equivalences, therefore, have isomorphic fundamental groups, hence, the restriction oreserves triviality of a covering. Since the restriction of $\widetilde{B} \to B$ to C is $\widetilde{C}_B$ is isomorphic to $\widetilde{C}_A$, it is trivial, therefore, $\widetilde{B}$ is trivial.

We claim that there is a way to define a trivialization $wtB = B \times F$ in such a way that the composition

$$C \times F = \widetilde{C}_A \xrightarrow{\theta} \widetilde{C}_B = C \times F$$

is identity.

This claim result from the fact that $g : C \to B$ induces a bijectiom on the connected components (recall that actually $C = \sqcup W_i,\ B = \sqcup V_i$ and the induces are the same).

In fact, choose a trivialization $\widetilde{B} = B \times F$ arbitrarily. This induces a trivialization of $\widetilde{C}_B$: $\widetilde{C}_B = C \times F$, which in turn gives a trivialization of $\widetilde{C}_A$ via θ. Now we have a pair of trivializations on $\widetilde{C}_A$. This pair differs by a collection of automorphisms α_i of F, *each one for each connected component* W_i. But since there is a bijection between the connected components of C and the connected components of B, we can use α_i to correct the trivialization of $\widetilde{B} \to B$. This yields the claim.

Finally, we immediately observe that the gluing the trivialized diagram of coverings (1) we get a trivial covering of X. □

4.4.6. From coverings to fundamental groups

We know from Week 3 that if X is a connected (good) topological space, it (not necessarily connected) coverings corresponds to sets with the action of the fundamental group $\pi_1(X, x)$.

Let now a diagram of cpnnected topological spaces $A \longleftarrow C \longrightarrow B$ be given. Choose a base point $x \in C$, put $X = A \sqcup^C B$ and consider the diagram of the fundamental groups

$$
\begin{tikzcd}
 & \pi_1(C,x) \arrow[dl] \arrow[dr] & \\
\pi_1(A,x) \arrow[dr] & & \pi_1(B,x) \arrow[dl] \\
 & \pi_1(X,x) &
\end{tikzcd}
\tag{2}
$$

which yields a group homorphism

$$\pi_1(A,x) \sqcup^{\pi_1(C,x)} \pi_1(B,x) \longrightarrow \pi_1(X,x).$$

We wish to prove this is an isomorphism.

Proposition 4.4.5 asserts that this homomorphism induces a bijection between isomorphism classes of sets F endowed with the action of the corresponding groups, $\pi_1(X,x)$ and $\pi_1(A,x) \sqcup^{\pi_1(C,x)} \pi_1(B,x)$. The following elementary lemma shows that this implies the homomorphism is a bijection.

Lemma. *Let $f : G \to H$ be a group homomorphism. For any H-set Y we denote $f^*(Y)$ the same set considered as a G-set via f. Assume that any G-set X is isomorphic to $f^*(Y)$ for some Y and that a map $Y_1 \to Y_2$ is a map of H-sets iff it is a map of G-sets. Then f is an isomorphism.*

Remark. The assumptions of the lemma say precisely that f *defines an equivalence between the categories of H-sets and of G-sets.*

Proof. The group H considered as an H-set with the action given by the left multiplication, satisfies the remarkable property

$$\mathrm{Hom}_H(H,Y) = Y.$$

therefore, by the assumption, $f^*(H)$ satisfies the similar remarkable property for G-sets. Thus, $f^*(H) = G$. This means that the action of G on H defined by f (the left action by $f(x)$, $x \in G$), is transitive and with the trivial stabilizer. Thus, f is a bijection.

$\square$

Home assignment.

1. Prove that the quotient of $\mathbb{C}$ by the group generated by one transformation

$$z \mapsto z + a$$

is isomorphic as a Riemann surface to $\mathbb{C}^*$.

2. Prove that the universal covering of the punctured disc cannot be $\mathbb{C}$.

3. Let Π be the fundamental group of a Riemann surface X having the upper half-plane as the universal covering space. Then, as we know, X is isomorphic to $\Pi\backslash\mathbb{H}$. Prove that two Riemann surfaces $X_1 = \Pi_1\backslash\mathbb{H}$ and $X_2 = \Pi_1\backslash\mathbb{H}$ are isomorphic iff the subgroups Π_1 and Π_2 in $PGL(2,\mathbb{R})$ are conjugate.*(Hint: use the lifting property of coverings)*

5. Week 6: Homology

Even though the new algebraic invariants of topological spaces, the homology, will not give a really new information in case of compact Riemann surfaces, it is worthwhile to know something about it.

5.1. **Homology of a simplicial complex.**

Let (V,S) be a simplicial complex with the set of vertices V and the collection of n-simplices $S_n \subset \begin{pmatrix} V \\ n+1 \end{pmatrix}$.

We will explain now how to assign to (V,S) a collection of abelian groups $H_n(S)$ called *the homology of S*. These groups will not depend on the choice of triangulation of a topological space (and are even isomorphic for homotopically equivalent spaces). We will not prove the invariance of the homology groups in general for two reasons. First of all, this is not a simple task. And, what is probably more important, the only interesting homology group for Riemann surfaces, H_1, can be easily expressed via the fundamental group.

5.1.1. Orientation

Orientation of a simplex is given by a choice of an order of its vertices. Two orderings of vertices define the same orientation if they differ by an even permutation.

In order to define the homology groups of S one has to, first of all, choose an orientation of each simplex.

By definition, an n-chain is an expression $\sum a_i\sigma_i$ where $a_i \in \mathbb{Z}, \quad \sigma_i \in S_n$, and only finite number of a_i is nonzero.

The collection of n-simplices form an abelial group (the free abelian group spanned by S_n). We denote it C_n or $C_n(S)$.

We will now construct a collection of maps $d : C_n \to C_{n-1}$ called differentials, so that the crucial property

$$d \circ d = 0$$

is satisfied.

5.1.2. The differential

In order to define the map $d : C_n \to C_{n-1}$ (called the differential) it is enough to determine

$$d(\sigma) = \sum_{\tau \in S_{n-1}} i(\sigma, \tau)\tau$$

for each $\sigma \in S_n$. Thus, we have just to determine the coefficients $i(\sigma, \tau) \in \mathbb{Z}$ (called *the incidence coefficients*).

Here is the prescription. If $\sigma \not\supset \tau$ as subsets of V, the incidence coefficient is zero.

Assume now that $\sigma = \{x_0, \ldots, x_n\}$ and $\tau = \sigma - \{x_i\}$. Assume that the numbering of x_i corresponds to the orientation of σ. Then, if the orientation of τ corresponds to the ordering $\{x_0, \ldots, \widehat{x_i}, \ldots, x_n\}$, we define $i(\sigma, \tau) = (-1)^i$; if the orientation of τ is opposite to the ordering $\{x_0, \ldots, \widehat{x_i}, \ldots, x_n\}$, then $i(\sigma, \tau) = -(-1)^i$.

For example, if $\sigma = \{x, y, z\}$ then

$$d(\sigma) = \{y, z\} - \{x, z\} + \{x, y\}.$$

5.1.3. **Lemma.** *One has $d^2 = 0$.*

Proof. It is enough to check that $d^2(\sigma) = 0$ for any $\sigma \in S_n$. Since $d\sigma$ is a sum of simplices od dimension $n-1$, $d^2\sigma$ is a linear combination of simplices of dimension $n-2$. Each simplex obtained from σ by erasing two its vertices appears twice in $d^2\sigma$; one has to check only that it appears with different signs.

iIf τ is obtained from σ by erasing the vertices i and j so that $i < j$, then τ appears once in the decomposition of $d^2\sigma$ as $(-1)^i(-1)^j\tau$, and for the second time as $(-1)^{j-1}(-1)^i\tau$. The signs are, obviously, different. □

5.1.4. Complex of abelian groups and its homology

A sequence of abelian groups C_i and homomorphisms $d : C_n \to C_{n-1}$ satisfying a condition $d^2 = 0$ is called a complex (of abelian group).

The notion of complex is one of the very basic notions of algebraic topology and homological algebra. Very often the following thing happens: one can assign to an interesting mathematical object (a topological space, a manifold, a complicated algebraic structure) a complex which is not unique but depends on various choices made (for instance, on a triangulation). However, one can extract from the complex a part which does not depend on the choices, and this part really characterized the object. This part is *the homology.*

Definition. Let

$$C_\bullet : \ldots \longrightarrow C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots$$

be a complex of abelian groups. Its n-th homology, denoted $H_n(C_\bullet)$, is the factor group Z_n/B_n where

$$Z_n = \{x \in C_n | d(x) = 0\}$$

and $B_n = d(C_{n+1}) \subseteq C_n\}$.

Note that $B_n \subset Z_n$ because of the property $d^2 = 0$.

5.2. **First calculations.** Let us calculate $H_0(S)$. A 0-chain is a (finite) linear combination of the vertices with integral coefficients. One has $Z_0 = C_0$ since there is no C_{-1}. Thus, we have to determine $B_0 = d(C_1)$. For any one-simplex $\sigma = \{x, y\}$ one has $d\sigma = \{y\} - \{x\}$. Thus, in H_0 we idetify the vertices connected by an edge. This proves the following

5.2.1. **Proposition.** *For any simplicial complex S its zeroth homology is the free abelian group spanned by the connected components of S. In particular, if S is connected, $H_0(S) = \mathbb{Z}$.*

We wish to say more in the connected case. The homology $H_0(S)$ is obtained as the quotient of the vector space $Z_0 = C_0$ spanned by the vertices modulo the subgroup generated by the expressions $x - y,\ x, y \in V$. Thus means that the 0-cycle $\sum a_i v_i,\ v_i \in V$ is a boundary iff $\sum a_i = 0$.

In general H_0 is the only obviously calculated homology. One can also add that if S is an n-dimensional simplicial complex, one has $H_i(S) = 0$ for $i > n$.

It is a good (and not absolutely trivial) exercise to directly prove that $H_i(\Delta^n) = 0$ for $i > 0$ when Δ^n is the stardard n-simplex consisting, by definition, of all nonempty subsets of the set of vertices $V = \{0, \ldots, n\}$.

We suggest to make this exercise at least for $n = 2$.

Another good exercise is (the simplicial model of) the circle: this is the complex consisting of three vertices and three edges.

Its first homology is easily seen to be $\mathbb{Z}$, generated by the one-cycle

$$\{x, y\} + \{y, z\} + \{z, x\}.$$

5.3. **Homology of 1-dimensional complex. Comparison to the fundamental group.** A one-dimensional simplicial complex is just what computer scientists call a graph: a set of vertices and a subset of pairs of vertices.

We assume that our one-dimensional complex is connected.

Let us first of all study the case our graph is a (finite) tree. We know that a tree can be defined recursively as a graph obtained from a tree (or a single point which is the smallest possible tree) by gluing an edge along a single vertex. Thus allows one to easily check the following

5.3.1. **Lemma.** *Any tree has a trivial first homology.*

Proof. The claim is obvious for a point, so we have the base for induction. Assume a tree T os obtained from a tree S adding an edge $\sigma = \{x, y\}$ with $x \in S$ (and $y \neq S$). If $z = \sum a_i \sigma_i$ is a one-cycle, the edge σ cannot appear in the expression with a nonzero coefficient a since otherwize dz will have ay as a summand.

Then z is a cycle for S, so by the induction hypothesis $z = 0$. $\square$

Note that the fundamental group of a tree is as well trivial. We can suggest two ways of proving this.

1. Any tree is contractible, that is is homotopy equivalent to a point. **The fundamental groups of homotopy equivalent spaces are isomorphic** — this very basic fact about fundamental groups we have not proven, so we have to leave it as an exercise. This implies $\pi_1(T)$ is trivial.

2. Using von Kampen theorem and the induction as for the homology.

Let us now compare the fundamental group and the homology group of a uconnected one-dimensional complex. Let S be such a comlplex. Choose a maximal subtree T in S. It contains all vertices of S and some of its edges.

Let $e_1, \dots, e_n$ be the edges **not** in T. We will identify not the fundamental group $\pi(S)$ with the free group generated by e_i, whereas the homology $H_1(S)$ will be identified with the free **abelian** group generated by e_i.

Choose any vertex x as a base point. Choose for each vertex $y \in S$ a path p_y connecting x with y. Note that this path can be represented by a one-chain in $C_1(S)$.

Any edge e_i with the ends y and z gives rise to a closed path starting at x, moving along p_y then e_i and then along $-p_z$ back to x. This closed path represents an element of the fundamental group $\pi_1(S)$ and it is easy to see that they are free generators of this group.

We can copy the above reasoning to calculate $H_1(S)$. The group of one-chains in T has no cycles. However, each edge e_i gives rize to a cycle $e_i + p_y - p_z$. Since $C_2(S) = 0$, there are no one-boundaries, and we get the following conclusion.

5.3.2. **Theorem.** *Let S be a one-dimensional simplicial complex. The group $\pi_1(S)$ is free, with free generators e_i (edges not in a chosen maximal subtree). The group $H_1(S)$ is the free abelian group generated by e_i.*

We see that for one-dimansional complex $H_1(S) = \Pi/[\Pi, \Pi]$ where $\Pi = \pi_1(X)$, X being the geometric realization of S.

5.3.3. **Remark.** It is very inconvenient that the homology of a topological space is defined in this course as a collection of groups assigned to a triangulation of X. This does not allow us to define a *canonical* isomorphism between H_1 and $\Pi/[\Pi, \Pi]$. It is worthwhile to *define* H_1 as the quotient of the fundamental group by the commutant, and present sthe definition via triangulation as a calculational tool.

5.4. **Adding simplices of dimension > 1.** Let S be a simplicial complet and let $S(n)$ be the subcomplex consistion of simplices of dimension $\leq n$. We can think of $S(2)$ as the complex obtained from $S(1)$ by attaching two-simplices $\sigma \in S_2$. Let us see what is happening to π_1 and to H_1 under this operation.

Van Kampen theorem gives an answer for the fundamental group. The fundamental group of $S \cup \sigma$ is the quotient of $\pi_1(S)$ by the normal subgroup generated by the class of the boundary of σ.

The similar formula for H_1 is obtained immediately by definition: attaching a two-simplex we do not change the group of one-cycles, but adding a new element $d\sigma$ to the boundary.

It is easy to see that attaching an n-simplex for $n > 2$ changes neither π_1 not H_1.

5.4.1. Homology of oriented surfaces

Recall that if X is a compact oriented surface of genus g, the group $\pi_1(X)$ is generated by $x_1, y_1, \ldots, x_g, y_g$, subject to the relation $\prod_{i=1}^{g}[x_i, y_i] = 1$.

The commutator subgroup of the free group is generated by all commutators $[x_i, x_j]$, $[x_i, y_j]$, $[y_i, y_j]$. In particular, it contains the product of $[x_i, y_i]$. Thus, we get

Proposition. *The first homology of the compact oriented surface of genus g is isomorphic to $\mathbb{Z}^{2g}$. As a free basis of this group one can take the classes of the edges x_i, y_i, $i = 1, \ldots, g$.*

5.4.2. The second homology

Recall that we use (without proof at the moment) the fact that any Riemann surface admits a triangulation.

Let S be the corresponding two-dimensional simplicial complex. We know that every edge belongs to precisely two triangles. It is easy to see that if one choose orientation of all two-simplices so that it corresponds to the orientation of the surface, the incidence coefficients of these two triangles at their common edge have different signs.

Therefore, a two chain $\sum a_i\sigma_i$ is a cycle iff **all** its coefficients a_i are proportional. Here one has to distinguish two cases.

- X is compact and therefore S is finite. Then the sum of all two-simplices is a cycle and it generates $H_2(X)$. In this case $H_2(X) = \mathbb{Z}$.
- X is not compact. Then S has an infinite number of two-simplices and there are no nonzero two-cycles. In this case $H_2(X) = 0$.

Home assignment.

1. Prove that the fundamental groups of homotopy equivalent spaces are isomorphic.
2. Using van Kampen theorem calculate $\pi_1(\mathbb{RP}^2)$ where $\mathbb{RP}^2$ is the (real) projective plane which can be presented as the disc whose opposite points on the boundary are identified.
3. Calculate $H_2(\mathbb{RP}^2)$.