RIEMANN SURFACES

3. Week 3. Topological structure of RS and of holomorphic maps

3.1. **Nonramified covering.** A map of topological spaces $f : X \to Y$ is called a nonramified covering if for each $y \in Y$ there exists a neighborhood U of y in Y and an isomorphism

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \times F \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle p_1} & \\ & U & \end{array} \tag{1}$$

where F is a discrete space, f is the restriction of the original map f and p_1 is the projection to the first component.

Note that any nonramified covering is automatically a local isomorphism: for any $x \in X$ there exists a neighborhood V of x and a neighborhood U of $f(x)$ so that f induces a homeomorphism between V and U (see Exercise 2).

Note that, according to the definition, if f is a nonramified covering, the function

$$y \in Y \mapsto |f^{-1}(y)|$$

is locally constant. Therefore, since Y is connected, it is constant. Its value (finite or infinite) is called the degree of f.

3.1.1. **Lemma.** *Let $f : X \to Y$ be a local homeomorphism. Then, if Y is a Riemann surface, there is a unique structure of Riemann surface on X such that f is holomorphic.*

Proof. Let $x \in X$ and let V be a neighborhood of x homeomorphic via f to a neighborhood $f(V) = U$ of $y = f(x)$. Choose a chart of Y containing y and contained in U. Its preimage under f defines a chart of X containing x.

If U_1 and U_2 are two charts of X constructed in the above way, the intersection $U_1 \cap U_2$ is mapped by f into the intersection of the images (it may be strictly less than the intersection of the images).

Therefore, since the images of U_i are compatible in Y, the charts U_i are compatible in X. □

We are willing to formulate conditions which would allow to get a riemann surface structure on Y from a Riemann surface X and a nonramified covering $f : X \to Y$.

We say that a covering $f : X \to Y$ is compatible with the complex structure on X if for any diagram (1) the homeomorphism between any pair of components of $f^{-1}(U)$ defined by f, is holomorphic.

3.1.2. **Lemma.** *Let X be a Riemann surface and let $f : X \longrightarrow Y$ be a nonramified covering compatible with the complex structure of X. Then there is a unique complex structure on Y such that f is holomorphic.*

Proof. For any U as in (1) define the atlas of U via homoemorphism of U with any component of $f^{-1}(U)$. So defined charts are automatically compatible. □

The following special case of Lemma 3.1.2 is especially useful.

3.1.3. **Proposition.** *Let X be a Riemann surface and let a group G acts on X by holomorphic isomorphisms. Assume that the action of G on X is free, that is for each $x \in X$ there is a neighborhood U containing x such that for each $g \neq 1$ in G $U \cap g(U) = \emptyset$. Then the quotient map*

$$p : X \longrightarrow G\backslash X$$

is a covering compatible with the complex structure of X. Therefore, the quotient $G\backslash X$ is Riemann surface.

3.1.4. **Example.** The quotient of $\mathbb{C}$ by the action of a lattice $L = \mathbb{Z} \oplus \mathbb{Z}\tau$ is obviously free, so the quotient is a Riemann surface.

3.1.5. Lifting property

Let $f : X \to Y$ be a nonramified covering, $x \in X,\ y = f(x)$. Let $\gamma : [0,1] \to Y$ be a continuous path in Y. We claim that γ can be uniquely lifted to a continuous path $\Gamma : [0,1] \to X$ satisfying the condition $\Gamma(0) = x$.

This is fairly standard: the set of points $t \in [0,1]$ such that the path $\gamma|_{[0,t]}$ can be uniquely lifted, is open and closed in $[0,1]$.

A generalization of this: Let Z be simply connected. Then any map $\gamma : Z \to Y$ sending a point $z \in Z$ to $y \in Y$ can be uniquely lifted to a map $\Gamma : Z \to X$ satisfying $\Gamma(z) = x$.

3.1.6. Proper maps

A continuous map $f : X \to Y$ of locally compact spaces is called *proper* if for any complact $C \subset Y$ the preimage $f^{-1}(C)$ is compact.

A covering is proper if and only if the preimage $f^{-1}(y)$ is a finite set (all preimages have the same cardinality since Y is connected).

If X is compact, any map $f : X \to Y$ is proper.

Lemma. *Let $f : X \to Y$ be a proper local isomorphism. Then f is a covering.*

Proof. Let $y \in Y$ and let $f^{-1}(y) = \{x_1, \dots, x_n\}$. Since f is a local isomorphism, there exist open neighborhoods U_i of x_i such that $f_{U_i} : U_i \to f(U_i)$ is a homeomorphism. Let $V = \cap f(U_i)$. We can replace U_i with $U_i \cap f^{-1}(V)$ so that for the new U_i we have $f(U_i) = V$.

Let us make a few reductions. First of all, we may assume V has a compact closure $\bar{V}$. Next, replace Y with $\bar{V}$ and X with $f^{-1}(\bar{V})$.

Then X is compact. We can cover X by the following open sets:

- U_i, $i = 1, \dots, n$.
- $f^{-1}(Y - Z)$ where Z runs through the set of compact neighborhoods of y.

Since X is compact,

$$X = \bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{m} f^{-1}(Y - Z_i) = \bigcup_{i=1}^{n} U_i \cup f^{-1}(Y - Z)$$

where $Z = \cap Z_i$ is also a compact neighborhood of y. Thus, replacing V by an open neighborhood of y inside Z, we get $f^{-1}(V) = \cup_{i=1}^{n} U_i \cap f^{-1}(V)$. The neighborhood V satisfies the required property. □

3.1.7. **Corollary.** *Let $f : X \longrightarrow Y$ be a nonconstant holomorphic map of Riemann surfaces. Assume that X is compact. Then (Y is as well compact and) the number of preimages of each point $y \in Y$ (counted with multiplicity) does not depend on y.*

The number of preimages is called *the degree of f*.

Proof. The set of critical points of f in X is discrete, therefore, finite. Let C be its image under f in Y. X is compact, thus f is proper, therefore the restriction

$$f : X - f^{-1}(C) \longrightarrow Y - C$$

is proper as well. It is a local isomorphism, therefore, a nonramified covering. Therefore, the number of preimages is constant for $y \in Y - C$.

Let us make sure that the same number of preimages (counting with multiplicities) has a point $y \in C$. Let $x \in f^{-1}(y)$. We know that there are local coordinates z near x and w near y so that f is written by the formula $w = z^n$. In the punctured neighborhood of y there are n preimages belonging to the neighborhood of x; the point y has one preimage x of multiplicity n. Thus, the number of preimages counting multiplicities is preserved. □

3.2. **Fundamental group.** The following definition makes sense for quite general topological spaces. Here is the requirement (definitely satisfied by manifolds):

We assume that the topological space X is locally simply connected. That means that any neighborhood of X contains a simply connected subneighborhood. Recall that a pathwise connected space X is called simply connected if for each pair of points $x, y \in X$ any two paths connecting them are homotopic:

for $\phi, \psi : [0,1] \to X$ with $\phi(0) = \psi(0) = x$, $\phi(1) = \psi(1) = Y$, there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that

$$F(0,t) = x, \quad F(1,t) = y, \quad F(s,0) = \phi(s), \quad F(s,1) = \psi(s).$$

We will now assign to a locally simply connected topological space X (and to a point $x \in X$) a group $\pi_1(X, x)$ which will be trivial once X is simply connected.

3.2.1. Loops. Product of loops

A continuous map $\phi : [0,1] \to X$ satisfying $\phi(0) = \phi(1) = x$ is called *a loop centered at* x. Given two loops ϕ, ψ centered at x, one defines their product $\phi\psi$ by the formula

$$\phi\psi(t) = \begin{cases} \phi(2t), & t \in [0, \frac{1}{2}] \\ \psi(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases} \tag{2}$$

A nuisance: the product so defined is not associative: the product $(\phi\psi)\chi$ passes the first two loops ϕ, ψ with the velocity 4, and the third loop χ with the velocity 2, whereas the product $\phi(\psi\chi)$ passes the first loop with the double velocity, and the loops ψ, χ with the velocity 4.

At least, the two products are homotopic.

3.2.2. **Theorem.** *The product of loops defined a meaningful operation on the homotopy classes of loops. The set of homotopy classes of loops is a group with respect to this operation.*

The group of homotopy classes of loops based at $x \in X$ is called *the fundamental group of* X and is denoted $\pi_1(X, x)$.

Proof. First of all, it is useful to check that two loops differing by a reparametrization, are homotopic.

In a more detail, let $u : [0,1] \to [0,1]$ be a monotone continuous map satisfying the condition $u(0) = 0$, $u(1) = 1$. We claim that any path $\phi : [0,1] \to X$ is homotopy equivalent to the composition ϕu (note that both paths have the same edges). In fact, the family of paths $t \mapsto \phi u_t$ connects ϕ with ϕu, where $u_t(s) = tu(s) + (1-t)s$.

This, in particular, imply that $\phi(\psi\chi)$ and $(\phi\psi)\chi$ are homotopic — since they differ only by a reparametrization.

Next we have to check (this is really easy and left as a homework) that the product of loops preserves homotopy of loops. Thus, an associative operation is defined on the set $\pi_1(X, x)$ of homotopy classes of loops.

The constant loop $i(s) = x$ is the unit of this operation, since the product $i\phi$ differs from ϕ by a reparametrization; finally, for any loop ϕ the loop $\psi(s) := \phi(1-s)$ is the (homotopy) inverse of ϕ since the composition $\phi\psi$ can be connected with i by the family

$$F(s,t) = \begin{cases} \phi(2st), & s \le \frac{1}{2}, \\ \phi(2(1-s)t), & s \ge \frac{1}{2}. \end{cases}$$

□

3.2.3. Examples: $X = \mathbb{C},\ \widehat{\mathbb{C}}$

The fundamental group of a simply connected space is trivial. The space $\mathbb{C}$ is clearly simply-connected. The Riemann sphere is also simply-connected: given two paths with common ends, throw out one point of $\widehat{\mathbb{C}}$ not belonging to the paths - and we will get a complex plane which is clearly simply-connected.

3.2.4. The circle and $\mathbb{C}^*$ The circle is the first example of a non-simply-connected space. For a continuous map $\phi : [0,1] \to S^1$ satisfying $\phi(0) = \phi(0) = 1 \in S^1$ can be described as a continuous map $\tilde{\phi} : [0,1] \to \mathbb{R}$ such that a$\tilde{\phi}(0) = 0,\ \tilde{\phi}(1) = 2\pi k$ for some $k \in \mathbb{Z}$. It is easy to see that homotopic paths correspond to the same value of k.

This allows one to identify $\pi_1(S)$ with the group $\mathbb{Z}$.

The Riemann surface $\mathbb{C}^* = \mathbb{C} - \{0\}$ has the same fundamental group as S^1.

3.3. **Universal covering.** There is a very intimate connection between the fundamental group and the nonramified coverings.

Let X be a connected locally simply connected space.

3.3.1. **Definition.** A map $f : \widetilde{X} \to X$ is called *a universal covering* if f is a nonramified covering and $\widetilde{X}$ is simply connected.

As we will see soon, a universal covering exist and is “almost unique”. We will explain later the meaning of the term “universal”.

Let $P(x,y)$ be the set of homotopy classes of paths from x to y. Thus, $\pi_1(X,x) = P(x,x)$.

3.3.2. Construction

Fix $x \in X$.

We define $\widetilde{X}$ as $\cup_{y\in X} P(x,y)$. The projection $p : \widetilde{X} \to X$ carries $P(x,y)$ to y. The fiber $p^{-1}(y)$ identifies with $P(x,y)$.

Let us define a topology on $\widetilde{X}$. Let $y \in X$ and let U be a simply connected neighborhood of y. Let a point of $p^{-1}(y)$ be represented by a path $\phi : [0,1] \to X$ connecting x with y. Since U is simply-connected, any point $z \in U$ is connected by a path ψ with y, *unique up to homotopy.* Thus the homotopy class of the product $\phi\psi$ is uniquely defined; the collection of points of $\widetilde{X}$ so defined is denoted U_ϕ.

The sets U_ϕ form a basis of the topology of $\widetilde{X}$.

Let us prove $\widetilde{X}$ satisfies the required properties. First of all, the map $p : \widetilde{X} \to X$ is obviously a covering. Let us check $\widetilde{X}$ is connected.

Let ϕ be a path connecting x to y. If U is a simply connected neighborhood of y, $p : U_\phi \to U$ is a homeomorphism. If now z is a point of the path ϕ belonging to U, the segment of ϕ between z and y lifts to U_ϕ. Compactness of $[0, 1]$ allows one to lift ϕ to a path connecting the class of the constant path with the class of ϕ in a finite number of steps.

Let us finally prove that $\widetilde{X}$ is simply connected. Since p is a covering and $\widetilde{X}$ is connected, any path connecting x with y in X can be uniquely lifted to a path in $\widetilde{X}$ starting at $i \in \widetilde{X}$.

This yields a one-to-one correspondence between the set $P(x, y)$ and the set of homotopy classes of paths in $\widetilde{X}$ connecting $i \in \widetilde{X}$ with a point over $y \in X$. Since the set of points over y also identifies with $P(x, y)$, any two paths in $\widetilde{X}$ from i to $\phi \in P(x, y)$, are homotopic. This proves that $\widetilde{X}$ is simply connected.

3.3.3. Universal property

Let $x \in X$ and let $p : \widetilde{X} \to X$ be defined as above. Let $q : X' \to X$ be a covering. Fix a point $x' \in X'$ over x. There are canonical one-to-one correspondences between:

1. Paths from x to y in X and paths from i to a point over y in $\widetilde{X}$.
2. Paths from x to y in X and paths from x' to a point over y in X'.

Comparing the above, we get the following

Proposition. *There is a unique map $r : \widetilde{X} \longrightarrow X'$ satisfying the conditions*

- $p = q \circ r$.
- $r(i) = x'$.

Note that the map $r : \widetilde{X} \to X'$ is automatically a covering. It is necessarily surjective if X' is connected.

Note as well that if X' is simply connected, r has to be an isomorphism. Therefore, $\widetilde{X}$ explicitly constructed above is (noncanonically) isomorphic to *any* universal covering.

3.3.4. Action of $\pi_1(X, x)$ on $\widetilde{X}$

Choose any point $x' \in \pi_1(X, x)$. By the proposition above, there exists a unique map $\widetilde{X} \to \widetilde{X}$ over X carrying i to x'. This defines an action of the fundamental group $\pi_1(X, x)$ on $\widetilde{X}$. The action is clearly free and the quotient $\widetilde{X}/\pi_1(X, x)$ identifies with X.

3.3.5. Classification of coverings: "Galois theory"

Let $q : Y \to X$ be a covering (and X and Y connected). Let $p : \widetilde{X} \to X$ be a universal covering. According to the above there exists a covering $r : \widetilde{X} \to Y$ such that $p = q \circ r$. This covering is uniquely defined by a choice of $y \in q^{-1}(x)$.

By definition, $r : \widetilde{X} \to Y$ is also a universal covering, so there is an action of $\pi_1(Y, y)$ on $\widetilde{X}$. Actually, $\pi_1(Y, y)$ is just a subgroup of $\pi_1(X, x)$ and the action is the restriction of the action of $\pi_1(X, x)$ on $\widetilde{(X)}$.

We have thus described a correspondence between (connected) coverings of X and subgroups of $\pi_1(X, x)$.

This correspondence is a close relative of the correspondence between field extensions and subgroups of Galois group in Galois theory.

Home assignment.

1. Let ϕ and ϕ' be two homotopic paths. Prove that the products $\phi\psi$ and $\phi'\psi$ are homotopic.
2. Prove that the action of $\pi_1(X, x)$ on $\widetilde{X}$ is given by the product of paths.
3. Prove that the fundamental groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic for connected X.
4. Prove that $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$. In particular, the fundamental group of an elliptic curve is $\mathbb{Z} \oplus \mathbb{Z}$.