

RIEMANN SURFACES

2. WEEK 2. BASIC DEFINITIONS

2.1. Smooth manifolds. Complex manifolds. Let X be a topological space. A (real) chart of X is a pair $(U, f : U \rightarrow \mathbb{R}^n)$ where U is an open subset of X and f establishes a homeomorphism of U with an open subset of \mathbb{R}^n .

The number n is called the (real) dimension of the chart, the open subset U of X is the domain of the chart.

Two charts $f : U \rightarrow \mathbb{R}^n$ and $g : V \rightarrow \mathbb{R}^m$ are compatible if the composition map (“change of coordinates”)

$$(1) \quad g \circ f^{-1} : f(U \cap V) \longrightarrow g(U \cap V)$$

is a diffeomorphism.

Note that any pair of disjoint charts ($U \cap V = \emptyset$) is compatible. If the intersection $U \cap V$ is nonempty, the compatible charts have the same dimension.

The meaning of the definitions is as follows. A chart $f : U \longrightarrow \mathbb{R}^n$ defines a coordinate system on U (a local coordinate system). A pair of compatible charts defines two coordinate systems on the intersection. Compatibility means that the passage from one coordinate system to the other is smooth.

2.1.1. Definition. A (smooth real) atlas on a topological space X is a collection of compatible smooth charts covering the whole X . A smooth manifold is a topological space endowed with a smooth atlas.

Usually, when one is talking about smooth manifolds, one assumes two extra conditions on the topological space X , in order to avoid some pathological possibilities. There they are.

- X is Hausdorff. This means that any two different points $x, y \in X$ have disjoint neighborhoods: there exist open neighborhoods $U \ni x, V \ni y$ such that $U \cap V = \emptyset$.
- X is countable at infinity: X is a union of a countable number of compacts.

2.1.2. Variations

If one requires that the transition functions $g \circ f^{-1}$ are of class C^k , this gives a definition of C^k -compatibility, C^k -atlas and C^k -manifold.

If one uses charts of type $f : U \longrightarrow \mathbb{C}^n$ instead of $f : U \longrightarrow \mathbb{R}^{2n}$ (which is practically the same), one is talking about complex charts of (complex) dimension n . In this case it makes sense to require a holomorphic version the notion of

compatibility. We will be mostly interested in the case of complex dimension one, where the notion of holomorphic function is known to us from the Complex Variables course.

We can now formulate the definition of the main object of our course.

2.1.3. Definition. A Riemann surface is a (connected) complex manifold of dimension one, Hausdorff and countable at infinity.

That is, a Riemann surface X has a collection of charts $f : U \longrightarrow \mathbb{C}$ covering the whole X so that the transition functions

$$g \circ f^{-1} : f(U \cap V) \longrightarrow g(U \cap V)$$

are holomorphic.

Two atlases on X are called equivalent if their union is an atlas (that is, if the charts of different atlases are compatible). We will not distinguish equivalent atlases.

2.1.4. Obvious examples

The complex plane \mathbb{C} is the first example of a RS. Its only chart is $U = \mathbb{C}$ with the identity map to \mathbb{C} .

The Riemann sphere $\widehat{\mathbb{C}}$ is the first example of a compact RS. Its atlas can be built from two charts: the identity map $\text{id} : \widehat{\mathbb{C}} - \infty \longrightarrow \mathbb{C}$ and the inverse $\phi : \widehat{\mathbb{C}} - \{0\} \longrightarrow \mathbb{C}$ defined as $\phi(z) = \frac{1}{z}$. Compatibility of the charts amounts to the fact that the function $w = \frac{1}{z}$ is a holomorphic bijection outside zero.

2.2. (Holomorphic) mappings between RS. Holomorphic and meromorphic functions. Let X and Y be two RS. A continuous map $f : X \rightarrow Y$ is called a holomorphic map (and we usually will not consider other maps between RS) if for each pair of charts $\phi : X \supset U \rightarrow \mathbb{C}$, $\psi : Y \supset V \rightarrow \mathbb{C}$, the composition

$$\phi(U \cap f^{-1}(V)) \xrightarrow{\phi^{-1}} U \cap f^{-1}(V) \xrightarrow{f} V \xrightarrow{\psi} \mathbb{C}$$

is holomorphic.

It is worthwhile to mention that this notion will not change if one replaces the atlases on X and on Y with equivalent (other) atlases.

The following special cases are of interest.

2.2.1. Holomorphic functions on a RS

Let X be a RS. A holomorphic map $f : X \rightarrow \mathbb{C}$ is called a *holomorphic function*. Let us make sure we get the notion we have expected to get.

By this definition, f is holomorphic iff for each chart $\phi : U \rightarrow \mathbb{C}$ of X the composition

$$\phi(U) \xrightarrow{\phi^{-1}} U \xrightarrow{f} \mathbb{C}$$

is holomorphic. In other words, a holomorphic function is a collection of compatible holomorphic functions on all charts.

2.2.2. Meromorphic functions on a RS

A holomorphic map $f : X \longrightarrow \widehat{\mathbb{C}}$ is called *a meromorphic function on X if it is not identically ∞* .

We leave as an exercise to prove that this definition is equivalent with the one given in the previous lecture:

A meromorphic function on X is a holomorphic function on $X - S$, S being a discrete subset of X , having only poles around the points of S .

The latter definition allows to claim that the set of meromorphic function on X forms a field. It is denoted $\text{Mer}(X)$.

2.3. Local structure of a map. Let $f : X \rightarrow Y$ be a non-constant holomorphic map (X is assumed to be connected). Let us show f is open that is that for any $U \subset X$ open the image $f(U)$ is open in Y .

A few reductions.

First of all, one can assume that U is small enough so that U belongs to a single chart of X and $f(U)$ belongs to a single chart of Y . In fact, any open set is a union of such open subsets, and f carries unions to unions.

Next, we claim that f cannot be constant at U . Since otherwise f will be constant in the whole chart containing U ; and if so, it will also be constant in all charts having a nonempty intersection with this chart, etc.

Since X is assumed to be connected, we will get f is constant on the whole X which contradicts to the assumptions.

Now we can talk about a single nonconstant holomorphic function $f : U \rightarrow \mathbb{C}$ defined at $U \subset \mathbb{C}$. By a standard theorem of Complex variables course f is open.

We can actually get a much more precise description of the local behavior of a holomorphic map f .

Let $z \in U$. Shifting the local coordinates in U and in \mathbb{C} by a constant, we can assume that $z = 0$ and $f(0) = 0$. Let

$$f(z) = c_n z^n + \sum_{i=n+1}^{\infty} c_i z^i, \quad c_n \neq 0,$$

be the Taylor expansion of f at 0.

We will assume for simplicity (and without loss of generality) that $c_n = 1$. Then $f(z) = z^n g(z)$ where $g(z) = 1 + \sum_{i=n+1}^{\infty} c_i z^{i-n}$. The function $h(z) = \sqrt[n]{g(z)}$ is holomorphic in a neighborhood of z (we assume it has value 1 at zero). The product $zh(z)$ vanishes at zero and has a nonzero derivative at zero, so it is invertible at a neighborhood of zero.

This means that if we choose another local parameter $w = zh(z)$, the map f will look as $f(w) = w^n$.

We have therefore proven that any nonconstant holomorphic map looks locally like the function $f(z) = z^n$.

2.3.1. Definition. We say that $f : X \rightarrow Y$ has ramification n at $x \in X$ if

$$f(z) = a_0 + a_n z^n + \dots$$

with $a_n \neq 0$ in local coordinates at a neighborhood of $x \in X$.

Note that the number n is one “almost everywhere”: if at some point the map f looks like $f(z) = z^n$, in the same coordinate system the value of the derivative $f'(z) = nz^{n-1}$ is nonzero outside $z = 0$.

Therefore, the set of points of X for which the number n is greater than zero, is discrete.

2.3.2. Proposition. *Let $f : X \rightarrow Y$ be a holomorphic map. Assume that X is compact. Then f is either constant or surjective.*

Note that if f is surjective, Y is necessarily compact.

Proof. If f is not constant, it is open. Therefore, $f(X)$ is open in T . Since the image of a compact set is compact, $f(X)$ has to be closed as well. Therefore, f is surjective and Y is compact. \square

2.3.3. Corollary. *Compact Riemann surfaces have no nonconstant holomorphic functions.*

2.4. Orientability. If $\phi : U \rightarrow \mathbb{R}^n$ and $\psi : V \rightarrow \mathbb{R}^n$ are two compatible charts having nonempty intersection, the transition map $\phi(U \cap V) \rightarrow \psi(U \cap V)$ is a diffeomorphism. This in particular means that the Jacobian of the transition map does not vanish. Since this is a real number, this means that it has a constant sign.

Two compatible charts are called *equally oriented* if the Jacobian of the transition map is positive.

An orientation of a manifold is a choice of an atlas consisting of equally oriented charts. There exist manifolds having no orientation! The simplest example is the Möbius band.

Fortunately for us, all Riemann surfaces have are oriented, considered as real two-dimensional manifolds.

In fact, any transition map is a holomorphic function of one complex variable. Let $z = x + iy$ and $f(z) = u(z) + iv(z)$ be a holomorphic function. Its Jacobian is the determinant of the matrix

$$(2) \quad \det \begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix} = u'_x v'_y - u'_y v'_x = (u'_x)^2 + (u'_y)^2 \geq 0.$$

We deduce therefore that

Riemann surfaces, considered as smooth manifolds, are orientable.

Home assignment.

1. Prove the equivalence of two definitions of meromorphic functions
 - A holomorphic map $X \rightarrow \widehat{\mathbb{C}}$
 - A holomorphic function outside a discrete subset $P \subset X$, having at most poles at the points of P .
2. Prove that a nonramified covering is a local homeomorphism.
3. Prove that the exponent $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$ is a covering. Prove that \mathbb{C}^* is isomorphic as Riemann surface to the quotient \mathbb{C}/L where $L = 2\pi i\mathbb{Z}$.