RIEMANN SURFACES

1. Week 1. Introduction: Examples, main problems, definitions

1.1. **Examples of RS. Holomorphic functions.** The following objects are Riemann surfaces.

1. The complex plane $\mathbb{C}$. The upper half-plane $\mathbb{H}$, or more generally, any open subset U in $\mathbb{C}$.

2. The Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

A more sophisticated example is given below (elliptic curve).

The most important property of a Riemann surface is that one can deal with holomorphic functions on it. In a neighborhood of each point one can choose a *local coordinate* which is a one-to one correspondence between the neighborhood and an open subset of $\mathbb{C}$.

For instance, the point $\infty \in \widehat{\mathbb{C}}$ has a local coordinate $w = \frac{1}{z}$. A function on a RS is holomorphic if it is presented locally ((that is, in each local coordinate) by an analytic function. For instance, a function $f : \widehat{\mathbb{C}} \to \mathbb{C}$ is holomorphic if its restriction to $\mathbb{C}$ is holomorphic, as well as its restriction to $\widehat{\mathbb{C}} - \{0\}$ written in the coordinate $w = \frac{1}{z}$.

Note: the notion is obviously good, but gives nothing for $\widehat{\mathbb{C}}$: any holomorphic function on the Riemann sphere is constant.

Recall:

Theorem. *(Liouville) A bounded holomorphic function on* $\mathbb{C}$ *is constant.*

This implies non-existence of nonconstant holomorphic functions on $\widehat{\mathbb{C}}$: any such function would be bounded as a function on $\mathbb{C}$, and, therefore, constant.

This is a very general phenomenon: we will see soon that there are no nonconstant holomorphic functions on *compact* Riemann surfaces.

1.2. **Meromorphic functions.** To have something interesting to study, we have to work with more general functions. These are, for instance, meromorphic functions. Recall that a meromorphic function on X is just a holomorphic function on a complement $X - S$ of X to a discrete subset S having at most poles at the points of S.

Recall that any isolated singular point x of a function f is of one of the following three types

- a removable singularity (Laurent series has no negative terms). In this case f extends to a function holomorphic at x.

- a pole (the Laurent series at x has finite number of negative terms).
- an essential singularity.

A holomorphic function on X can be viewed as a (holomorphic) map $f : X \to \mathbb{C}$. Furthermore, a meromorphic function on X can be viewed as a holomorphic map $f : X \to \widehat{\mathbb{C}}$. This differs a meromorphic function from a function having an essential singularity: the latter can not be extended to the sungular point in no sense.

Very often the amount of meromorphic functions on a Riemann surface is not very big and not very small — that is is worth studying.

1.2.1. Meromorphic functions on $\widehat{\mathbb{C}}$.

The result below is well-known from the Complex Variable course.

Lemma. *Meromorphic functions on $\widehat{\mathbb{C}}$ are just the rational functions in z.*

Proof. The rational functions are certainly meromorphic. Let f be a meromorphic function on $\widehat{\mathbb{C}}$. The set of poles of a meromorphic function is discrete, therefore, finite since $\widehat{\mathbb{C}}$ is compact.

In $z_1, \ldots, z_n \in \mathbb{C}$ are the poles of f of degrees $d_1, \ldots, d_n$ then $p := f \cdot \prod(z - z_i)^{d_i}$ has no poles in $\mathbb{C}$ and has at most a pole at ∞. This means that p is a polynomial, so that $f = \frac{p}{\prod(z-z_i)^{d_i}}$ is a rational function. □

The meromorphic functions on a Riemann surfaces X form a field denoted $\mathrm{Mer}(X)$. We have just proven that $\mathrm{Mer}(\widehat{\mathbb{C}}) = \mathbb{C}(z)$ is the field of rational functions of one variable. In particular, it is infinite-dimensional as a vector space.

If we wish to get some numerical information about $\widehat{\mathbb{C}}$, we can ask what is the dimension of the vector space of the meromorphic functions f having poles of degres $\leq d_i$ at given points $z_i,\ i = 1, \ldots, n$.

In the case of $\widehat{\mathbb{C}}$ these numbers can be easily calculated.

The answer is $\prod_i (d_i + 1)$, see Exercise 1.

1.2.2.

One of important general questions we will solve in the course: how does this result generalizes to general Riemann surfaces?

Even the mere existence of nonconstant meromorphic functions is an important and nontrivial result. In this course we will prove that any Riemann surface X admits a nonconstant meromorphic function. Recall that such function can be seen as a holomorphic map $f : X \to \widehat{\mathbb{C}}$. Moreover, this map has very nice topological property: it is a ramified covering. To explain the latter term, let us mention that a typical example of a ramified covering is the map $f : \mathbb{C} \to \mathbb{C}$ given by the formula $f(z) = z^n$.

1.3. **Analytic continuation.** Historically, the first Riemann surfaces have been defined as Riemann surfaces of a germ of analytic function.

Recall that any power series $f = \sum_{i=0}^{\infty} a_i(z-z_0)^i$ having a nonzero radius of convergence $r > 0$ defines an analytic function on the disc $\{z||z-z_0| < r\}$. For any point z_1 in this disc one can rewrite this function as a power series of $z - z_1$ with the radius of convergence at least $r - |z_1 - z_0|$ — but may be greater than that. This allows one to extend an analytic function defined initially by a power series to a domain greater then the original disc.

One proceeds as follows. We call a *function element* a pair (D, f) where D is a disc and f a power series around the center of D whose radius of convergence is (at least) the radius of D. A function element (D_1, f_1) is called an (immediate) analytic continuation of (D, f) if the center of D_1 belongs to D and the analytic functions f_1 and f coincide in $D_1 \cap D$.

Let $\gamma : [0,1] \longrightarrow \mathbb{C}$ be a continuous path in $\mathbb{C}$ with $a = \gamma(0), \quad b = \gamma(1)$. Analytic continuation of a function element along γ is a collection $t \mapsto (D_t, f_t)$ of analytic elements such that D_t is a disc with center at $\gamma(t)$ and for any $s < t$ such that $\gamma([s,t]) \subset D_s$ the pair (D_t, f_t) is an immediate analytic continuation of (D_s, f_s). One can easily prove (using compactness of a segment and continuily of the radius of convergence of f_t) that any analytic continuation along a path can be equally acomplished by a finite sequence of immediate analytic continuations.

Let now (D, f) be an analytic element and let (D_α, f_α) be the collection of all analytic elements which can be obtained from (D, f) by analytic continuation. We can assign to (D, f) the following space: it is the quition of the disjoint union of D_α by the equivalence relation generated by pairs of immediate analytic continuations.

The resulting space is locally isomorphic to $\mathbb{C}$; this is what is called the Riemann surface of (D, f).

In this way one constructs Riemann surfaces of some standard multivalued functions. Just two examples:

1.3.1. $w = \sqrt{z}$

If $z = re^{i\phi}$ then $w = \pm\sqrt{r}e^{\frac{i\phi}{2}}$.

In real analysis we were able to choose a positive value of the root; no this becomes impossible as there is no notion of "positivity" for complex numbers. However, one can choose a branch of the function on any simply connected domain U of $\mathbb{C}$ not containing the origin. In particular, one can choose $U = \mathbb{C} - \mathbb{R}_{\geq 0}$ and agree for $z = re^{i\phi}$ with $\phi \in]0, 2\pi[$ that $w = \sqrt{r}e^{\frac{i\phi}{2}}$.

One can go further and choose another copy $U_- = \mathbb{C} - \mathbb{R}_{\geq 0}$ and agree to define $w = \sqrt{r}e^{\frac{i\phi}{2}}$.

Note that the values of w at U are not compatible when we approach to positive real semi-line from above or from below. The same is true for values of w at U_-.

However, the values of w at U above the semi-line is compatible to the values of w at U_- below the semi-line. Thus, if we glue U with U_- so that the upper part of U is glued to the lower part of U_- and vice versa, we will get a univalent function w on the union of U with U_- (and with the real positive semi-line $\mathbb{R}_{>0}$).

This is an example of what one can get gluing together series expansions at different discs.

1.3.2. $w = \ln(z)$ Similarly to the above, if $z = re^{i\phi}$, one has

$$\ln(z) = \ln(r) + i\phi + 2\pi ik, \quad k \in \mathbb{Z}.$$

Now, in order to get a univalent function, we have to take a lot of copies $U_i, \quad i \in \mathbb{Z}$, and glue the lower part of U_i to the upper part of U_{i+1} for all i.

1.3.3. Example: Riemann surface of an algebraic function.

The first among two examples above can be generalized (and rerormulated) as follows.

Let $z = f(w)$ be a polynomial function of degree n. We know that for each value of $z \in \mathbb{C}$ there are precisely n (not necessarily different) values of w satisfying the equation. Thus, we can look at w as an n-valued function of z.

Let $X \in \mathbb{C}^2$ be the set of pairs (z, w) satisfying the equation. This set is called *an algebraic curve* given by the polynomial f (it would look like a curve if $\mathbb{C}$ were replaced with $\mathbb{R}$).

The projection $p : X \longrightarrow \mathbb{C}$ along the first coordinate send (z, w) to z. For a given z there are n points (counting with the multiplicities) in the preimage. Let us describe the points where the preimage consists of n *distinct* points. We know from the course of Algebra that the polynomial $f(w) - z$ has a multiple root w iff w is also a root of the derivative $f'(w)$. This means that w is a critical point (and $z = f(w)$ is then called a critical value of f).

We claim that X is almost precisely what one obtains by gluing together all elements of the analytic functions as explained at the beginning of the subsection. In fact, if we choose a non-critical value $z \in \mathbb{C}$ of f and $(z, w) \in X$, one has a germ of analytic function $w(z)$ in the neighborhood of (z, w). The corresponding Taylor expansion around any point (z, w) will converge in any disc with the center at z do not containing critical values of f. Thus, as a result of gluing all analytic elements we will get $X - p^{-1}(\textit{critical values of } f)$. In the case $f = z^2$ we get precisely $U \cup U_- \cup \mathbb{R}_{>0}$.

One can prove that there is a canonical way to compactify X, so that the resulting compacr Riemann surface $\widehat{X}$ maps to $\widehat{\mathbb{C}}$.

Moreover, an important result (which we hope to prove in this course) says that any compact RS is a RS of an algebraic function.

1.4. **Elliptic curves.** Fix $\tau \in \mathbb{C} \setminus \mathbb{R}$. Let $L = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$. The points of L are the vertices of a tiling of $\mathbb{C}$ with parallelograms obtained by shifting of "the fundamental parallelogram" defined as the convex hull of the points

$$0, 1, \tau, \tau + 1.$$

Define $T := \mathbb{C}/L$ — the quotient of the set $\mathbb{C}$ by the action of L given by the formula

$$\rho(\ell, z) = z + \ell, \ \ell \in L, \ z \in \mathbb{C}.$$

As a set, T can be presented as the fundamental parallelogram described above, with identified edges: $[0, 1]$ with $[\tau, \tau + 1]$ and $[0, \tau]$ with $[1, \tau + 1]$. All vertices are identified into one vertex.

One has a canonical map $\pi : \mathbb{C} \to T$ sending $z \in \mathbb{C}$ to its equivalence class. T is a topological space: a subset U of T is called open iff its preimage $\pi^{-1}(U)$ is open in $\mathbb{C}$.

The quotient space T is a Riemann surface (we cannot prove this formally since we do not have yet a formal definition of RS). Functions of T are just the functions on $\mathbb{C}$ invariant with respect to the action of L:

$$f(z) = f(z + \ell) \text{ for all } \ell \in L.$$

The Riemann surface T is called *the elliptic curve*. We will see later on that for different τ the corresponding elliptic curves are in general non-isomorphic.

We can immediately deduce that T has no nonconstant holomorphic functions: such a function would give a bounded holomorphic function on $\mathbb{C}$ which is impossible. It is very interesting to study meromorphic functions on T.

1.4.1. Meromorphic functions on T: elementary properties.

A meromorphic function f on T corresponds to a meromorphic function on $\mathbb{C}$ invariant under L. In other words, this is a meromorphic function having two noncollinear periods: 1 and τ. A classical name for such meromorphic function: *an elliptic function*.

Since T is compact, f has a finite number of poles in T; in other words, the corresponding function on $\mathbb{C}$ has a final number of poles inside the fundamental parallelogram (if there are poles at the boundary, we should choose only one representative in each equivalence class).

Let $z_1, \ldots, z_n$ be these poles and let $d_i, \ i = 1, \ldots, n$ be the degree of the pole of f at z_i. The degree of f is, by definition, the sum $\sum_{i=1}^n d_i$.

Recall that if a function f is holomorphic in a punctured neighborhood of a point z then its residue at z is defined as

$$\mathrm{Res}_z(f) = \frac{1}{2\pi i} \int f(w) dw,$$

where the integral is taken along a small circle around z.

Equivalently, $\mathrm{Res}_z(f)$ is the coefficient a_{-1} of the Laurent expansion of f around the point z.

Lemma. *One has*

$$\sum_{i=1}^{n} \mathrm{Res}_{z_i}(f) = 0$$

for any meromorphic function f on T.

Proof. Making a change of variables if necessary, we can assume that all poles z_i are strictly inside the fundamental parallelogram. Then the sum of the integrals is equal to the integral of f along a closed curve in the fundamental parallelogram containing all the poles. This integral obviously vanishes. □

As an immediate result of the above, we deduce that there are no meromorphic functions on T of degree one.

Another easy but interesting result.

Lemma. *Let f be a meromorphic function on T of degree d. Than f assumes any value $c \in \mathbb{C}$ precisely d times.*

Proof. Replacing f with $f - c$ we reduce Lemma to the claim that the number of zeroes of f equals the number of poles d.

Thus becomes a standard fact of Complex Variable course: one should apply the Residue Theorem to the logarithmic derivative $\frac{f'}{f}$. □

1.4.2. Weierstrass function $\wp$

We are looking for a meromorphic function $\wp$ having a degree two pole at 0 and no other poles in the fundamental parallelogram. This means that $\wp(z) = \frac{1}{z^2} + \alpha(z)$ where $\alpha(z)$ is a meromorphic function having no poles in the fundamental parallelogram. The above condition defines $\wp$ up to a constant (since the difference between the two should be holomorphic on T). Therefore, we can make everything unique if we require that $\alpha(0) = 0$. The latter means that $\wp(z) = \frac{1}{z^2} + z\beta(z)$ where β is a meromorphic function on $\mathbb{C}$ holomorphic inside the fundamental parallelogram.

The first suggestion could be

$$\sum_{\ell \in L} \frac{1}{(z-\ell)^2}.$$

This sum does not unfortunately converge. In order to get convergence, as well as the condition $\alpha(0) = 0$, we put

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \ell \in L} \left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2} \right). \tag{1}$$

Of course, one has to prove the above series converges. Moreover, now it becomes less obvious that $\wp(z)$ is an elliptic function (that is has periods 1 and τ). We leave these two claims for a while.

Recall that $\wp(z) = \frac{1}{z^2} + z\beta(z)$ where β has no poles in the fundamental parallelogram. Let us calculate the derivative. We get

$$\wp'(z) = -2\sum_{\ell\in L} \frac{1}{(z-\ell)^3}. \tag{2}$$

Thus, $\wp'(z)$ is an elliptic function of degree three; it has, therefore, three zeroes in the fundamental parallelogram. One can easily guess these zeroes: since $\wp'$ is an odd function, these are $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1}{2}+\frac{\tau}{2}$.

Consider now an elliptic function

$$f(z) = (\wp(z)-\wp(\frac{1}{2}))(\wp(z)-\wp(\frac{\tau}{2}))(\wp(z)-\wp(\frac{1}{2}+\frac{\tau}{2})). \tag{3}$$

The function f has degree six, and has three zeroes at $z = \frac{1}{2}, \frac{\tau}{2}$ and $\frac{1}{2}+\frac{\tau}{2}$. Moreover, the zeroes have multiplicity (at least) two, since the derivative $\wp'$ vanishes at these points.

This means that the function $\frac{f(z)}{(\wp'(z))^2}$ is holomorphic, and therefore, is constant. We have proven the following

1.4.3. **Theorem.** *One has*

$$(\wp')^2 = 4(\wp-e_1)(\wp-e_2)(\wp-e_3)$$

where $e_1 = \wp(\frac{1}{2})$, $e_2 = \wp(\frac{\tau}{2})$ *and* $e_3 = \wp(\frac{1}{2}+\frac{\tau}{2})$.

Proof. The only thing to check is the value of the constant. Since $(\wp')^2 \sim \frac{4}{z^6}$ and $f \sim \frac{1}{z^6}$ in the neighborhood of $z=0$, the constant is equal to 4. □

We are now able to describe all meromorphic functions on T.

Let f be a meromorphic function on T. Assume first of all that f is even, that is $f(z) = f(-z)$. In this case the set of poles and the set of zeroes of f are symmetric with respect to 0. Let

$$z_1,\dots,z_k,-z_1,\dots,-z_k$$

be the set of zeroes of f, each one appearing the number of times according to its multiplicity. Let

$$p_1,\dots,p_k,-p_1,\dots,-p_k$$

be the set of poles of f. Then the function

$$g(z) = \frac{\prod_{i=1}^k(\wp(z)-\wp(z_i))}{\prod_{i=1}^k(\wp(z)-\wp(p_i))} \tag{4}$$

has the same poles and the same zeroes as f (recall that $\wp$ is even!). Thus, f/g is holomorphic, and, therefore, constant.

Let now f be a general meromorphic function on T. One can present f as a sim f_0+f_1 where f_0 is even and f_1 is odd. Therefore, f_0 is a rational function of $\wp$, and $f_1/\wp'$ is as well a rational function of $\wp$. We have deduced the following description of meromorphic functions on T.

1.4.4. **Theorem.** *Let $T=\mathbb{C}/L$ where $L=\mathbb{Z}\oplus\mathbb{Z}\cdot\tau$, $\tau\in\mathbb{C}-\mathbb{R}$. Let $\wp$ be the Weierstrass function and $\wp'$ be its derivative. Then any meromorphic function f on T can be uniquely presented as*

$$f=g(\wp)+h(\wp)\wp'$$

where g and h are rational functions.

1.5. **Convergence of the series (1).** The problem of convergence of the series (1) is not so trivial. The simplest way is the following.

First of all, the series (2) converges absolutely and uniformly at any compact not containing the points $\ell\in L$ to a holomorphic function $\delta(z)$. The function $\frac{2}{z^3}+\delta(z)$ is holomorphic also at 0. Define now

$$\wp(z)=\frac{1}{z^2}+\int_0^z\left(\frac{2}{w^3}+\delta(w)\right)dw. \tag{5}$$

The following easy result from the Complex Variables course shows one can deduce a presentation (1) from the integral expression (5).

1.5.1. **Theorem.** *Let $f_i(z)$ be continuous functions in $D\subseteq\mathbb{C}$ so that $f(z)=\sum_{i=1}^\infty f_i(z)$ converges uniformly in D. Let γ be a path in D. Then*

$$\int_\gamma f(z)dz=\lim_{n\to\infty}\int_\gamma f_i(z)dz.$$

Let us show how one proves $\wp(z)$ is elliptic. The function $\delta=\wp'$ is obviously elliptic. This implies that $\wp(z+\ell)-\wp(z)$ is constant for any $\ell\in L$. The rest is left as an exercise.

Home assignment.

1. What is the dimension of the vector space of the meromorphic functions f on $\widehat{\mathbb{C}}$ having poles of degres $\le d_i$ at given points $z_i,\ i=1,\dots,n$?
2. Using the integral presentation (5) deduce
   - That $\wp(z)$ is an even function.
   - That $\wp(z+\ell)-\wp(z)$ is constant for any $\ell\in L$

Deduce from the properties above that $\wp$ is elliptic.

3. Define a map $T\longrightarrow\widehat{\mathbb{C}}$ using the Weierstrass function $\wp$. Prove that any point $z\in\widehat{\mathbb{C}}$ has two (counting with multiplicities) preimages. Which points of $\widehat{\mathbb{C}}$ have have only one preimage?