3. Week 3. Topological structure of RS and of holomorphic maps

3.1. Nonramified covering. A map of topological spaces $f : X \to Y$ is called a nonramified covering if for each $y \in Y$ there exists a neighborhood $U$ of $y$ in $Y$ and an isomorphism

$$f^{-1}(U) \xrightarrow{\cong} U \times F$$

where $F$ is a discrete space, $f$ is the restriction of the original map $f$ and $p_1$ is the projection to the first component.

Note that any nonramified covering is automatically a local isomorphism: for any $x \in X$ there exists a neighborhood $V$ of $x$ and a neighborhood $U$ of $f(x)$ so that $f$ induces a homeomorphism between $V$ and $U$ (see Exercise 2).

Note that, according to the definition, if $f$ is a nonramified covering, the function

$$y \in Y \mapsto |f^{-1}(y)|$$

is locally constant. Therefore, since $Y$ is connected, it is constant. Its value (finite or infinite) is called the degree of $f$.

3.1.1. Lemma. Let $f : X \to Y$ be a local homeomorphism. Then, if $Y$ is a Riemann surface, there is a unique structure of Riemann surface on $X$ such that $f$ is holomorphic.

Proof. Let $x \in X$ and let $V$ be a neighborhood of $x$ homeomorphic via $f$ to a neighborhood $f(V) = U$ of $y = f(x)$. Choose a chart of $Y$ containing $y$ and contained in $U$. Its preimage under $f$ defines a chart of $X$ containing $x$.

If $U_1$ and $U_2$ are two charts of $X$ constructed in the above way, the intersection $U_1 \cap U_2$ is mapped by $f$ into the intersection of the images (it may be strictly less than the intersection of the images).

Therefore, since the images of $U_i$ are compatible in $Y$, the charts $U_i$ are compatible in $X$. □

We are willing to formulate conditions which would allow to get a riemann surface structure on $Y$ from a Riemann surface $X$ and a nonramified covering $f : X \to Y$. 

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We say that a covering $f : X \to Y$ is compatible with the complex structure on $X$ if for any diagram (1) the homeomorphism between any pair of components of $f^{-1}(U)$ defined by $f$, is holomorphic.

3.1.2. Lemma. Let $X$ be a Riemann surface and let $f : X \to Y$ be a nonramified covering compatible with the complex structure of $X$. Then there is a unique complex structure on $Y$ such that $f$ is holomorphic.

Proof. For any $U$ as in (1) define the atlas of $U$ via homomorphism of $U$ with any component of $f^{-1}(U)$. So defined charts are automatically compatible. □

The following special case of Lemma 3.1.2 is especially useful.

3.1.3. Proposition. Let $X$ be a Riemann surface and let a group $G$ acts on $X$ by holomorphic isomorphisms. Assume that the action of $G$ on $X$ is free, that is for each $x \in X$ there is a neighborhood $U$ containing $x$ such that for each $g \neq 1$ in $G$ $U \cap g(U) = \emptyset$. Then the quotient map

$$p : X \to G\backslash X$$

is a covering compatible with the complex structure of $X$. Therefore, the quotient $G\backslash X$ is Riemann surface.

3.1.4. Example. The quotient of $\mathbb{C}$ by the action of a lattice $L = \mathbb{Z} \oplus \mathbb{Z}\tau$ is obviously free, so the quotient is a Riemann surface.

3.1.5. Lifting property

Let $f : X \to Y$ be a nonramified covering, $x \in X$, $y = f(x)$. Let $\gamma : [0, 1] \to Y$ be a continuous path in $Y$. We claim that $\gamma$ can be uniquely lifted to a continuous path $\Gamma : [0, 1] \to X$ satisfying the condition $\Gamma(0) = x$.

This is fairly standard: the set of points $t \in [0, 1]$ such that the path $\gamma|_{[0,t]}$ can be uniquely lifted, is open and closed in $[0, 1]$.

A generalization of this: Let $Z$ be simply connected. Then any map $\gamma : Z \to Y$ sending a point $z \in Z$ to $y \in Y$ can be uniquely lifted to a map $\Gamma : Z \to X$ satisfying $\Gamma(z) = x$.

3.1.6. Proper maps

A continuous map $f : X \to Y$ of locally compact spaces is called proper if for any compact $C \subset Y$ the preimage $f^{-1}(C)$ is compact.

A covering is proper if and only if the preimage $f^{-1}(y)$ is a finite set (all preimages have the same cardinality since $Y$ is connected).

If $X$ is compact, any map $f : X \to Y$ is proper.

Lemma. Let $f : X \to Y$ be a proper local isomorphism. Then $f$ is a covering.
Proof. Let \( y \in Y \) and let \( f^{-1}(y) = \{x_1, \ldots, x_n\} \). Since \( f \) is a local isomorphism, there exist open neighborhoods \( U_i \) of \( x_i \) such that \( f_{U_i} : U_i \to f(U_i) \) is a homeomorphism. Let \( V = \bigcap f(U_i) \). We can replace \( U_i \) with \( U_i \cap f^{-1}(V) \) so that for the new \( U_i \) we have \( f(U_i) = V \).

Let us make a few reductions. First of all, we may assume \( V \) has a compact closure \( \bar{V} \). Next, replace \( Y \) with \( \bar{V} \) and \( X \) with \( f^{-1}(\bar{V}) \).

Then \( X \) is compact. We can cover \( X \) by the following open sets:

- \( U_i, i = 1, \ldots, n \).
- \( f^{-1}(Y - Z) \) where \( Z \) runs through the set of compact neighborhoods of \( y \).

Since \( X \) is compact,

\[
X = \bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{m} f^{-1}(Y - Z_i) = \bigcup_{i=1}^{n} U_i \cup f^{-1}(Y - Z) \]

where \( Z = \bigcap Z_i \) is also a compact neighborhood of \( y \). Thus, replacing \( V \) by an open neighborhood of \( y \) inside \( Z \), we get \( f^{-1}(V) = \bigcup_{i=1}^{n} U_i \cap f^{-1}(V) \). The neighborhood \( V \) satisfies the required property. \( \square \)

3.1.7. Corollary. Let \( f : X \to Y \) be a nonconstant holomorphic map of Riemann surfaces. Assume that \( X \) is compact. Then \( Y \) is as well compact and the number of preimages of each point \( y \in Y \) (counted with multiplicity) does not depend on \( y \).

The number of preimages is called the degree of \( f \).

Proof. The set of critical points of \( f \) in \( X \) is discrete, therefore, finite. Let \( C \) be its image under \( f \) in \( Y \). \( X \) is compact, thus \( f \) is proper, therefore the restriction

\[
f : X - f^{-1}(C) \to Y - C
\]

is proper as well. It is a local isomorphism, therefore, a nonramified covering. Therefore, the number of preimages is constant for \( y \in Y - C \).

Let us make sure that the same number of preimages (counting with multiplicities) has a point \( y \in C \). Let \( x \in f^{-1}(y) \). We know that there are local coordinates \( z \) near \( x \) and \( w \) near \( y \) so that \( f \) is written by the formula \( w = z^n \). In the punctured neighborhood of \( y \) there are \( n \) preimages belonging to the neighborhood of \( x \); the point \( y \) has one preimage \( x \) of multiplicity \( n \). Thus, the number of preimages counting multiplicities is preserved. \( \square \)

3.2. Fundamental group. The following definition makes sense for quite general topological spaces. Here is the requirement (definitely satisfied by manifolds):

We assume that the topological space \( X \) is locally simply connected. That means that any neighborhood of \( X \) contains a simply connected subneighborhood. Recall that a pathwise connected space \( X \) is called simply connected if for each pair of points \( x, y \in X \) any two paths connecting them are homotopic:
for \( \phi, \psi : [0, 1] \to X \) with \( \phi(0) = \psi(0) = x, \phi(1) = \psi(1) = Y \), there exists a continuous map \( F : [0, 1] \times [0, 1] \to X \) such that
\[
F(0, t) = x, \quad F(1, t) = y, \quad F(s, 0) = \phi(s), \quad F(s, 1) = \psi(s).
\]

We will now assign to a locally simply connected topological space \( X \) (and to a point \( x \in X \)) a group \( \pi_1(X, x) \) which will be trivial once \( X \) is simply connected.

### 3.2.1. Loops. Product of loops

A continuous map \( \phi : [0, 1] \to X \) satisfying \( \phi(0) = \phi(1) = x \) is called a loop centered at \( x \). Given two loops \( \phi, \psi \) centered at \( x \), one defines their product \( \phi \psi \) by the formula
\[
\phi \psi(t) = \begin{cases} 
\phi(2t), & t \in [0, \frac{1}{2}] \\
\psi(2t - 1), & t \in \left[\frac{1}{2}, 1\right].
\end{cases}
\]

A nuisance: the product so defined is not associative: the product \( (\phi \psi) \chi \) passes the first two loops \( \phi, \psi \) with the velocity 4, and the third loop \( \chi \) with the velocity 2, whereas the product \( \phi(\psi \chi) \) passes the first loop with the double velocity, and the loops \( \psi, \chi \) with the velocity 4.

At least, the two products are homotopic.

### 3.2.2. Theorem. The product of loops defined a meaningful operation on the homotopy classes of loops. The set of homotopy classes of loops is a group with respect to this operation.

The group of homotopy classes of loops based at \( x \in X \) is called the fundamental group of \( X \) and is denoted \( \pi_1(X, x) \).

**Proof.** First of all, it is useful to check that two loops differing by a reparametrization, are homotopic.

In a more detail, let \( u : [0, 1] \to [0, 1] \) be a monotone continuous map satisfying the condition \( u(0) = 0, \ u(1) = 1 \). We claim that any path \( \phi : [0, 1] \to X \) is homotopy equivalent to the composition \( \phi u \) (note that both paths have the same edges). In fact, the family of paths \( t \mapsto \phi u_t \) connects \( \phi \) with \( \phi u \), where \( u_t(s) = tu(s) + (1 - t)s \).

This, in particular, imply that \( \phi(\psi \chi) \) and \( (\phi \psi) \chi \) are homotopic — since they differ only by a reparametrization.

Next we have to check (this is really easy and left as a homework) that the product of loops preserves homotopy of loops. Thus, an associative operation is defined on the set \( \pi_1(X, x) \) of homotopy classes of loops.

The constant loop \( i(s) = x \) is the unit of this operation, since the product \( i \phi \) differs from \( \phi \) by a reparametrization; finally, for any loop \( \phi \) the loop \( \psi(s) := \phi(1 - s) \) is the (homotopy) inverse of \( \phi \) since the composition \( \phi \psi \) can be connected
with \( i \) by the family
\[
F(s, t) = \begin{cases} 
\phi(2st), & s \leq \frac{1}{2}, \\
\phi(2(1-s)t), & s \geq \frac{1}{2}.
\end{cases}
\]

\( \square \)

3.2.3. **Examples:** \( X = \mathbb{C}, \hat{\mathbb{C}} \)

The fundamental group of a simply connected space is trivial. The space \( \mathbb{C} \) is clearly simply-connected. The Riemann sphere is also simply-connected: given two paths with common ends, throw out one point of \( \hat{\mathbb{C}} \) not belonging to the paths - and we will get a complex plane which is clearly simply-connected.

3.2.4. **The circle and \( \mathbb{C}^* \)** The circle is the first example of a non-simply-connected space. For a continuous map \( \phi : [0, 1] \to S^1 \) satisfying \( \phi(0) = \phi(0) = 1 \in S^1 \) can be described as a continuous map \( \tilde{\phi} : [0, 1] \to \mathbb{R} \) such that \( a \tilde{\phi}(0) = 0, \tilde{\phi}(1) = 2\pi k \) for some \( k \in \mathbb{Z} \). It is easy to see that homotopic paths correspond to the same value of \( k \).

This allows one to identify \( \pi_1(S) \) with the group \( \mathbb{Z} \).

The Riemann surface \( \mathbb{C}^* = \mathbb{C} - \{0\} \) has the same fundamental group as \( S^1 \).

3.3. **Universal covering.** There is a very intimate connection between the fundamental group and the nonramified coverings.

Let \( X \) be a connected locally simply connected space.

3.3.1. **Definition.** A map \( f : \tilde{X} \to X \) is called a universal covering if \( f \) is a nonramified covering and \( \tilde{X} \) is simply connected.

As we will see soon, a universal covering exist and is “almost unique”. We will explain later the meaning of the term “universal”.

Let \( P(x, y) \) be the set of homotopy classes of paths from \( x \) to \( y \). Thus, \( \pi_1(X, x) = P(x, x) \).

3.3.2. **Construction**

Fix \( x \in X \).

We define \( \tilde{X} \) as \( \cup_{y \in X} P(x, y) \). The projection \( p : \tilde{X} \to X \) carries \( P(x, y) \) to \( y \). The fiber \( p^{-1}(y) \) identifies with \( P(x, y) \).

Let us define a topology on \( \tilde{X} \). Let \( y \in X \) and let \( U \) be a simply connected neighborhood of \( y \). Let a point of \( p^{-1}(y) \) be represented by a path \( \phi : [0, 1] \to X \) connecting \( x \) with \( y \). Since \( U \) is simply-connected, any point \( z \in U \) is connected by a path \( \psi \) with \( y \), unique up to homotopy. Thus the homotopy class of the product \( \phi\psi \) is uniquely defined; the collection of points of \( \tilde{X} \) so defined is denoted \( U_\phi \).
The sets $U_\phi$ form a basis of the topology of $\tilde{X}$.

Let us prove $\tilde{X}$ satisfies the required properties. First of all, the map $p : \tilde{X} \to X$ is obviously a covering. Let us check $\tilde{X}$ is connected.

Let $\phi$ be a path connecting $x$ to $y$. If $U$ is a simply connected neighborhood of $y$, $p : U_\phi \to U$ is a homeomorphism. If now $z$ is a point of the path $\phi$ belonging to $U$, the segment of $\phi$ between $z$ and $y$ lifts to $U_\phi$. Compactness of $[0, 1]$ allows one to lift $\phi$ to a path connecting the class of the constant path with the class of $\phi$ in a finite number of steps.

Let us finally prove that $\tilde{X}$ is simply connected. Since $p$ is a covering and $\tilde{X}$ is connected, any path connecting $x$ with $y$ in $X$ can be uniquely lifted to a path in $\tilde{X}$ starting at $i \in \tilde{X}$.

This yields a one-to-one correspondence between the set $P(x, y)$ and the set of homotopy classes of paths in $\tilde{X}$ connecting $i \in \tilde{X}$ with a point over $y \in X$. Since the set of points over $y$ also identifies with $P(x, y)$, any two paths in $\tilde{X}$ from $i$ to $\phi \in P(x, y)$, are homotopic. This proves that $\tilde{X}$ is simply connected.

### 3.3.3. Universal property

Let $x \in X$ and let $p : \tilde{X} \to X$ be defined as above. Let $q : X' \to X$ be a covering. Fix a point $x' \in X'$ over $x$. There are canonical one-to-one correspondences between:

1. Paths from $x$ to $y$ in $X$ and paths from $i$ to a point over $y$ in $\tilde{X}$.
2. Paths from $x$ to $y$ in $X$ and paths from $x'$ to a point over $y$ in $X'$.

Comparing the above, we get the following

**Proposition.** There is a unique map $r : \tilde{X} \to X'$ satisfying the conditions

- $p = q \circ r$.
- $r(i) = x'$.

Note that the map $r : \tilde{X} \to X'$ is automatically a covering. It is necessarily surjective if $X'$ is connected.

Note as well that if $X'$ is simply connected, $r$ has to be an isomorphism. Therefore, $\tilde{X}$ explicitly constructed above is (noncanonically) isomorphic to any universal covering.

### 3.3.4. Action of $\pi_1(X, x)$ on $\tilde{X}$

Choose any point $x' \in \pi_1(X, x)$. By the proposition above, there exists a unique map $\tilde{X} \to \tilde{X}$ over $X$ carrying $i$ to $x'$. This defines an action of the fundamental group $\pi_1(X, x)$ on $\tilde{X}$. The action is clearly free and the quotient $\tilde{X}/\pi_1(X, x)$ identifies with $X$. 
3.3.5. Classification of coverings: “Galois theory”

Let \( q : Y \to X \) be a covering (and \( X \) and \( Y \) connected). Let \( p : \tilde{X} \to X \) be a universal covering. According to the above there exists a covering \( r : \tilde{X} \to Y \) such that \( p = q \circ r \). This covering is uniquely defined by a choice of \( y \in q^{-1}(x) \).

By definition, \( r : \tilde{X} \to Y \) is also a universal covering, so there is an action of \( \pi_1(Y, y) \) on \( \tilde{X} \). Actually, \( \pi_1(Y, y) \) is just a subgroup of \( \pi_1(X, x) \) and the action is the restriction of the action of \( \pi_1(X, x) \) on \( \tilde{X} \).

We have thus described a correspondence between (connected) coverings of \( X \) and subgroups of \( \pi_1(X, x) \).

This correspondence is a close relative of the correspondence between field extensions and subgroups of Galois group in Galois theory.

**Home assignment.**

1. Let \( \phi \) and \( \phi' \) be two homotopic paths. Prove that the products \( \phi \psi \) and \( \phi' \psi \) are homotopic.

2. Prove that the action of \( \pi_1(X, x) \) on \( \tilde{X} \) is given by the product of paths.

3. Prove that the fundamental groups \( \pi_1(X, x) \) and \( \pi_1(X, y) \) are isomorphic for connected \( X \).

4. Prove that \( \pi_1(X \times Y) \) is isomorphic to \( \pi_1(X) \times \pi_1(Y) \). In particular, the fundamental group of an elliptic curve is \( \mathbb{Z} \oplus \mathbb{Z} \).