INTRODUCTION TO LIE GROUPS

1. Part 1: Introduction. Examples. Topological groups. $GL(n, \mathbb{R})$

1.1. **Historical remarks.** Lie is just a name.

Sophus Lie (1842–1899) developed the theory of continuous transformation groups (now: Lie groups) in the end of 19 century.

The theory of finite groups has been developed approximately at the same time (slightly earlier). Sylow was Lie's friend that taught Galois theory at Christiania (Oslo) university and proved foundational Sylow theorems.

1.2. **Examples.**

A Lie group is a group that has some extra geometric / topological properties. It takes time to grasp the formal definition — we start with some easy examples.

- $(\mathbb{R}^n, +)$.
- $GL(n, \mathbb{R})$ is a subset of matrices, that is a subset of $\mathbb{R}^{n^2}$. So this is not just a group but also a topological space.
- $SO(2, \mathbb{R})$ is a group of rotations of the 2-dimensional euclidean space, but also a circle.
- $SU(2)$, the collection of unitary matrices of determinant 1. A general element of $SU(2)$ has form

$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$$

  with $a, b, c, d \in \mathbb{R}$ satisfying the condition $a^2+b^2+c^2+d^2 = 1$. Thus, $SU(2)$ is a three-dimensional sphere.

1.3. **Topological groups.**

We will start with a somewhat easier notion of a topological group.

A topological group is a topological space G endowed with a structure of group, that is a map $m : G \times G \to G$ satisfying the following properties (describing compatibility of two structures).

1. m is a continuous map. Recall that to understand this we should remember that $G \times G$ is also a topological space where open subsets are unions of the subsets $U \times V$ where U, V are open in G.
2. The map $G \to G$ carrying g to g^{-1} is also continuous.

1.3.1. Let us show that $GL(n,\mathbb{R})$ is a topological group. The multiplication is given by matric multiplication which is given by simple formulas $\sum_k a_{ik}b_{kj}$. This is obviously a continuous function.

The passage to inverse has also an explicit formula including $1/det(A)$ and the adjoint matrix A' whose entries are the minors. Since the determinant is a continuos function and $x \mapsto 1/x$ is continuous when $x \neq 0$, this proves continuity of the inverse for $GL(n,\mathbb{R})$.

1.3.2. *Exercise.* Prove that if G is a topological group and H is a subgroup of G then H is a topological group in the induced topology. Prove that the closure $\bar{H}$ is a subgroup in G.

1.3.3. *Exercise.* Let G be a topological group and $U \subset G$ an open neighborhood of $1 \in G$. Prove that there exists an open neighborhood V of 1 such that $V \cdot V \subset U$.

1.3.4. The following is a very general claim for those who like abstract nonsense.

As we see, in order to define topological groups we use the notion of topological space and continuous map (these two notions together give rise to a category, the category of topological spaces), as well as the notion of direct product (that can also be defined in categorical terms). So, given any category $\mathcal{C}$ with finite products, one defines a group object G in $\mathcal{C}$ as an object G in $\mathcal{C}$ together with a morphism $m : G \times G \to G$ (product), $e : t \to G$ (the unit, where t is the final object of $\mathcal{C}$ that exists as the product of the empty set of objects) and $i : G \to G$ (inverse) satisfying the following properties:

- m is associative that is two maps $G \times G \times G \to G$, $m \circ (\mathrm{id} \times m)$ and $m \circ (m \times \mathrm{id})$, coincide.
- The compositions $G \to G \times G \stackrel{m}{\to} G$ where the first arrow is $\mathrm{id} \times e$ or $e \times \mathrm{id}$, is identity. This is the unit axiom.
- The composition $G \to G \times G \stackrel{m}{\to} G$ coincides with $G \to t \stackrel{e}{\to} G$. Here the first arrow is either $\mathrm{id} \times i$ or $i \times \mathrm{id}$.

In this way one can get many meaningful notion of groups with extra structures, including Lie groups ($\mathcal{C}$ is the category of smooth manifolds), group schemes, algebraic groups, analytic groups etc.

1.3.5. Lie groups are, by definitions, groups that are simultaneously groups and smooth manifolds, the structures beling compatible in a way similar to the compatibility of the structures in the notion of topological group.

For instance, $GL(n)$ is an open subset of $Mat(n,\mathbb{R})$ that is $\mathbb{R}^{n^2}$. The maps $m : GL(n) \times GL(n) \to GL(n)$ and the inverse $GL(n) \to GL(n)$ are in fact smooth (we know what is smoothness for a map defined at an open subset of $\mathbb{R}^N$ to $\mathbb{R}^M$). This is what makes $GL(n)$ a Lie group. To define Lie groups in general, we should know well what is the natural context for the notion of smooth map.

This is the context of smooth manifold. So, we will study some smooth manifold theory.

1.4. **Smooth manifolds.** A smooth manifold M is a topological space for which one can talk about smooth functions $f : U \to \mathbb{R}$ defined at an open set $U \subset M$. To talk about smoothness, it is nice to have coordinates, at least local coordinates. This leads to a definition of chart.

1.4.1. **Definition.**
- Let $U \subset M$ be an open subset. A chart on U is a continuous map $f : U \to \mathbb{R}^n$ for some n such that f defines a homeomorphism of U with $f(U)$ that is an open subset of $\mathbb{R}^n$.
- Two charts, $f : U \to \mathbb{R}^n$ and $g : V \to \mathbb{R}^m$ are compatible if the bijection $g \circ f^{-1} : f(U \cap V) \to g(U \cap V)$ is a diffeomorphism (that is is given by an invertible matrix of smooth functions, so that the inverse matrix has also smooth entries).

Note that if $U \cap V = \emptyset$ then any charts on U and V are compatible. If $U \cap V \neq \emptyset$ and the charts are compatible then necessarily $m = n$.

1.4.2. **Definition.** An atlas on a topological space M is a collection of compatible charts $f_\alpha : U_\alpha \to \mathbb{R}^n$ that covers M, $M = \cup U_\alpha$.

1.4.3. **Definition.** A smooth manifold M is a Hausdorff topological space with a countable base (of open subsets) with a chosen atlas. Two atlases on the same space M are called compatible if their union is also an atlas.

Note that if M is connected, the charts have all the same dimension called the dimension of M and denoted by $\dim(M)$. In general $\dim(M)$ is a locally constant function on M (constant on each component).

1.5. **Smooth maps. Product.** In order to define Lie groups, we still need two components. We have to define a smooth map between smooth manifolds. And we have to define the direct product of two smooth manifolds.

We do not distinguish manifold structures on the same space M defined by compatible atlases. Informally, his means that equivalent atlases determine the same manifold. More formally, we can say this in two ways: 1) We automatically add to the atlas all charts compatible to it. 2) After we define smooth maps between smooth manifolds, we will see that for a topological space endowed with two equivalent atlases, the identity map is an invertible smooth map (a diffeomorphism).

1.5.1. **Definition.** A continuous map $f : M \to N$ is smooth if for any pair of charts

$$a : U \to \mathbb{R}^m, \; b : V \to \mathbb{R}^n$$

with $U \subset M, V \subset N$, the map

$$a(U \cap f^{-1}(V)) \to \mathbb{R}^n$$

given by the composition $b \circ f \circ a^{-1}$ is smooth.

1.5.2. *Smooth functions.* In particular, given a manifold M and $U \subset M$ an open subset, we know what is a smooth function $f : U \to \mathbb{R}$. The set of smooth functions on U is a commutative ring denoted $C^\infty(U)$.

1.5.3. *Exercise.* A chart $a : U \to \mathbb{R}^n$ is compatible with an atlas on M iff for each open $V \subset U$ it establishes a bijection between $C^\infty(V)$ and $C^\infty(a(V))$, that is $f : a(V) \to \mathbb{R}$ is smooth iff the composition $f \circ a$ belongs to $C^\infty(V)$.

This means that the smooth structure on M is uniquely defined by the assignment $U \mapsto C^\infty(U)$.

1.5.4. *Exercise.* A map $f : X \to Y$ between two smooth manifolds is smooth iff for any open $U \subset Y$ and any $\phi \in C^\infty(U)$ the composition $f^{-1}(U) \xrightarrow{f} U \xrightarrow{\phi} \mathbb{R}$ is in $C^\infty(f^{-1}(U))$.

1.5.5. Given two smooth manifolds M and N, their cartesian product $M \times N$ has a structure of manifold. The charts for $M \times N$ are pairs of charts: if $a : U \to \mathbb{R}^n$ and $b : V \to \mathbb{R}^m$ are charts for $U \subset M$ and for $V \subset N$ respectively, $a \times b : U \times V \to \mathbb{R}^{m+n}$ gives a chart for $M \times N$. If $M = \cup U_\alpha$ and $N = \cup V_\beta$, $M \times N = \cup U_\alpha \times V_\beta$. Thus, we get an atlas for $M \times N$.

The projections $M \times N \to M$ and $M \times N \to N$ are obviously smooth.

1.5.6. **Lemma.** *Let M, N, K be three manifolds. A smooth map $K \to M \times N$ is uniquely defined by the pair of it compositions with the projections.*

Proof. Exercise. ☐

We are now ready to give the main definition of our course.

1.5.7. **Definition.** A Lie group is a manifold G together with a binary operation $m : G \times G \to G$ that is a smooth map converting G into a group, so that the map $G \to G$ carrying $x \in G$ to x^{-1}, is smooth.

Recall that all charts of a connected manifold have the same dimension. It is called the dimension of a manifold. In general, if a manifold in not connected, different components may have different dimensions.

1.5.8. **Lemma.** *All components of a Lie group G have the same dimension.*

Proof. Since the multiplication is a smooth map, for any $g \in G$ the left multiplication by g, $L_g : G \to G$ is a smooth map. It is invertible, so it is a diffeomorphism. This implies that the dimension of the component of 1 is equal to the dimension of the component of $g \in G$. ☐

Zero-dimensional Lie groups are just the discrete (countable) groups.

1.5.9. *Lie group homomorphism.* Let G, H be Lie groups. A map $f : G \to H$ is a Lie group homomorphism if it is a group homomorphism and a smooth map. It is a Lie group isomorphism if it is a group isomorphism and a diffeomorphism.

Example: $f : \mathbb{R} \to U(1)$ given by the formula $f(x) = \exp(ix)$.

2. Tangent space. Submanifold. Lie subgroup

2.1. **Tangent space.** We understand well what is a tangent vector to a curve in $\mathbb{R}^n$. Given a smooth curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ with $\gamma(0) = x$, a tangent vector $\dot{\gamma}(0)$ is a vector with components $(\dot{\gamma}_1, \ldots, \dot{\gamma}_n)$.

How to define the tangent space to a manifold M at a point x? Look at an example: $S^2 \subset \mathbb{R}^3$. Tangent vectors to S^2 at x belong to $\mathbb{R}^3$. In order to describe which vectors of $\mathbb{R}^3$ belong to $T_x(S^2)$, we can do the following: choose a smooth curve $\gamma : (-\epsilon, \epsilon) \to S^2$ with $\gamma(0) = x$. The composition of γ with the embedding to $\mathbb{R}^3$ is a differentialble map $(-\epsilon, \epsilon) \to \mathbb{R}^3$ and the tangent vector is $\dot{\gamma}(0)$.

This leads us to the following definition of $T_x(M)$ that does not require the embedding of M into $\mathbb{R}^n$.

2.1.1. **Definition.**
1. Two (smooth) curves $\gamma, \gamma' : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = \gamma'(0) = x$ are called equivalent if in a certain chart containing x (or, equivalently, in any chart) they define the same tangent vector at x.
2. We define $T_x(M)$ as the set of equivalence classes of curves as above.

Note that $T_x(M)$ is a vector space, but we do not know yet how to see this.

Here is a way. Choose a chart $a : U \to \mathbb{R}^n$ containing x. Then the assignment $\gamma \mapsto \frac{d}{dt}(a \circ \gamma)(0)$ defines a bijection from $T_x(M)$ to $\mathbb{R}^n$. Different charts define different bijections, but they differ by the Jacobi matrix:

if $b : U \to \mathbb{R}^n$ is another chart (we assume it is defined on the same open subset, otherwise we will replace it with the intersection), the bijections from $T_x(M)$ to $\mathbb{R}^n$ differ by the Jacobi matrix at $a(x)$ of the transformation $b \circ a^{-1} : a(U) \to b(U)$.

The above reasoning leads us to an alternative definition of the tangent space $T_x(M)$.

2.1.2. **Definition.** A tangent vector $v \in T_x(M)$ is, by this definition, a collection, for each chart $a : U \to \mathbb{R}^n$, of $v_a \in \mathbb{R}^n$, compatible in the following sense:

For any pair of charts $a : U \to \mathbb{R}^n$, $b : U \to \mathbb{R}^n$ one has $v_b = J(b \circ a^{-1})(a(x))v_a$, where J denotes the Jacobi matrix.

2.1.3. *Examples.* Since the manifolds we care most are Lie groups, let us calculate tangent spaces at 1 for different groups we mentioned before.

- Let $G = GL(n, \mathbb{R})$. This is an open subset of the space of $n \times n$-matrices, so, obviously, $T_1(GL(n)) = M_{n \times n}$.
- Let $G = SO(n, \mathbb{R})$ and let γ be a curve in G with $\gamma(0) = 1$. Then we can write $\gamma(s) = 1 + sA + o(s^2)$ where $A \in M_{n \times n}$ and we want to

describe possible values for A. Obviously $\gamma(s)^t = 1 + sA^t + o(s^2)$ so $1 = \gamma(s)\gamma(s)^t = 1 + s(A + A^t) + o(s^2)$ which implies that A is skew-symmetric. In the opposite direction, if A is skew-symmetric, $\gamma(s) = e^{sA}$ is orthogonal (remember Linear algebra?) and $e^{sA} = 1 + sA + o(s^2)$. We have proven that $T_1(G)$ in this case is the vector space of skew-symmetric matrices.

2.1.4. *Tangent map.* Let $f : M \to N$ be a smooth map, $x \in M$, $y = f(x)$. Any curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x$ gives rise to a curve $f \circ \gamma$ with $f \circ \gamma(0) = y$. It is easy to verify that equivalent curves are carried to equivalent curves, so this defines a map of tangent spaces

$$Tf : T_x(M) \to T_y(N).$$

It is easy to verify that this is a linear map. In the cse M and N are open subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$, Tf is the linear map given by the Jacobi matrix.

One has

2.1.5. **Lemma.**

$$T(g \circ f) = Tg \circ Tf.$$

This is actually what is called in the basic Analysis course the chain rule.

2.2. **Submanifold.**

2.2.1. **Definition.** A smooth map $f : M \to N$ is called immersion if, for any $x \in M$, the tangent map $T_x(f) : T_xM \to T_{f(x)}N$ is injective.

Note that immersion is not necessarily one-to-one. Example: a self-intersecting smooth curve in $\mathbb{R}^2$.

2.2.2. **Definition.** A subset M of a manifold N is an immersed submanifold if M is endowed with a structure of manifold such that the embedding $M \to N$ is an immersion.

2.2.3. *Example.* Note that the topology on M does not need to be induced from the topology on N. For example, let $N = S^1 \times S^1$. We define the map $f : \mathbb{R} \to N$ by the formula $f(x) = (x(\mod \mathbb{Z}), \pi x(\mod \mathbb{Z}))$. The map f is one-to-one, so its image is an immersed manifold. It is dense in N, so the preimage of any small disc in N is the disjoint union of infinite number of open segments in $\mathbb{R}$. This means that the topology on $\mathbb{R}$ is not induced from that on N.

2.2.4. **Definition.** An immersed submanifold M of a manifold N is called an embedded submanifold if the topology of M is induced from N.

An obvious example: an embedding of S^n into $\mathbb{R}^{n+1}$.

2.2.5. *Implicit function theorem.* Recall

Theorem. *(Implicit function theorem) Let $F : U \subset \mathbb{R}^{m+n} \to \mathbb{R}^n$ be a smooth map (given by a collection of n smooth functions of m+n variables). Assume that the Jacobi matrix $J = (\frac{\partial F_i}{\partial x_j})^{i=1,\dots,n}_{j=m+1,\dots,m+n}$ is invertible at a point $(a,b) \in \mathbb{R}^{m+n}$ and $f(a,b) = c \in \mathbb{R}^n$. Then there exist open neighborhoods $\mathbb{R}^m \supset V \ni a$, $\mathbb{R}^n \supset W \ni b$ and a unique smooth function $f : V \to W$ such that for $(x,y) \in V \times W$*

$$F(x,y) = c \Leftrightarrow y = f(x).$$

2.2.6. A smooth map $f : M \to N$ is called submersion at $x \in M$ if the tangent map $T_x f : T_x(M) \to T_{f(x)}(N)$ is surjective.

Fix $y \in N$ and define $X = f^{-1}(y) \subset M$. assuming that f is a submersion at any $x \in X$, the Implicit function theorem defines a structure of a closed submanifold on X.

Step 1 A chart near x. Choose a chart $a : U \to \mathbb{R}^n$ for M near x and a (small) chart $b : W \to \mathbb{R}^m$ near y. We get a map $b \circ f \circ a^{-1} : a(U) \to \mathbb{R}^m$ with a surjective Jacobian at $a(x)$. By the IFT $a(U)$ contains an open subset of the form $V \times W$ where V is a neighborhood of $\mathbb{R}^{n-m}$, W a neighborhood in $\mathbb{R}^m$ so that $\mathbb{R}^n = \mathbb{R}^{n-m} \oplus \mathbb{R}^m$ and there is a unique map $\phi : V \to W$ such that $(x, x') \in X$ iff $x' = \phi(x)$. The image of the map $(\mathrm{id}, \phi) : V \to V \times W$ is therefore an open neighborhood of x in X. Its inverse gives therefore a chart for X at x. Note that the dimension of the chart is $n - m$ as expected.

Step 2. We have to verify that the charts so constructed are compatible.

Step 3. The description of the charts for X given above implies that the tangent map $T_x X \to T_x M$ is an embedding (as it is given as the graph of $T(\phi) : TV \to TW$). Since the composition $T_x X \to T_x M \to T_y N$ is the tangent of the constant map, it is zero. Therefore $T_x X$ identifies with the kernel of $T_x f$.

2.2.7. The converse of the above also holds. Given an embedded submanifold $X \hookrightarrow M$, for any $x \in X$ there exists an open neighborhood U of x in M and a submersion $p : U \to N$ with $p(x) = y$ for which $X \cap U = p^{-1}(y)$.

We leave this claim without proof.

In particular, any embedded submanifold X of M is locally closed in the sense of the following lemma-definition.

2.2.8. **Lemma.** *For a subset X of a topological space M the following conditions are equivalent.*

1. *X is an intersection of an open and a closed subset of M.*
2. *X is open in its closure $\bar{X}$.*
3. *For any $x \in X$ there exists an open neighborhood U of x in M such that $X \cap U$ is closed in U.*

Proof. This is an easy exercise in general topology. $\square$

A subset X of a topological space M satisfying the above properties is called locally closed.

3. Elementary properties of Lie groups

3.1. **Connected component of 1.** Let G be a Lie group. We denote by G^0 the connected component of 1.

3.1.1. **Proposition.** *G^0 is a normal subgroup. This is a Lie group.*

Proof. A continuous image of a connected set is connected. The image of $G^0 \times G^0$ under the multiplication map is connected and contains 1, so it belongs to G^0. For any $g \in G$ the image of G^0 with respect to the map $x \mapsto gxg^{-1}$ is connected and contains 1, so belongs to G^0, so G^0 is a normal subgroup. This is obviously a Lie group. □

3.2. **Neighborhood of 1.**

3.2.1. **Theorem.** *Let G be a connected Lie group and let U be any neighborhood of 1. Then U generates G as an abstract group.*

Proof. Denote H the abstract subgroup of G generated by U. This is an open set as for any $h \in H$ the product hU is an open neighborhood of h in G. This implies that any coset gH is an open subset of G. Since the cosets are disjoint and open, they are also closed:

$$H = G - \cup_{g \notin H} gH.$$

Thus, H is both open and closed in G. Since G is connected and $H \neq \emptyset$, $G = H$. □

3.2.2. **Corollary.** *Let $f : G \to H$ be a Lie group homomorphism for which $T_1 f : T_1(G) \to T_1(H)$ is surjective. Assume that H is connected. Then f is surjective.*

Proof. Due to the inverse function theorem the image of f contains a neighborhood of $1 \in H$. Thus, it coincides with H. □

3.3. **Lie subgroup.**

3.3.1. **Definition.** A Lie subgroup of a Lie group G is a subgroup H that is simultaneously an immersed submanifold.

3.3.2. **Definition.** A closed Lie subgroup of a Lie group G is a subgroup H that is simultaneously an embedded submanifold.

Note that the terminology is slightly misleading: it is not clear from the definition that a closed Lie subgroup is necessarily closed as a subgroup!

Let us prove this now.

3.3.3. **Proposition.** *A closed Lie subgroup of a Lie group is in fact a closed subgroup.*

Proof. Let H be a closed Lie subgroup of G. Since H is an embedded submanifold of G, it is locally closed that is open in its closure $\bar{H}$. By Exercise 1.3.2, $\bar{H}$ is a (topological) subgroup of G and H is its open subgroup. Then H is also a closed subgroup of $\bar{H}$. Since H is dense in $\bar{H}$, $H = \bar{H}$. $\square$

3.4. **The space of cosets. Factor group.** We wish to repeat some parts of elementary group theory with Lie groups replacing discrete groups.

3.4.1. **Theorem.** *Let G be a Lie group and H a closed Lie subgroup. Then the set of cosets $G/H = \{gH | g \in G\}$ acquires a unique structure of a manifold so that the map $G \to G/H$ is smooth.*

The notion of quotient is always defined for topological spaces: if X is a topological space and R is an equivalence relation on X, X/R is defined as the topological space whose points are the equivalence classes, and the topology is defined by the condition

$U \subset X/R$ is open iff its preimage in X is open.

In particular, if G is a topological group and H is a subgroup that is not closed, G/H cannot be Hausdorff (as it has nonclosed points). This means that the requirement in the above theorem for H to be a closed Lie subgroup is reasonable.

Proof. Denote $\rho : G \to G/H$ the canonical projection and let us endow G/H with the quotient topology. For any open set $U \subset G/H$ we define $C^\infty(U)$ as the set of functions $f : U \to \mathbb{R}$ such that $f \circ \rho \in C^\infty(\rho^{-1}(U))$. There is at most one structure of smooth manifold on G/H with so defined collection of smooth efunctions.

To verify the existence of this smooth structure, it is enough to produce a covering collection of charts for G/H having the indicated above set of smooth functions. Then these charts will be automatically compatible and so they will define an atlas.

Let $\dim(G) = n$, $\dim(H) = m$. In Lemma 3.4.2 below we will find a (small) embedded connected submanifold U of G containing 1 and having dimension $n - m$ such that the map $m : U \times H \to G$ induced by the multiplication is an open embedding. Without loss of generality we can assume that U admits a global chart $a : U \to \mathbb{R}^{n-m}$.

The map $gU \times H \to G$ under multiplication is also an open embedding having $g \in G$ in its image. Its image $g\bar{U}$ in G/H is diffeomorphic to U so we get a chart for $g\bar{U}$. $\square$

3.4.2. **Lemma.** *Let H be a closed subgroup of a Lie group G. There exists a submanifold U of G with $1 \in U$ such that the multiplication $m : U \times H \to G$ is an open embedding.*

Proof. Step 1. Choose a submanifold U_0 of G containing 1 such that $T_1G = T_1U_0 \oplus T_1H$. This cal be done as follows. Choose an open neighborhood W of 1 at G with a chart $b : W \to \mathbb{R}^n$. The image $b(W \cap H)$ is an embedded submanifold in $b(W)$ that is open in $\mathbb{R}^n$. Choose a vector subspace V of $\mathbb{R}^n$ such that $T_{b(1)}(b(W \cap H)) \oplus V = \mathbb{R}^n$. We can chose a small neighborhood V_0 of 0 in V such that $b(1) + V_0$ intersects with $b(W \cap H)$ at one point $b(1)$ only. Then the submanifold $U_0 := b^{-1}(b(1) + V_0)$ satisfies the required properties.

Step 2 The map

$$U_0 \times H \to G$$

induced by the multiplication, induces an isomorphism of the tangent spaces at 1 (verify this!), so it induces an isomorphism of the tangent spaces at a neighborhood of 1. Now IFT asserts that there exist neighborhoods of 1, V_0 in H and U_0 (maybe, smaller than the original U_0) such that $U_0 \cap H = \{1\}$ and the multiplication map induces an open embedding

$$U_0 \times V_0 \to G. \tag{1}$$

We denote $W_0 = U_0V_0$. This is a neighborhood of 1 and, by Exercise 1.3.3, there exist $U_1 \subset U_0$ and $V_1 \subset V_0$ such that for $W_1 = U_1V_1$ one has $W_1W_1 \subset W_0$. Without loss of generality we can assume $W_1 = W_1^{-1}$ (otherwise take the intersection).

We now claim that the map

$$U_1 \times H \to G$$

induced by the multiplication is an embedding. Otherwise we would have $x, y \in U_1$ so that $y = xh$ with $h \in H$. Then $h = x^{-1}y$ belongs to $U_1U_1 \subset W_0 \cap H = V_0$ and then $y = xh$ contradicts the injectivity of (1).

□

3.4.3. *Quotient maps.* A smooth map $p : X \to Y$ is called *a quotient map* if the following conditions are fulfilled.

1. $U \subset Y$ is open iff $p^{-1}(U)$ is open in X.
2. $f : U \to \mathbb{R}$ is smooth iff $f \circ p$ is a smooth function on $p^{-1}(U)$.

For example, a map $\rho : G \to G/H$ constructed above is a quotient map.

Quotient maps enjoy a very special property.

3.4.4. **Proposition.** *Given a commutative diagram*

$$
\begin{tikzcd}
X \arrow[rr, "p"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}
\tag{2}
$$

where f and p are smooth and p is a quotient map, g has to be smooth.

Proof. Exercise. ☐

3.4.5. *Locally trivial fibrations.* A special case of a quotient map is given by a locally trivial fibration of smooth manifolds. Here is the definition.

Definition.
1. A map $p : X \to Y$ of smooth manifolds is called a trivial fibration with fiber Z (also a smooth manifold) if there is a diffeomorphism $\theta : X \to Y \times Z$ such that p is the composition of θ with the natural projection $Y \times Z \to Y$.
2. A map $p : X \to Y$ is a locally trivial fibration with fiber Z if there exists an open covering $Y = \cup U_i$ such that $p^{-1}(U_i) \to U_i$ is a trivial fibration with fiber Z.

3.4.6. *Exercises.* 1. Any locally trivial fibration is a quotient map.

2. If $p_i : X_i \to Y_i$, $i = 1, 2$, are locally trivial fibrations then $X_1 \times X_2 \to Y_1 \times Y_2$ is a locally trivial fibration.

3. Verify that the canonical map $G \to G/H$ is a locally trivial fibration with the fiber H.

3.4.7. **Proposition.** *Let H be a normal closed Lie subgroup. Then G/H admits a natural structure of a Lie group.*

Proof. We have a manifold G/H that also has a structure of a group. Let us verify that the structure maps

$$m : G/H \times G/H \to G/H, \ i : G/H \to G/H$$

are smooth. This can be easily proven using Proposition 3.4.7. In fact, the map $G \times G \to G/H \times G/H$ is a locally trivial fibration so the composition $G \times G \to G \to H$ factors through a smooth map $G/H \times G/H \to G/H$. ☐

3.4.8. *Example.* The groups $PGL(n, \mathbb{R}) := GL(n, \mathbb{R})/Z$ are Lie groups. Here Z is the group of scalar matrices.

3.5. **The action of a Lie group on a manifold.**

3.5.1. **Definition.** Let G be a Lie group and let M be a smooth manifold. An action of G on M is a smooth map

$$R : G \times M \to M$$

defining an action of G on M as an abstract set-theoretical action.

The latter means that

- $R(1, m) = m$.
- $R(gh, m) = R(g, R(h, m))$.

As for any action of a group on a set, it makes sense to talk about orbits of the action and stabilizers.

3.5.2. **Definition.**
- Two ponts $x, y \in M$ belong to the same orbit if for some $g \in G$ $y = g(x)$.
- For $x \in M$ its stabilizer is $\texttt{Stab}_G(x) = \{g \in G | g(x) = x\}$.

3.5.3. **Lemma.** *A stabilizer is a closed subgroup.*

Proof. This directly follows from the fact that the map $G \times M \to M$ is continuous. ☐

3.5.4. *Homogeneous spaces.* A smooth manifold on which a Lie group G acts transitively, is called a homogeneous space. We see that a homogeneous space can be presented as G/H where G is a Lie group and H a closed Lie subgroup.

3.5.5. *Exercise.*

- Identify S^n with the quotient of $O(n+1, \mathbb{R})$ by $O(n, \mathbb{R})$.
- Identify S^{2n+1} with the quotient of $U(n+1)$ by $U(n)$.

3.5.6. *Exercise.* The set of all m-dimensional vector subspaces in $\mathbb{R}^n$ is called the (real) Grassmannian $Gr(m, n)$. Prove that $GL(n, \mathbb{R})$ acts transitively on $Gr(m, n)$. Use this to define a smooth structure on $Gr(m, n)$. What is its dimension? Note that $Gr(1, n)$ is also called the projective $n-1$-space.

3.5.7. *Exercise.* A *lattice* in $\mathbb{C}$ is an abelian subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$ that generates $\mathbb{C}$ over $\mathbb{R}$ (so that $\text{Span}_{\mathbb{Z}}(1, \pi)$ is not a lattice). Define on the space M of lattices in $\mathbb{C}$ a structure of smooth manifold as follows. Define a transitive action of $GL(2, \mathbb{R})$ on M, find a stabilizer of a certain $x \in M$ and verify that it is a closed Lie subgroup. What is the dimension of M?

The following two results are more serious. We will prove them later.

3.5.8. **Proposition.**
1. *Any stabilizer* $\texttt{Stab}(x)$ *is a closed Lie subgroup of* G*, so that the quotient* $G/\texttt{Stab}(x)$ *has a natural structure of a smooth manifold.*
2. *For any* $x \in M$ *the natural map* $G/\texttt{Stab}(x) \to M$ *is an injective immersion whose image is the orbit of* x *(it needs not be closed).*

Here is an example of an action for which an orbit is an immersed but not embedded submanifold.

Take (as many times before) $M = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ and define the action of $\mathbb{R}$ on M by the formula

$$R(x,(y,z)) = (y+\alpha x, z+\beta x) \mod \mathbb{Z}^2,$$

where $\frac{\alpha}{\beta} \notin \mathbb{Q}$. Each orbit of such action is dense in M.

3.5.9. *Examples: left action of G on itself.* Also right action and adjoint action.

The left action of G on itself is defined by the formula

$$\ell : G \times G \to G,\ \ell(g,h) = gh.$$

Similarly, the right action is defined by the formula

$$r : G \times G \to G,\ r(g,h) = hg^{-1}.$$

The adjoint action is composed of the two:

$$\mathrm{Ad} : G \times G \to G,\ \mathrm{Ad}(g,h) = ghg^{-1}.$$

3.5.10. *Representations.* A finite dimensional representation of a Lie group G is a Lie group homomorphism $\rho : G \to GL(V)$ where V is a (usually complex) finite dimensional vector space.

In other words, a representation is given by a smooth map

$$R : G \times V \to V$$

such that V is endowed with an obvious structure of a manifold and $R(g,-) : V \to V$ is $\mathbb{C}$-linear for any $g \in G$.

4. Closed linear groups

4.1. **Exponent and logarithm.** A very big class of Lie groups appears as closed subgroups of $GL(n,\mathbb{R})$. We define *a closed linear group G* as a closed subgroup of $GL(n,\mathbb{R})$ (as a topological group).

Later on we will see that a closed linear group is automatically a closed Lie subgoup of $GL(n,\mathbb{R})$.

4.1.1. *Exercise.* Prove that $GL(n,\mathbb{R})$ is a closed subgroup of $GL(n,\mathbb{C})$. Prove that $GL(n,\mathbb{C})$ is a closed subgroup of $GL(2n,\mathbb{R})$.

4.1.2. *Example.* The subgroup of diagonal invertible matrices $T \subset GL(n,\mathbb{R})$ is a closed linear group isomorphic to $(\mathbb{R}^*)^n$.

The group of upper triangular invertible matrices $B \subset GL(n,\mathbb{R})$ is a closed linear group.

4.1.3. *Example.* Similarly, the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} | x, y, z \in \mathbb{R} \right\} \tag{3}$$

is a closed linear group. Note that $Z \subset H$ consisting of the matrices with $x = y = 0$ and $z \in \mathbb{Z}$, is a discrete subgroup of the center of H. Therefore, that the quotient H/Z is also a Lie group. We will see later that H/Z is not a linear group.

4.1.4. *Exponent of a matrix.* Recall that if $X \in M_n(\mathbb{R})$ then

$$\exp(X) = \sum_{n \geq 0} \frac{X^n}{n!}$$

is convergent and satisfies the following properties.

- $\exp(X + Y) = \exp(X) \cdot \exp(Y)$ provided X and Y commute.
- $\exp(X)\exp(-X) = 1$, in particular, $\exp(X) \in GL_n(\mathbb{R})$.

For $A \in M_n(\mathbb{R})$ the map

$$\phi_A : \mathbb{R} \to GL_n(\mathbb{R})$$

carrying $t \in \mathbb{R}$ to $\exp(tA)$ is a homomorphism of Lie groups. Its image is called the one-parametric subgroup defined by A. One has $\dot{\phi}_A(0) = A$, so A is the tangent at 1 defined by the curve ϕ_A.

4.1.5. **Lemma.** *Let $\phi : \mathbb{R} \to GL_n(\mathbb{R})$ be a homomorphism of Lie groups and let $A = \dot{\phi}(0)$. Then $\phi = \phi_A$.*

Proof. Let us calculate $\phi'(t) \in M_n(\mathbb{R})$. This is

$$\phi'(t) = \lim_{s \to 0} \phi(t + s)/s = \phi(t) \lim_{s \to 0} \phi(s)/s = \phi(t) \cdot A.$$

The same holds for $\phi_A : \mathbb{R} \to GL_n(\mathbb{R})$. Now the theorem on (existence and) uniqueness of a solution to ODE implies the result. $\square$

The map $\exp : M_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is smooth since it is given by a converging power series. Its tangent map $T_0 \exp$ is identity, so this is a local diffeomorphism. The inverse map is defined in a neighborhood of $1 \in GL_n(\mathbb{R})$. it is given by the logarithm functor

$$\log(X) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(X - 1)^k}{k}$$

that converges for $||X - 1|| < 1$. Note: for a real argument, log can be uniquely defined as a function inverse to exp. This is already wrong for a complex argument: the function inverse to exp is not uniquely defined, as $exp(z) = \exp(z + 2\pi i)$.

Maximum one can do is to choose a branch in a neighborhood of a point. A standard choice is to choose a power expansion of log at $z = 1$ that gives the value $\log(1) = 0$:

$$\log(z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(z-1)^k}{k}.$$

We can use the same power series expansion for the matrix argument. Similarly to the complex argument, the series converges for X having morm < 1.

Recall that the norm of an operator is the maximal absolute value of its (complex) eigenvalues.

4.1.6. **Lemma.** 1. *One has*

$$\exp(A) = \lim_{n\to\infty} (1 + A/n)^n.$$

2. *One has*

$$\prod_{k=1}^{n} (1 + A_k/n^2) = 1 + O(\frac{1}{n})$$

if $||A_k|| < C$.

Proof. Standard: compare, for instance, the coefficients of the power expansion. □

4.2. **Lie algebra of a closed linear group.** Let $G \subset GL_n(\mathbb{R})$ be a closed linear group. Its Lie algebra is defined as

$$\mathfrak{g} = \{X \in M_n(\mathbb{R}) | \exp(tX) \in G \ \forall t \in \mathbb{R}\}.$$

4.2.1. *Example.* Let $G = SL_n(\mathbb{R})$. This is a closed subgroup of $GL_n(\mathbb{R})$ as it is given inside GL_n by one (continuous) equation $\det(A) = 1$. Let us calculate the corresponding Lie algebra. $\det\exp(tX) = 1$ is equivalent to $\exp\operatorname{tr} tX = 1$ that is $\operatorname{tr} X = 0$. The Lie algebra of $SL_n(\mathbb{R})$ is denoted $\mathfrak{sl}_n(\mathbb{R})$. This is the space of matrices of trace 0.

The following result is very important.

4.2.2. **Lemma.** $A \in \mathfrak{g}$ *if and only if there exists a smooth curve* $\gamma : (-\epsilon, \epsilon) \to GL_n(\mathbb{R})$ *with values in* G *such that* $A = \gamma'(0)$.

Proof. The only if part is obvious. Let us assume $A = \gamma'(0)$ and deduce that $A \in \mathfrak{g}$, that is, that $\exp(At) \in G$ for all t. We will use the condition that G is a closed subgroup and we will present a sequence of elements in G converging to $\exp(At)$ for any given t. It is sufficient to do this for all $t < \epsilon$ as $\exp(At) = \exp(A\frac{t}{n})^n$ and one can always find n so that $\frac{t}{n} < \epsilon$. Then

$$\gamma(t/m)^m = (1 + At/m + O(t^2/m^2))^m = (1 + A(t/m))^m (1 + O(t^2/m^2))^m,$$

`(the last equality requires an explanation)` and so

$$\lim_{m\to\infty} \gamma(t/m)^m = \exp(At).$$

□

4.2.3. **Proposition.** *$\mathfrak{g}$ is a vector subspace of $M_n(\mathbb{R})$. If $X, Y \in \mathfrak{g}$ then $[X, Y] := XY - YX \in \mathfrak{g}$.*

Proof. 1. Let us show $\mathfrak{g}$ is a vector subspace of $M_n(\mathbb{R})$. First of all, $\mathfrak{g}$ is closed under the scalar multiplication. In effect, if $A \in \mathfrak{g}$, $\exp(tA) \in G$ for all t. This implies that $\exp(t\lambda A) \in G$ for all λ and all t, that is that $\lambda A \in \mathfrak{g}$. We will now prove that $\mathfrak{g}$ is closed under addition.

2. Let $A, B \in \mathfrak{g}$. Define $\gamma(t) = \exp(At)\exp(Bt)$. We have $\gamma'(0) = A + B$. By 4.2.2 this proves that $A + B \in \mathfrak{g}$.

3. Similarly we define, for fixed $s \in \mathbb{R}$

$$\gamma(t) = \exp(At)\exp(Bs)\exp(-At)\exp(-Bs).$$

One has $\gamma'(0) = A - \exp(Bs)A\exp(-Bs)$ so $A - \exp(Bs)A\exp(-Bs) \in \mathfrak{g}$. This gives a path in $\mathfrak{g}$ that has form $s[A, B] + O(s^2)$. Dividing by s, we get a sequence of points in $\mathfrak{g}$ converging to $[A, B]$. This proves the claim. □

The following theorem was proven by John von Neumann in 1929.

4.2.4. **Proposition.** *Any closed linear group is a closed Lie subgroup of $GL_n(\mathbb{R})$.*

Proof. Let G be a closed subgroup of $GL(n, \mathbb{R})$. Define the Lie algebra of G as above,

$$\mathfrak{g} = \{X \in M_n(\mathbb{R}) | \exp(tX) \in G\ \forall t \in \mathbb{R}\}.$$

We will find a neighborhood U of $0 \in M_n(\mathbb{R})$ and a neighborhood V of $1 \in GL(n, \mathbb{R})$, together with a diffeomorphism $\Phi : U \to V$, such that Φ induces a bijection between $\mathfrak{g} \cap U$ and $G \cap V$.

This will give a neightborhood V of $1 \in GL(n, \mathbb{R})$ where $V \cap G$ is a submanifold. This will imply that G is a closed Lie subgroup of $GL(n, \mathbb{R})$.

Choose an inner product on $M_n(\mathbb{R})$. We have $M_n(\mathbb{R}) = \mathfrak{s} \oplus \mathfrak{g}$ where $\mathfrak{s} = \mathfrak{g}^\perp$. We define

$$\Phi : M_n(\mathbb{R}) \to GL(n, \mathbb{R})$$

by the formula $\Phi(s + x) = \exp(s)\exp(x)$ where $s \in \mathfrak{s}$ and $x \in \mathfrak{g}$. Note that Φ is similar to the exponent map but not quite as $\exp(s)\exp(x) = \exp(s + x)$ only if $sx = xs$. Let us verify however that $\Phi : M_n(\mathbb{R}) \to GL(n, \mathbb{R})$ induces the identity $T_0\Phi = \mathrm{id}_{M_n(\mathbb{R})}$. In fact, choose $s + x \in M_n(\mathbb{R})$ and consider the image under Φ of the line $t \mapsto ts + tx$. We have

$$\Phi(ts + tx) = \exp(ts)\exp(tx) = 1 + ts + tx + O(t^2),$$

so $\frac{d}{dt}\Phi(ts + tx) = s + x$ is the image of $s + x$ under $T_0(\Phi)$.

By the inverse function theorem, Φ induces a diffeomorphism between an open neighborhood U_0 of $0 \in M_n(\mathbb{R})$ and an open neighborhood V_0 of $1 \in GL(n, \mathbb{R})$. We are not yet done as there might exist $h \in H \cap V_0$ that are not of the form $\Phi(x)$, $x \in \mathfrak{g} \cap U_0$. We will prove that one can find smaller neighborhoods $U \subset U_0$ and $V \subset V_0$ that fulfill the mentioned property.

Let us choose a collection of neighborhoods U_k of $0 \in M_n(\mathbb{R})$ forming the basis of topology (for instance, choosing U_k to be open balls of radius $1/k$). Let $V_k = \Phi(U_k)$ and assume, to the contrary, that $V_k \cap G \neq \Phi(U_k \cap \mathfrak{g})$. By definition of $\mathfrak{g}$, $\Phi(U_k \cap \mathfrak{g}) \subset V_k \cap G$, so, according to the assumption, for every k there is $s_k + x_k \in U_k$ such that $s_k \neq 0$ and $\exp(s_k)\exp(x_k) \in G$, that is $\exp(s_k) \in G$. Put $y_k = \frac{s_k}{||s_k||}$. The collection of points y_k belongs to the unit sphere $S \subset \mathfrak{s}$; it is infinite, so there is a subsequence converging to $y \in S$; we will denote this subsequence by y_k.

We will now prove that $\exp(ty) \in G$ for any $t \in \mathbb{R}$. This would lead to contradiction as this would mean that $y \in \mathfrak{g}$. Recall that $s_k \in U_k$ so $||s_k||$ tends to zero. Choose $m_k \in \mathbb{Z}$ such that

$$m_k||s_k|| \leq t < (m_k + 1)||s_k||.$$

Then $\exp(s_k)^{m_k} = \exp(m_k||s_k||y_k) \longrightarrow \exp(ty)$. Since $\exp(s_k) \in G$ and G is closed, $\exp(ty) \in G$.

$\square$

As Eli Cartan pointed out in 1930, literally the same proof works for general Lie groups: any closed subgroup of a Lie group is a closed Lie subgroup. To make sense of the above proof in the general context, one will have to define the general notion of exponent for Lie groups. We will do this later.

4.3. **Representations of Lie algebras.** We defined a (finite dimensional) representation of a Lie group as a Lie group homomorphism $G \to GL(V)$ (V is a real, or, more often, a complex vector spaces). If $G \subset GL(V)$ is a closed linear group, by definition its Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ that is just the Lie algebra of matrices with the operation $[A, B] = AB - BA$.

This observation leads to the following general definition.

4.3.1. **Definition.** Let $\mathfrak{g}$ be a Lie algebra. A representation of $\mathfrak{g}$ in a vector space V is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. In other words, a representation of $\mathfrak{g}$ is given by a bilinear map

$$\mathfrak{g} \times V \to V,\ (x, v) \mapsto x(v),$$

such that $[x, y](v) = x(y(v)) - y(x(v))$.

We will later see that any finite dimensional representation of a Lie group defines a finite dimensional representation of the corresponding Lie algebra.

5. Classical groups

As a special case of the theory presented above, we describe a collection of Lie groups called the classical groups.

These are groups $GL(n,\mathbb{F}), SL(n,\mathbb{F})$ as well as their subgroups of linear transformations preserving a certain non-degenerate form.

5.1. **Setup.** Let V be a finite dimensional vector space over a field $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Let $\phi : V\times V\to F$ be a non-degenerate form on V.

We require ϕ to be bilinear in case $\mathbb{F}=\mathbb{R}$. We require it to be bilinear or sesquilinear if $\mathbb{F}=\mathbb{C}$. We will also add a symmetricity or anti-symmetricity condition.

We define $G(\phi)$ as the subgroup of $GL(V)$ (or of $SL(V)$) preserving the form ϕ. According to the general theory, $G(\phi)$ is a closed subgroup of $GL(V)$, so it is a Lie subgoup. Moreover, the corresponding Lie algebra $\mathfrak{g}(\phi)$ consists of the endomorphisms $A : V\to V$ such that $\exp(tA)\in G(\phi)$ for all $t\in\mathbb{R}$.

Let us study when is this condition fulfilled.

We already know that $\exp(A)\in SL(V)$ iff $\mathrm{tr}(A)=0$. Let us describe when $\exp(A)$ preserves the form ϕ.

5.2. **Examples.**

5.2.1. Let $\mathbb{F}=\mathbb{C}$ and $\phi : V\times V\to\mathbb{C}$ be a symmetric bilinear form. All nondegenerate symmetric bilinear forms over $\mathbb{C}$ are equivalent, so we can think that ϕ is the standard form $\phi(x,y)=\sum x_iy_i$.

The corresponding closed linear group is the group of orthogonal matrices that is the collection of $A\in GL(n,\mathbb{C})$ satisfying $AA^t=I$. This group is denoted $O(n,\mathbb{C})$ (the complex orthogonal group). Its Lie algebra $\mathfrak{o}(n,\mathbb{C})$ consists of matrices X such that $\exp(sX)\exp(sX)^t=1$ for all s. This is equivalent to the condition $X+X^t=0$. Thus, $\mathfrak{o}(n,\mathbb{C})$ is the space of skew-symmetric matrices.

It is easy (but not necessary as we already know this in general) to verify that this is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})=M_n(\mathbb{C})$.

5.2.2. In the case $\mathbb{F}=\mathbb{R}$ there are non-equivalent symmetric bilinear forms. If the form is positively definite, we get the (real) orthogonal group

$$O(n,\mathbb{R})=\{A\in GL(n,\mathbb{R})|AA^t=I\}$$

and the corresponding Lie algebra $\mathfrak{o}(n,\mathbb{R})$ consisting of antisymmetric matrices. If we choose another form (Sylvester theorem classifies them all), we get *indefinite* orthogonal groups $O(p,q)$ and their Lie algebras $\mathfrak{o}(p,q)$. Note that $O(p,q)$ has four connected components if $p,q>0$, see discussion below. As usual, $SO(p,q)=O(p,q)\cap SL(n)$. This group consists of two components if $p,q>0$. The connected component of 1 of $O(p,q)$ is denoted by $SO^+(p,q)$.

5.2.3. The intersection $SL(n,\mathbb{F})\cap O(n,\mathbb{F})$ is denoted $SO(n,\mathbb{F})$. By the general theorem, Lie algebra of an intersection of closed linear groups is intersection of their Lie algebras. It is easy to see that $\mathfrak{o}(n,\mathbb{F})\subset\mathfrak{sl}(n,\mathbb{F})$ so that the groups $O(n,\mathbb{F})$ and $SO(n,\mathbb{F})$ have the same Lie algebras. It is easy to see that, similarly to the real case, $SO(n,\mathbb{C})$ is the connected component of 1 in $O(n,\mathbb{C})$.

5.2.4. Let $\phi : V\times V\to\mathbb{C}$ be an inner product. This is a form that is antilinear in the first argument and linear in the second. It is symmetric in a skewed sense: $\phi(w,v)=\overline{\phi(v,w)}$.

In an orthogonal basis one has $\phi(x,y)=\sum\bar{x}_iy_i$.

The linear transformations preserving a fixed inner product, is called unitary. The corresponding group, called the unitary group, is

$$U(n)=\{A\in GL(n,\mathbb{C})|AA^*=I\},$$

where, as usual, $A^*=\overline{A^t}$.

The corresponding Lie algebra $\mathfrak{u}(n)$ consists of matrices X such that $\exp(tX)$ is unitary for all $t\in\mathbb{R}$. One has

$$\exp(tX)^{-1}=\exp(-tX),$$

so $\exp(tX)\in U(n)$ for all $t\in\mathbb{R}$ iff $tX^*+tX=0$ which implies that $X^*+X=0$. Thus, $\mathfrak{u}(n)$ consists of skew-Hermitean matrices:

$$\mathfrak{u}(n)=\{X\in GL(n,\mathbb{C})|X^*=-X\}.$$

5.2.5. As before, we define $SU(n)=SL(n,\mathbb{C})\cap U(n)$. This is a closed linear group with Lie algebra

$$\mathfrak{su}(n)=\mathfrak{u}(n)\cap\mathfrak{sl}(n,\mathbb{C}).$$

Note that this is a smaller algebra than $\mathfrak{u}$ as the trace of a skew-Hermitean matrix needs not be zero.

5.2.6. It is interesting to study the special cases for small n.

For instance, $U(1)$ is the group of complex number of modulus 1. The homomorphism

$$\det : GL(n,\mathbb{F})\to\mathbb{F}^*$$

is a Lie group homomorphism for $\mathbb{F}=\mathbb{R},\mathbb{C}$ with the kernel $SL(n,\mathbb{F})$. Its restriction gives the following surjective homomorphisms.

$$\det : O(n,\mathbb{F})\to O(1,\mathbb{F})=\{\pm1\}$$

with the kernel $SO(n,\mathbb{F})$,

$$\det : U(n)\to U(1)$$

with the kernel $SU(n)$.

5.2.7. Let now V be an even-dimensional vector space over $\mathbb{F}$ and let

$$\phi : V \times V \to \mathbb{F}$$

be a non-degenerate antisymmetric bilinear form. One can choose a basis $V = \mathrm{Span}\{v_i, w_i, i = 1, \ldots, n\}$ such that for $v = \sum x_i v_i + \sum y_i w_i$ and $v' = \sum x'_i v_i + \sum y'_i w_i$, one has

$$\phi(v, v') = \sum (x_i y'_i - x'_i y_i).$$

The group $Sp(2n, \mathbb{F})$ consists of the matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying the conditions

$$A^t C = C^t A; \quad B^t D = D^t B; \quad A^t D = 1 + C^t B.$$

5.2.8. *Exercise.* Calculate the Lie algebra of $Sp(2n)$. In other words, find matrices X such that $\exp(sX) \in Sp(2n)$ for all $s \in \mathbb{R}$.

5.3. **Small ranks.** It is worth noting that

- $Sp(2, \mathbb{F}) = SL(2, \mathbb{F})$.
- $SO(2, \mathbb{R}) = U(1)$ is (topologically) a circle.
- $SU(2)$ is a 3-dimensional sphere.

5.4. $SU(2)$ **and** $SO(3)$**.** In this subsection we will define a very interesting Lie group homomorphism

$$\pi : SU(2) \to SO(3)$$

and will study its properties.

5.4.1. Recall that $SU(2)$ consists of 2×2 complex matrices of form $\begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix}$ satisfying the condition $a^2 + b^2 + c^2 + d^2 = 1$.

To define the homomorphism π, we have to define first of all a real 3-dimensional representation of $SU(2)$. Below we will do precisely this.

First of all, let us define an associative algebra over $\mathbb{R}$ of dimension 4 called *the quaternion algebra*. It has a basis $\{1, i, j, k\}$ over $\mathbb{R}$, $\mathbb{H} = \mathrm{Span}_{\mathbb{R}}\{1, i, j, k\}$, with the multiplication defined by the formulas

- $i^2 = j^2 = k^2 = -1$.
- $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.

The algebra (that is, an associative ring containing $\mathbb{R}$ in its center) $\mathbb{H}$ is very interesting. It contains the complex numbers as the set of expressions $a + bi$.

Let us look a little bit closer to the quaternions. Let us prove, for instance, that any nonzero quaternion is invertible. The proof is actually the same as for the complex numbers. Given $q = a + bi + cj + dk \in \mathbb{H}$, we define $\bar{q} = a - bi - cj - dk$. An easy calculation shows $q\bar{q} = a^2 + b^2 + c^2 + d^2$ si that, unless $q = 0$, $q\bar{q}$ is a positive real number. Thus, if we define $|q| = \sqrt{q\bar{q}}$, we get the formula $q^{-1} = \frac{1}{|q|^2}\bar{q}$.

Recall that $\{z \in \mathbb{C} | |z| = 1\} = U(1)$. We will now show that

$$SU(2) = \{q \in \mathbb{H} | |q| = 1\},$$

both as abstract groups and as smooth manifolds.

The isomorphism assigns to a quaternion $q = a + bi + cj + dk$ of absolute value 1 the matrix $\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$. It is straighforward to verify that the product of quaternions corresponds to the product of unitary matrices.

Now, any $q \neq 0$ define an automorphism π_q of $\mathbb{H}$ by the formula

$$\pi_q(x) = qxq^{-1}.$$

One has $|\pi_q(x)| = |x|$. Moreover, the space of pure quaternions $\mathbb{H}_0 = \mathrm{Span}\{i, j, k\}$ is invariant under π_q. Therefore, one has the action of $SU(2)$ on the three-dimensional real space $\mathbb{H}_0$ by orthogonal transformations.

This yields a group homomorphism $\pi : SU(2) \to SO(3)$. Its kernel is the set of quaternions of absolute value 1 commuting with all pure (and therefore, all) quaternions.

It is an easy exercise to verify that $\mathrm{Ker}\phi = \{\pm 1\}$.

5.4.2. *π is surjective.* The groups $SU(2)$ and $SO(3)$ are both three-dimensional. The tangent map $T_1\pi$ is therefore an isomorphism, therefore, π is a local isomorphism. This implies that a neighborhood of 1 in $SO(3)$ belongs to the image of π. Therefore, π is surjective.

In the next section we will study in more detail Lie group homomorphisms with discrete kernel.

5.5. **Connected components.** If G is a Lie group and G^0 is the connected component of 1, The quotient G/G^0 is a discrete group. This group identifies with the set of connected components of G.

The set of connected components of a (reasonable) topological space, for instance, of a manifold, is denoted $\pi_0(X)$. We have just verified that, if G is a Lie group, $\pi_0(G)$ is a group. In this subsection we will descuss the group of components of classical groups.

The following result is useful in determining $\pi_0(X)$.

5.5.1. **Lemma.** *Let $p : X \to Y$ be a locally trivial fibration with fiber F. If F and Y are connected, X is also connected.*

Proof. We will discuss a more precise claim in the next section. This one can be proven as an exercise. □

We know that if H is a closed Lie subgroup of a Lie group G then the natural map $G \to G/H$ is a locally trivial fibration with fiber H. This implies that if a Lie group G acts transitively on a connected manifold X with connected stabilizer, then G is connected.

5.5.2. Appying the results of Exercise 3.5.5, we can deduce that $U(n)$ are connected and $O(n)$ have two connected components, $SO(n)$ being the connected component of 1 in $O(n)$.

5.5.3. By QR decomposition (see Linear algebra course) any real invertible matrix A can be uniquely presented as a product QR where Q is orthogonal and R is upper triangular with positive entries at the diagonal. This implies that $GL(n,\mathbb{R})$ has two connected components, exactly as $O(n,\mathbb{R})$.

5.5.4. Let $O(p,q;\mathbb{R})$ be the real group preserving the quadratic form

$$\sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2,$$

This group consists of matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ satisfying the conditions

$$A^tA = I + C^tC;\ A^tB = C^tD;\ D^tD = I + B^tB.$$

The matrix C^tC represents a non-negative self-adjoint operator. This means that A^tA is positively definite hense A is invertible. Similarly D is invertible. This yields (at least) four connected components for $O(p,q)$, determined by the signs of $\det(A)$, $\det(D)$.

6. Covering spaces. Simply connected groups

A part of this section is just a chapter in elementary topology. In it we define the notion of covering of a topological space. If X is a smooth manifold, any its covering is also a smooth manifold.

There always exists a universal covering of a (good) connected topological space. If X is a connected Lie group, any its covering is also a Lie group. A Lie group and it covering have the same Lie algebra.

6.1. **Covering. Universal covering.**

6.1.1. **Definition.** A (continuous) map $\pi : Y \to X$ of topological spaces is called a covering if for any $x \in X$ there exists a neighborhood $U \ni x$ and a set F (considered as a dicrete topological space) together with and homeomorphism $\theta : \pi^{-1}(U) \to U \times F$ such that $p_1 \circ \theta = \pi$.

6.1.2. *Examples.*

- The map $f : \mathbb{C}^* \to \mathbb{C}^*$, $f(z) = z^n$, is a covering.
- The map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^n$ is not a covering.
- The map $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering.

In what follows we will assume that X is a manifold. In fact the requirement should be much weaker (it is enough for X to be locally simply connected), but we will not see more general topological spaces.

Here is the most important property of covering spaces.

6.1.3. **Proposition.** *Let $\pi : Y \to X$ be a covering. Given $y \in Y$, $x = \pi(y)$ and $f : [0,1] \to X$ a continuous map with $f(0) = x$, there exists a unique continuous map $g : [0,1] \to Y$ such that $f = \pi \circ g$ and $g(0) = y$.*

Proof. For any $t \in [0,1]$ there exists an open set $U_t \subset X$ such that $f(t) \in U_t$ and a homeomorphism $\theta_t : \pi^{-1}(U_t) \to U_t \times F_t$ as in the definition of covering.

Since $[0,1]$ is compact, there is a finite number of points $t_1, \ldots, t_n$ such that $f([0,1]) \subset \cup U_i$ where $U_i := U_{t_i}$. We can now show by induction in $k = 1, \ldots, n$ that the restriction $f : [0, t_k] \to X$ uniquely lifts to $g : [0, t_k] \to Y$. This is very easy and is left as an exercise. $\square$

6.1.4. *Connectedness. Path-connectedness.* A topological space X is connected if it has no nontrivial decomposition $X = U_1 \cup U_2$ as a union of open disjoint subsets.

X is called path-connected if any two points $x, y \in X$ can be connected by a continuous path: there exists $f : [0,1] \to X$ such that $x = f(0),\ y = f(1)$.

Any path-connected space is connected: assume, on the contrary that X is path connected but $X = U_1 \cup U_2$ is a nontrivial decomposition into disjoint union of open subsets. Choose $x \in U_1$ and $y \in U_2$ and left $f : [0,1] \to X$ be a path connecting x with y. Then $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are two nonempty disjoint open subsets of $[0,1]$. Contradiction.

For our kind of spaces the two notions of connectedness coincide. Let $x \in X$ and define U as the set of all $y \in X$ that can be connected to x with a path. Then it is easy to see that U is both open and closed. (We use here that any point of X has a neighborhood that is path connected. The is obviously so for manifolds).

6.1.5. *Homotopy. Simply-connected spaces.* Two maps $f, g : [0,1] \to X$ are called *homotopic* if $x := f(0) = g(0)$ and $y := f(1) = g(1)$ and there exists $h : [0,1] \times K \to X$ such that

- $h(0, -) = f,\ h(1, -) = g$
- $h(-, 0) = x,\ \ h(-, 1) = y.$

Note: this is a special kind of homotopy, the one "preserving the ends".

A connected space X is called *simply connected* if for any two maps $f, g : [0,1] \to X$ with $x := f(0) = g(0),\ y := f(1) = g(1)$ there exists a homotopy between f and g.

6.1.6. *Examples.*

1. $X = \mathbb{R}^n$ is simply connected, for one can define
$$h(s,t) = sf(t) + (1-s)g(t).$$
2. $X = S^1$ the circle is not simply connected as the path around the circle cannot be contracted to a point.
3. $X = S^n$, $n > 1$, is simply connected. Here is the proof: $X - \{x\}$ is homeomorphic to $\mathbb{R}^n$ is contractible. For any two paths f, g with common ends let us find $x \in X$ that does not belong to them. Choose a homotopy between f and g inside $X - \{x\}$. This, in particular, with give a homotopy in X.

Here is a continuation of Proposition 6.1.3.

6.1.7. **Proposition.** *Let $\pi : Y \to X$ be a covering, $f, f' : [0,1] \to X$ two paths with $f(0) = f'(0) = x$, $f(1) = f'(1) = y$. Let g, g' be the liftings of f and f' with $g(0) = g'(0) = z$. $\pi(z) = x$. Then, if f and f' are homotopic, $g(1) = g'(1)$ and the paths g and g' are homotopic.*

Proof. Shortly: For any $(s,t) \in [0,1] \times [0,1]$ there is $U_{s,t} \subset X$ open so that the preimage $\pi^{-1}(U_{s,t})$ is homeomorphic to a product. Now we can divide the square $[0,1] \times [0,1]$ into N^2 small squares so that each one of them belongs to a certain $U_{s,t}$. We can now reduce the lifting of h to consecutive lifting of the restriction of h to one of the N^2 small squares. This is easy. $\square$

6.1.8. *Universal cover.* Let now X be a connected space. We will construct a simply connected space $\tilde{X}$ together with a covering $\pi : \tilde{X} \to X$. We will see later that this covering satisfies a certain inversal property. This is why we will call it a universal cover of X.

6.1.9. *Construction of $\tilde{X}$.* Choose $x \in X$ and define a set $\tilde{X} = Z/\sim$, a quotient of Z by an equivalence relation, as follows.

- Z is the set of continuous maps $f : [0,1] \to X$ with $f(0) = x$.
- The paths $f, g \in Z$ are equivalent iff $f(1) = g(1)$ and f and g are homotopic, that is, there exists $h : [0,1] \times [0,1] \to X$ satisfying the properties listed in 6.1.5.

The map $\pi : \tilde{X} \to X$ carries f to $f(1)$. We will now define a topology on $\tilde{X}$ so that π becomes a covering and $\tilde{X}$ is simply-connected. For any $y \in X$ choose a simply-connected neighborhood U_y (here we use that the topology of X is good enough). The set $F := \pi^{-1}(y)$ identifies with the set homotopy classes of paths connecting x to y. We define the map
$$\pi^{-1}(U_y) \to U_y \times F$$

by a pair of maps; the first one is the restriction of π, whereas the map $\pi^{-1}(U_y) \to F$ carries any path from x to $z \in U_y$ to its composition with a path in U_y from z to y.

It is easy to verify that the map so defined is a bijection. We define topology on $\pi^{-1}(U_y)$ so that this ibjection becomes a homeomorphism.

Let, finally, prove that $\tilde{X}$ is simply connected. Any two paths connecting two points y and z in $\tilde{X}$ give rise to two paths connecting $\pi(y)$ with $\pi(z)$ in X. Composing them with a path from x to $\pi(y)$ defined (up to homotopy) by y, we get two paths from x to $\pi(z)$ that lift to the same point z in $\tilde{X}$. This means that they are homotopic. This implies that two original paths from y to z are homotopic.

6.2. **Universal property.** Let X be connected, $x \in X$ and let $\pi : \tilde{X} \to X$ be the universal covering of X constructed as above (note that the construction slightly depends of the choice of x). We denote by $x_0 \in \tilde{X}$ the class of the constant path from x to x.

Let now $\rho : Y \to X$ be a covering.

6.2.1. **Proposition.** *There is a one-to-once correspondence between maps $f : \tilde{X} \to Y$ over X (that is, satisfying $\pi = \rho \circ f$) and the points $y \in \rho^{-1}(x)$. The correspondence assigns to $f : \tilde{X} \to Y$ the image $f(x_0)$.*

Proof. For any $z \in \tilde{X}$ choose a path h connecting x_0 to z. Its image in X will connect x with $\pi(z)$. There is a unique lifting of this path to a path in Y connecting y with a certain point that will be now declared the image of z. The construction does not depend on the choice of the path connecting x_0 to z as $\tilde{X}$ is simply connected. ☐

6.2.2. **Corollary.** *There is a one-to-one correspondence between the set of maps $\tilde{X} \to \tilde{X}$ over X and the set $\pi^{-1}(x)$. Each such map is a homeomorphism; Therefore, the set $\pi^{-1}(x)$ acquires a group structure where x_0 is the unit element. This is the group of homotopy classes of paths (loops) connecting x with itself.*

6.2.3. **Definition.** The fundamental group $\pi_1(X, x)$ is defined as the group of automorphisms of $\tilde{X}$ or as the group of homotopy classes of loops at x in X.

In the case when X is not connected, $\pi_1(X, x)$ is defined as $\pi_1(X_0, x)$ where X_0 is the connected component of x in X.

The fundamental group of X, as defined above, depends on the choice of a base point $x \in X$. What is the connection between $\pi_1(X, x)$ and $\pi_1(X, y)$? There is no connection if x and y belong to different components of X. Otherwise we can assume that X is connected. In this case there is a path γ connecting x to y. Any loop α around x defines a loop $\gamma \circ \alpha \circ \gamma^{-1}$ around y. This is obviously an isomorphism of groups $\pi_1(X, x)$ and $\pi_1(X, y)$. Note that a path γ' homotopic to γ defines the same isomorphism of the fundamental groups; non-homotopic paths may define different isomorphisms.

The same reasoning can be reformulated in terms of universal coverings.

Given two universal coverings $\tilde{X}$ and $\tilde{X}'$ of X, a homeomorphism $\tilde{X} \to \tilde{X}'$ is uniquely given by an image of $x_0 \in \tilde{X}$ in $\tilde{X}'$. If $\tilde{X}$ is obtained by the above construction with the base point $x \in X$ and $\tilde{X}'$ by the same construction with the base point y, The choice of a point over x in $\tilde{X}'$ is equivalent to a choice of a homotopy class of a path connecting y to x.

6.3. **Fundamental group and coverings.** Let X be a connected topological space, $x \in X$. We can now present a full classification of coverings of X, in terms of the fundamental group $\pi_1(X, x)$.

Recall that, given a group G and a set F, an action of G on F is a group homomorphism $\rho : G \to \mathrm{Aut}(F)$. Equivalently, this is a map $r : G \times F \to F$ satisfying the following properties.

- $r(1, x) = x$.
- $r(g, r(h, x)) = r(gh, x)$.

A map $a : F_1 \to F_2$ of G-sets is called a G-map if for any $g \in G$ and any $f \in F_1$ one has $g(a(f)) = a(g(f))$.

Given a covering $p : Y \to X$, we assign to it the set $F = p^{-1}(x)$ on which the group $G = \pi_1(X, x)$ acts is follows. Given $g \in G$ represented by a loop $\gamma : x \to x$, and $f \in F$, there is a unique lifting of γ that starts at f; its end will be denoted $g(f)$.

6.3.1. **Theorem.** *The construction described above establishes an bijection between the (isomorphism classes) of G-sets and (isomorphism classes of) coverings. Any G-map $F_1 \to F_2$ defines a map of the corresponding covernings of X.*

It is interesting to compare properties of coverings with the properties of the corresponding G-sets.

6.3.2. **Proposition.** *A coverning $p : Y \to X$ is connected iff the corresponding G-set F is transitive, that is for any $f, f' \in F$ there exists $g \in G$ such that $f' = g(f)$.*

6.3.3. **Corollary.** *Let $p : Y \to X$ be a connected covering of a connected space X, $y \in Y$, $x = p(y)$. The fundamental group $\pi_1(Y, y)$ identifies with the stabilizer of $y \in p^{-1}(x)$ under the action of $\pi_1(X, x)$.*

6.4. **Some fundamental groups.** Recall that a map $f : X \to Y$ is called a homotopy equivalence if there exists $g : Y \to X$ so that the compositions $f \circ g$ and $g \circ f$ are homotopic to identity.

6.4.1. **Proposition.** *Let $f : X \to Y$ be a homotopy equivalence. Then the map $\pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.*

□

6.4.2. *Some basic examples.* Since $\mathbb{R}^n$ is homotopy equivalent to a point, it is simply connected. Next, the n-dimensional sphere S^n is simply connected for $n > 1$.

Since $\exp : \mathbb{R} \to U(1)$ is a universal covering, $\pi_1(U(1)) = \mathbb{Z}$. We know that topologically $U(1) = S^1$. Since S^1 is homotopy equivalent to $\mathbb{R}^*$, one has $\pi_1(\mathbb{R}^*) = \mathbb{Z}$.

6.4.3. $SL(2, \mathbb{R})$. Any $A \in SL(2, \mathbb{R})$ has a unique presentation as $A = QR$ where $Q \in SO(2, \mathbb{R})$ and $\mathbb{R}$ is upper-triangular with positive elements on the diagonal.

We leave as an exercise to verify the following.

- The group of upper-triangular matrices with positive elements on the diagonal is contractive (that is, homotopy equivalent to a point).
- $SL(2, \mathbb{R})$ is homotopy equivalent to $SO(2, \mathbb{R})$.

Thus, $\pi_1(SL(2, \mathbb{R})) = \mathbb{Z}$.

6.4.4. *The group $SL(2, \mathbb{C})$.* Similarly to the above, for any $A \in SL(2, \mathbb{C})$ there exists a unique decomposition $A = QR$ where $Q \in SU(2)$ and R is upper-triangular with positive real entries at the diagonal.

This implies, similarly to the above, that $\pi_1(SL(2, \mathbb{C})) = \pi_1(SU(2))$.

Since $SU(2)$ is topologically S^3, the group $SL(2, \mathbb{C})$ is simply-connected.

6.5. Coverings of Lie groups.

6.5.1. *Coverings of manifolds.* First of all, let $p : Y \to X$ be a covering and let X be a manifold.

6.6. **Lemma.** *There is a unique manifold structure on Y such that p is a smooth map.*

Proof. For any $x \in X$ there exists an open $U \ni x$ such that $p^{-1}(U) \to F \times U$ is a homeomorphism over U. This implies that there is a unique smooth structure on $p^{-1}(U)$. Since for any two open subsets U, V of X the smooth structures on them are compatible, the smooth structures on $p^{-1}(U)$ and $p^{-1}(V)$ are also compatible. □

6.6.1. *Group structure on $\tilde{G}$.* Let G be a Lie group and let $\pi : \tilde{G} \to G$ be a universal covering. Choose $u \in \tilde{G}$ such that $\pi(u) = 1$. We will now define a group structure on $\tilde{G}$ so that $\tilde{G}$ becomes a Lie group with the unit element u and π becomes a Lie group homomorphism.

Assuming G to be connected, π is surjective, so that $K = \mathrm{Ker}(\pi)$ is a discrete normal subgroup of $\tilde{G}$. Lemma 6.6.2 below than shows that the elements of K commute with the elements of $\tilde{G}$.

The multiplication $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ is defined as follows. Given $x_1, x_2 \in \tilde{G}$, choose paths $\gamma_i : [0,1] \to \tilde{G}$ connecting u with x_i, and multiply their images $\pi \circ \gamma_i$ in G. We get a new path $\delta : [0,1] \to G$ defined by the formula $\delta(s) = \pi(\gamma_1(s))\pi(\gamma_2(s))$ and we lift it to a path $\tilde{\delta}$ such that $\pi(\tilde{\delta}) = \delta$.

In this way we define a map $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ satisfying the following properties.

- $\pi \circ \tilde{m} = m \circ (\pi \times \pi)$.
- $\tilde{m}$ is continuous.

Now $\tilde{m}$ is automatically smooth. In fact, the map $\pi : \tilde{G} \to G$ is a local diffeomorphism. This allows one to chose charts for $(g,h) \in \tilde{G} \times \tilde{G}$ and for $gh \in \tilde{G}$ diffeomorphic to their images in $G \times G$ and in G. Tis prove smoothness of $\tilde{m}$.

6.6.2. **Lemma.** *Let G be a connected Lie group and K a discrete normal subgroup of G. Then for all $g \in G$ and $k \in K$ one has $gk = kg$.*

Proof. Let $\gamma : [0,1] \to G$ connect 1 with g. The map $s \mapsto \gamma(s)^{-1} k \gamma(s)$ is continuous with values in K, therefore, constant. Thus, $g^{-1}kg = k$. □

6.7. **Exact sequence of a locally trivial fibration.** Let $p : Y \to X$ be a locally trivial fibration with connected X. Fix $y \in Y$, $x = p(y)$, and we think of (Y,y) and (X,x) as pointed spaces. We denote $F = p^{-1}(x)$. We have a sequence of pointed spaces

$$(F,y) \xrightarrow{q} (Y,y) \xrightarrow{p} (X,x).$$

For a pointed set (X,x) we denote $\pi_0(X,x)$ the pointed set of connected components of X (a pointed set is a set with a chosen point in it, in this case the component of x). A continuous map of pointed sets induces a map of pointed π_0. This yields a sequence

$$\pi_0(F,y) \to \pi_0(Y,y) \to \pi_0(X,x) = *$$

of pointed sets. We similarly have a sequence of homomorphisms

$$\pi_1(F,y) \to \pi_1(Y,y) \to \pi_1(X,x)$$

of fundamental groups. We will say that a sequence of maps of pointed sets

$$\ldots \to (S_{n+1}, s_{n+1}) \xrightarrow{f_{n+1}} (S_n, s_n) \xrightarrow{f_n} (S_{n-1}, s_{n-1}) \to \ldots$$

is exact if for any n the image of f_{n+1} coincides with $f_n^{-1}(s_{n-1})$.

6.7.1. **Theorem.** *There is an exact sequence of pointed sets*

$$\pi_1(F,y) \to \pi_1(Y,y) \to \pi_1(X,x) \xrightarrow{\partial} \pi_0(F,y) \to \pi_0(Y,y) \to \pi_0(X,x) = *.$$

Proof. We will only construct the map ∂. An interested reader can find the proof in any book on basic algebraic topology.

Let $p : Y \to X$ be a locally trivial fibration, $y \in Y$ and $x = p(y)$. It is easy to see that any path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ can be lifted (non-uniquely but uniquely up to homotopy) to a path $\delta : [0,1] \to Y$ such that $\delta(0) = y$. Now, for any $g \in \pi_1(X,x)$ we define $\partial(g)$ as follows: we represent g with a path γ as above, and define $\partial(g)$ as the connected component of $\delta(1) \in F$. $\square$

6.7.2. **Remark.** It is worthwhile to add that $\pi_1(X,x)$ acts on the set $\pi_0(F)$ so that the map ∂ defined in the proof is deduced from this action, $\partial(g) = g([y])$, where $[y]$ is the component of $y \in F$. Using this action, one describes $\pi_0(Y)$ as the factor $\pi_0(F)/\pi_1(X,x)$.

The above exact sequence is often used to calculate fundamental groups of spaces.

6.8. **Example of a nonlinear Lie group.** We will now prove that the group H/Z defined in 4.1.3, is not linear.

6.8.1. *Representations of a circle.* For any $n \in \mathbb{Z}$ we define a one-dimensional complex representation ρ_n of the group $U(1)$ by the formula

$$\rho_n(z) = z^n \in \mathbb{C}^*.$$

For the time being we accept, without proof, the following result.

Theorem. *Any finite-dimensional representation of $U(1)$ decomposes as a sum of ρ_n.*

Let $r : G \to GL(V)$ be a complex finite-dimensional representation of G. We will prove that V is a sum of trivial representations. This will imply, in particular, that G is not linear.

First of all, the group G contains in its center $U(1) = \mathbb{R}/\mathbb{Z}$. We denote $V_n = \{v \in V | r(z)v = z^n v\}$. The theorem formulated above implies that $V = \oplus V_n$ (most of V_n are zero, of course). One can verify that V_n is a subrepresentation of V, that is that for any $g \in G$ one has $r(g)(V_n) \subset V_n$. We will now prove that $V_n = 0$ for $n \neq 0$. In fact, a calculation shows that any element g of G with $x = y = 0$ is a commutator. Therefore, $\det(r(g)) = 1$. On the other hand, for $v \in V_n$ $\det(r(g)) = z^{nd}$ where $z \in U$ represents g and $d = \dim V_n$. This imiplies the claim.

7. Vector fields. Lie algebra of a Lie group.

We start with a general notion of a vector field on a manifold. We continue with studying invariant vector fields on a Lie group.

7.1. **Vector fields.**

7.1.1. Let U be an open subset in $\mathbb{R}^n$. By definition, all tangent spaces $T_x(U)$, $x \in U$, identify with $\mathbb{R}^n$. A (smooth) vector field on U is, by definition, an assignment $x \in U \mapsto V_x \in T_x(U)$ defined by a smooth function $V : U \to \mathbb{R}^n$.

Any vector field Φ on U defines an operator on $C^\infty(U)$ carrying f to the function sending x to the directional derivative of $f'_{\Phi(x)}$. In particular, choosing the base $x_1, \ldots, x_n$ of $\mathbb{R}^n$, the constant vector field $\Phi(x) := x_i$ defines the operator $\frac{\partial}{\partial x_i}$ on $C^\infty(U)$. A general vector field given by the components $\Phi_1, \ldots, \Phi_n$, defines the derivation

$$\sum \Phi_i \frac{\partial}{\partial x_i} : C^\infty(U) \to C^\infty(U). \tag{4}$$

The space of vector fields on U is just the set of expressions (4). We denote it by $\mathtt{Vect}(U)$. This is a free module over $C^\infty(U)$ with the basis $\frac{\partial}{\partial x_i}$.

7.1.2. *Base change.* What happens to our formulas for vector fields under the change of coordinates? Assume we have $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ and a smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$ given by the matrix of smooth functions $y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)$ establishing a diffeomorphism of U with V.

One can (try to) use the tangent maps $T_x f : T_x U \to T_{f(x)} V$ to construct a vector field on V from a vector field on U. Of course, one should make a calculation to prove that the resulting assignment will give a (smooth) vector field. It is easy to see that the vector field $\frac{\partial}{\partial x_i}$ is sent, under the tanget map, to

$$\frac{\partial}{\partial x_i} = \sum \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j.}$$

Note that these are the standard calculus formulas.

`Note that, in general, a smooth map` $f : U \to V$ `does not induce a map` $\mathtt{Vect}(U) \to \mathtt{Vect}(V)$`. This was possible only because in our case` $f : U \to V$ `was a diffeomorphism.`

7.1.3. Let now M be a smooth manifold, U an open subset in M. A vector field on U is an assignment $x \in U \mapsto V(x) \in T_x(U)$ that is smooth in any chart of U.

The set of vector fields on U is denoted $\mathtt{Vect}(U)$. This is a vector space. Obviously, $\mathtt{Vect}(U)$ is always a module over $C^\infty(U)$ This module is not necessarily free as we have no global coordinate system to get a basis consisting of $\frac{\partial}{\partial x_i}$.

7.1.4. *Vector fields as derivations.* A linear map $d : C^\infty(M) \to C^\infty(M)$ is called a derivation if it satisfies the Leibniz rule:

$$d(fg) = d(f)g + fd(g).$$

Any endomorphism defined by a vector field is a derivation. This can be verified at any chart where the result follows from the "usual" Leibniz rule

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}.$$

It turns out that the converse is also true. We will use it without proof as interpretation of vector fields as derivations is very convenient.

The following exercise makes us belive that the above claim may be correct.

7.1.5. *Exercise.* Let $A = k[x_1, \dots, x_n]$ be the polynomial ring over a field k. Let Φ be a k-linear derivation of A. Prove that

$$\Phi = \sum \Phi_i \frac{\partial}{\partial x_i}$$

where $\Phi_i = \Phi(x_i) \in A$.

7.1.6. In is easy to see that if $\theta : M \to N$ is a diffeomorphism, it induces an isomorphism $T\theta : \mathtt{Vect}(M) \to \mathtt{Vect}(N)$. Here are the details.

Given $x \in M$ and $y = \theta(x) \in N$ we have an isomorphism $T\theta_x : T_x(M) \to T_y(N)$. Thus, given a vector field Φ on M, we get an assignment $y \mapsto T_x(\Phi(x)) \in T_y(N)$. It remains to prove that this assignment is given by smooth functions in any chart of N. This is immediate since θ is a diffeomorphism and Φ is given by smooth functions. Let us describe $T\theta$ in terms of derivations. First of all, θ induces an isomorphism of the algebras of smooth functions

$$\theta_* : C^\infty(M) \to C^\infty(N)$$

carrying $f \in C^\infty(M)$ to the function $\theta_*(f)$ defined by the formula $\theta(f)(y) = f(\theta^{-1}(y))$. Now, given a derivation $\delta : C^\infty(M) \to C^\infty(M)$, we define $T\theta(\delta)$ by the formula

$$T\theta(\delta)(f) = \theta_*(\delta(\theta_*^{-1}(f))),$$

where f is a smooth function on N. In other words,

$$T\theta(\delta)(f)(y) = \theta_*(\delta(\theta_*^{-1}(f)))(y) = \delta(\theta_*^{-1}(f)))(\theta^{-1}(y)).$$

7.2. **Lie bracket on the vector fields.** There is a binary operation on the collection of vector fields $\mathtt{Vect}(M)$ that satisfies the properties similar to the bracket $[X, Y] = XY - YX$ on matrices.

7.2.1. *Bracket in terms of derivations.* Given two vector fields $X, Y \in \mathtt{Vect}(M)$, denote by d_X, d_Y the corresponding derivations. We define $[d_X, d_Y] : C^\infty(M) \to C^\infty(M)$ as the endomorphism given by the formula

$$[d_X, d_Y](f) = d_X(d_Y(f)) - d_Y(d_X(f)).$$

It is easy to see that this is also a derivation. Therefore, it is given by a vector field denoted by $[X, Y]$.

7.2.2. Let us write down the explicit formulas for $\mathtt{Vect}(U)$ where U is an open subset of $\mathbb{R}^n$. Let $X=\sum f_i\frac{\partial}{\partial x_i}$ and $Y=\sum g_i\frac{\partial}{\partial x_i}$. Then

$$[X,Y](\phi)=\sum_{i,j}(f_i\frac{\partial g_j}{\partial x_i}-g_i\frac{\partial f_j}{\partial x_i})\frac{\partial \phi}{\partial x_j},$$

so

$$[X,Y]=\sum_{i,j}(f_i\frac{\partial g_j}{\partial x_i}-g_i\frac{\partial f_j}{\partial x_i})\frac{\partial}{\partial x_j}.$$

7.3. **Lie algebras.** We will now give a definition of a Lie algebra.

7.3.1. **Definition.** A Lie algebra is a vector space $\mathfrak{g}$ endowed with a bilinear operation $x,y\mapsto[x,y]$ satisfying the following properties.

- $[x,y]=-[y,x]$.
- $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ (Jacobi identity).

Notes: The first condition is often replaced with the condition $[x,x]=0$ that is equivalent if $1+1\neq 0$ (as in the present course).

7.3.2. *Exercise.* Let A be an associative algebra. Then the operation $[a,b]=ab-ba$ defines on A a structure of a Lie algebra.

For instance, $M_n(\mathbb{R})$ is a Lie algebra with respect to the bracket operation.

7.3.3. *Exercise.* Verify that $\mathfrak{sl}_n(\mathbb{R})$ is a Lie subalgebra of $M_n(\mathbb{R})$.

7.3.4. *Exercise.* Verify that the space $\mathtt{Vect}(M)$ of vector fields on a manifold is a Lie algebra with respect to the bracket. More generally, verify that the set of derivations of any algebra A (that is, a vector space endowed with a bilinear operation) is a Lie algebra with respect to the bracket defined as $[\delta,\delta']=\delta\circ\delta'-\delta'\circ\delta$.

7.4. **Vector fields on a Lie group.** Let G be a Lie group. Left multiplication by $g\in G$ defines a diffeomorphism $L_g:G\to G$ that induces a linear map

$$T_g:=T_{L_g}:\mathtt{Vect}(G)\to\mathtt{Vect}(G).$$

7.4.1. A vector field X is called *left-invariant* if $T_g(X)=X$ for any g. Let $X_g\in T_g(G)$ be the component of X at g. Left invariance of X means that $X_{gh}=T_g(X_h)$ for any $g,h\in G$.

In particular, a left-invariant vector field on G is uniquely determined by it component $X_1\in T_1(G)$. Conversely, any choice of $X_1\in T_1(G)$ allows one to define $X_g:=T_g(X_1)\in T_g(G)$.

7.4.2. **Lemma.** *For any $X_1\in T_1(G)$ the assignment $X_g:=T_g(X_1)$ defines a left-invariant vector field.*

Proof. This is a smooth vector field by smoothness of $m : G \times G \to G$. It is left-invariant as $T_g(X_h) = T_g(T_h(X_1)) = T_{gh}(X_1) = X_{gh}$. □

7.4.3. **Lemma.** *Let X, Y be left-invariant vector fields. Then $[X, Y]$ is also left-invariant.*

Proof. It is convenient to use the interpretation of vector fields in terms of derivations. A vector field X on G is left-invariant iff for any $g \in G$ and for any $f \in C^\infty(G)$ one has

$$X(g_*(f)) = g_*(X(f)).$$

A direct computation shows that if X and Y satisfy this property, then $[X, Y]$ also satisfies it. □

7.4.4. **Definition.** Lie algebra $\mathtt{Lie}(G)$ of a Lie group G is the Lie algebra of left-invariant vector fields on G. As proven above, it identifies, as a vector space, with the tangent space $T_1(G)$.

7.4.5. *Example.* Let us make everything explicit for the case $G = GL_n(\mathbb{R})$. A vector field on G is given by a smooth function $f : GL_n \to M_n$. Left multiplication $L_g : GL_n \to GL_n$ induces an endomorphism $T_g : M_n \to M_n$ also given by left multiplication by g. Thus, an invariant vector field defined by $X \in M_n = T_1(GL_n)$ is the assignment $g \mapsto gX$.

7.5. **Vector fields and flows.** Vector fields on manifolds have two complementary manifestation. The one interprets a vector field on M as a derivation of the algebra of functions $C^\infty(M)$. On the other way, vector fields can be integrated, to define flows, as follows.

7.5.1. *Flows.* Given a vector field X on a manifold M, and a point $x \in M$, one can integrate it, that is find $\epsilon > 0$ and a unique curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\dot\gamma(t) = X_{\gamma(t)}$. This defines (locally, that is, in a neighborhood of x and for small ϵ) a map carrying x to a point $\gamma(t) =: \Phi_t^X(x)$. The meaning of the indices in the notation: $\Phi_t^X(x)$ is the result of the movement along the vector field X during the time t, starting from $x \in M$.

7.5.2. *Bracket in terms of flows.* Given two vector fields X and Y, the value of $[X, Y]$ at $x \in M$ can be described as follows. We define

$$\phi(s, t) = \Phi_t^{-Y} \circ \Phi_s^{-X} \circ \Phi_t^{Y} \circ \Phi_s^{X}(x).$$

Here is a standard fact in differential geometry whose proof we give for completeness.

Proposition. *The value of $[X, Y]$ at $x \in M$ can be calculated as $\frac{\partial^2 \phi}{\partial s \partial t}(0, 0)$.*

Proof. Let us, first of all, present the connection between two interpretation of a vector field: as something defining a flow and as an operator on functions. Given a vector field X with a flow Φ^X, and given a function f, one has $X(f)(x) = \frac{d}{dt}(f(\Phi_t^X(x)))(0)$. Let now X, Y be two vector fields and let the function $\phi : \mathbb{R}\times\mathbb{R} \to M$ be defined as above. We will prove that for any function $f \in C^\infty(M)$ and for each $x \in M$ one has

$$\frac{\partial^2}{\partial s\partial t}(f \circ \phi)(0,0) = X(Y(f))(x) - Y(X(f))(x).$$

The map ϕ is defined as a composition of four flows applied to x, so that two of them depend of s and two of them depend of t. This is inconvenient. For this technical reason we prefer to define a (more general) function ψ as follows.

$$\psi(s, s', t, t') = \Phi_{t'}^{-Y} \circ \Phi_{s'}^{-X} \circ \Phi_t^Y \circ \Phi_s^X(x),$$

so that $\phi(s,t) = \psi(s,s,t,t)$ and therefore

$$\frac{\partial^2}{\partial s\partial t}(f \circ \phi)(0,0) = (\frac{\partial^2}{\partial s\partial t} + \frac{\partial^2}{\partial s\partial t'} + \frac{\partial^2}{\partial t\partial s'} + \frac{\partial^2}{\partial s'\partial t'})(f\circ\psi)(0,0,0,0).$$

Note that we chose the order of derivation that is more convenient for calculation; for instace, $\frac{\partial^2}{\partial t\partial s'} = \frac{\partial}{\partial t}\frac{\partial}{\partial s'}$. We present below the example of the calculation of the first summand.

$$\begin{aligned}\frac{\partial}{\partial t}(f \circ \Phi_{t'}^{-Y} \circ \Phi_{s'}^{-X} \circ \Phi_t^Y \circ \Phi_s^X(x)) = \frac{\partial}{\partial t}((\Phi_{t'}^{-Y} \circ \Phi_{s'}^{-X})_*(f) \circ \Phi_t^Y \circ \Phi_s^X(x)) = \\ Y((\Phi_{t'}^{-Y} \circ \Phi_{s'}^{-X})_*(f)(\Phi_s^X(x)),\end{aligned} \tag{5}$$

so the second derivative yields

$$\frac{\partial^2}{\partial s\partial t}(f \circ \Phi_{t'}^{-Y} \circ \Phi_{s'}^{-X} \circ \Phi_t^Y \circ \Phi_s^X(x)) = X \circ Y((\Phi_{t'}^{-Y} \circ \Phi_{s'}^{-X})_*(f)(x),$$

whose value at $(0,0,0,0)$ gives $X \circ Y(f)(x)$. The rest of the summands are

$$-X((\Phi_{s'}^{-X}\Phi_t^Y)_*Y(f))(x),\ -YX((\Phi_{t'}^{-Y})_*(f))(\Phi_s^X(x)) \text{ and } XY(f)(\Phi_t^Y \circ \Phi_s^X(x)).$$

Evaluating at $(0,0,0,0)$ and summing up, we get the required answer. $\square$

Remark. Note that the first partial derivatives $\frac{\partial\phi}{\partial s}(0,0)$ and $\frac{\partial\phi}{\partial t}(0,0)$, as well as $\frac{\partial^2\phi}{\partial s^2}(0,0)$ and $\frac{\partial^2\phi}{\partial t^2}(0,0)$, vanish as $\phi(s,0) = \phi(0,t) = x$. Therefore, the mixed second derivative is the only nontrivial term of second degree in s,t.

7.5.3. *The exponential map.* Let $X \in \mathtt{Lie}(G)$ and let $v_X \in \mathtt{Vect}(G)$ be the corresponding left-invariant vector field. By the general construction given above we get an integral line

$$\gamma : (-\epsilon, \epsilon) \to G$$

in a neighborhood of 0, such that $\dot\gamma(t) = v_X(\gamma(t)) = \gamma(t)(X)$. Let us show that this map extends to the whole $\mathbb{R}$. In fact, since v_X is invariant, the left translation

$L_{\gamma(t)}$ for $t \in (-\epsilon, \epsilon)$ carries a solution of the ODE to a solution. Because of the uniqueness of the solution, they coincide on the intersection, that is

$$\gamma(s+t) = \gamma(s)\gamma(t).$$

It makes sense to call it $\exp_X$. Obviously, $\exp_X(t) = \exp_{tX}(1)$ so we call it $\exp(tX)$.

This defines a smooth map $\exp : \mathfrak{g} \to G$. Let us calculate its derivative at 0, $T\exp(0)$. One can easily see that $T\exp(0) = \mathrm{id}_{\mathfrak{g}}$. Therefore, $\exp : \mathfrak{g} \to G$ is a local diffeomorphism.

7.5.4. *Exercise.* Verify that in case of $G = GL_n$ the map exp defined above is given by the matrix exponent.

7.5.5. We can now use 7.5.2 to find a formula for the bracket $[X, Y]$ for $X, Y \in T_1(G)$. We define $\phi(s,t) = \exp(-tY)\exp(-sX)\exp(tY)\exp(sX)$. Then for varying s the derivative $\frac{\partial\phi}{\partial t}$ gives a map from $(-\epsilon, \epsilon) \to T_1(G)$ whose derivative $\frac{\partial^2\phi}{\partial s\partial t}$ is a vector in $T_1(G)$. This is the value of $[X, Y]$. In other words, `up to higher degree terms,` one has

$$\exp(-tY)\exp(-sX)\exp(tY)\exp(sX) \sim \exp(st[X, Y]).$$

7.5.6. *Exercise.* Prove that the Lie bracket of invariant vector fields on GL_n is given by the formula $[X, Y] = XY - YX$.

7.6. **More on connection between a Lie group and its Lie algebra.** A Lie group homomorphism $f : G \to H$ induces a linear map $T_1 f : T_1(G) \to T_1(H)$.

7.6.1. **Theorem.** *The linear map $T_1 f$ is a Lie algebra homomorphism.*

Proof. We have to verify that $T_1 f$ preserves the Lie bracket. This follows from the description of the Lie bracket given above and the fact that $f : G \to H$ carries $\exp(X) \in G$ to $\exp(Y) \in H$ where $Y = T_1(f)(X)$. The latter follows from the uniqueness of the solution of ODE. □

Recall that a finite dimensional representation of a Lie group G in a vector space V is just a Lie group homomorphism $G \to GL(V)$ (note that V can be real or complex vector space). Since the Lie algebra of $GL(V)$ is just the Lie algebra of endomorphism of V (we denote it by $\mathfrak{gl}(V)$ when it is considered as a Lie algebra), we deduce that any finite dimensional representation of G induces a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. It makes sense to give the following definition.

7.6.2. **Definition.** A finite dimensional representation (real or complex) of a Lie algebra $\mathfrak{g}$ is a homomorphism of Lie algebras

$$\mathfrak{g} \to \mathfrak{gl}(V).$$

In other words, a representation of $\mathfrak{g}$ is an assignment, for each $X \in \mathfrak{g}$, of an operator $\rho(X) \in \mathrm{End}(V)$, such that

$$\rho([X,Y]) = \rho(X) \circ \rho(Y) - \rho(Y) \circ \rho(X).$$

7.6.3. *Example: Adjoint action.* Recall the adjoint action of a Lie group G on itself given by the formula $\mathrm{Ad}_g(h) = ghg^{-1}$. Since $\mathrm{Ad}_g(1) = 1$, this defines an action of G on $\mathfrak{g} = \mathtt{Lie}(G)$. This is a representation of G called *the adjoint representation*. According to above, it induces a representation of $\mathfrak{g}$ on $\mathfrak{g}$ that is also called the adjoint representation and denoted $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. We will now show that the adjoint representation is given by the formula

$$\mathrm{ad}(X)(Y) = [X,Y].$$

In effect, to calculate $\mathrm{ad}(X)(Y)$, one should choose a curve in G with tangent X at 1, for instance, $\exp(sX)$, calculate $\mathrm{Ad}_{\exp(sX)}(Y)$ and take the derivative. This means that one should derive twice, in t and in s, the expression

$$\mathrm{Ad}_{\exp(sX)}(\exp(tY)) = \exp(sX)\exp(tY)\exp(-sX).$$

The result can be deduced from the calculation of the mixed derivatives of ψ presented in 7.5.2.

7.6.4. An action of a Lie group G on a manifold M can hardly be presented as a homomorphism from G to the group of diffeomorphisms of M — for the simple reason that this group is infinitely dimensional and so is not (formally) a Lie group. However, one can easily see that the Lie algebra $\mathtt{Vect}(M)$ of vector fields on M plays the role of the Lie algebra of the "not-so-Lie group" of diffeomorphisms of M.

Let G be a Lie group and let $\mathfrak{g} = T_1(G)$ be the Lie algebra of G. Given an action

$$m : G \times M \to M,$$

any choice of $x \in M$ defines a smooth (orbit) map $m_x : G \to M$ given by the formula $m_x(g) = m(g,x)$. Its tangent map gives $T_1(G) \to T_x(M)$. This defines a linear map $Lie(m) : \mathfrak{g} \to \mathtt{Vect}(M)$ that carries $X \in \mathfrak{g}$ to the vector field whose x-component is $T_1 m_x(X)$.

We have the following analog of Theorem 7.6.1.

7.6.5. **Theorem.** *The map* $\mathtt{Lie}(m) : \mathfrak{g} \to \mathtt{Vect}(M)$ *is an* `anti-homomorphism` *of Lie algebras, that is* $\mathtt{Lie}(m)([X,Y]) = -[\mathtt{Lie}(m)(X), \mathtt{Lie}(m)(Y)]$.

Note that a map $f : \mathfrak{g} \to \mathfrak{h}$ is an anti-homomorphism of Lie algebras if and only if $-f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

7.6.6. **Remark.** The minus sign in the formulation of the theorem requires an explanation. Let us illustrate it with the following example. Let $M = G$ and the action be the standard left action, that is $m : G \times G \to G$ is the multiplication. Let us calculate the map $\mathtt{Lie}(m) : \mathfrak{g} \to \mathtt{Vect}(G)$. The g-component of $\mathtt{Vect}(m)(X)$ is the image of X under the map $T_1(G) \to T_g(G)$ induced by the right multiplication by g. Thus, $\mathtt{Lie}(m) : \mathfrak{g} \to \mathtt{Vect}(G)$ carries $\mathfrak{g}$ to right-invariant vector fields. Recall that we identified $\mathfrak{g}$ with the space of left-invariant vector fields. If we denote by $\mathfrak{g}^R$ the set of right-invariant vector fields on G, both $\mathfrak{g}$ and $\mathfrak{g}^R$ indentify with $T_1(G)$, but they acqure different Lie algebra structures from $\mathtt{Vect}(G)$. One can show that the structures differ by sign, see Exercise below.

7.6.7. *Exercise.* Prove that the identity map $\mathfrak{g} \to \mathfrak{g}^R$ is an anti-isomorphism of Lie algebras, $[X, Y] = -[X, Y]_R$. *Hint: The map $i : G \to G$ carrying g to g^{-1} carries left-invariant vector fields to right-invariant vector fields.*

Proof of the theorem. The proof is very similar to the proof of7.6.1. We use the description of the bracket of vector fields given in 7.5.2. For a fixed $X \in \mathfrak{g}$ we have to describe the flow Φ_t^X on M defined by the vector field v^X whose y-component for $y \in M$ is given by the formula $v_y^X = T_1 m_y(X)$. The map $m_{g(x)} : G \to M$ is the composition

$$G \overset{R_g}{\to} G \overset{m_x}{\to} M,$$

so the integral lines of the flow Φ^X starting from $y := g(x)$ are images of the integral lines of the flow defined by the right-invariant vector field on G defined by X, the one given by the formula $g \mapsto T_1(R_g)(X)$. Now the claim follows from Remark 7.6.6. □

7.7. **Closed subgroups of Lie groups.** Theorem of von Neumann 4.2.4 claims that any closed subgroup of $GL_n(\mathbb{R})$ is a closed Lie subgroup. We have now all necessary tools to deduce the following result of Elie Cartan (1930).

7.7.1. **Theorem.** *Any closed subgroup of a Lie group G is in fact a closed Lie subgroup.*

Proof. The proof follows the original proof by von Neumann. Recall the main steps of the proof.

Let G be a Lie group and H a closed subgroup in it. Let $\mathfrak{g}$ be the Lie algebra of G. We define $\mathfrak{h} \subset \mathfrak{g}$ as the collection of X such that $\exp(tX) \in H$ for all t (this is the definition we used for linear groups, but with a slightly different meaning for the exponent). We prove that $\mathfrak{h}$ is a Lie subalgebra, and then that in a certain neighborhood U of 1 the intersection $H \cap U$ lies in the image of $\exp : \mathfrak{h} \to G$. □

We can now prove Proposition 3.5.8 describing the orbits of an Lie group action on a manifold.

7.7.2. *Proof of 3.5.8.* Let G be a Lie group acting on a manifold M. Let $x \in M$. The stabilizer $H := \mathtt{Stab}_G(x)$ is a closed subgroup. Therefore, by Cartan theorem, it is a closed Lie subgroup. This implies that there is a natural structure of manifold on the quotient G/H that is, as a set, coincides with the orbit of G containing $x \in M$. By universal property of the quotient, see 3.4.4, the orbit map $G/H \to M$ is a smooth map. It remains to verify that this map is an immersion, that is, to prove that the tangent map is injective. The tangent space of G/H at 1 identifies with $\mathfrak{g}/\mathfrak{h}$. Recall that the orbit of x is defined by the map $m_x : G \to M$ carrying $g \in G$ to $g(x) \in M$. Look at the tangent map $Tm_x : \mathfrak{g} \to T_x(M)$. We have to verify that if $Tm_x(X) = 0$ then $X \in \mathfrak{h}$. In fact, If $Tm_x(X) = 0$, the integral line Φ_t^X in M is constant, therefore $\exp(tX) \in H$. By definition this implies $X \in H$.

7.7.3. **Corollary.** *Let $f : G \to H$ be a Lie group homomorphism and let* $\mathtt{Lie}(f) : \mathfrak{g} \to \mathfrak{h}$ *be the induced homomorphism of Lie algebras. Then $K = \mathrm{Ker}(f)$ is a closed normal Lie subgroup of G with Lie algebra* $\mathrm{Ker}(\mathtt{Lie}(f))$.

Proof. The homomorphism f defines a left action of G on H as the composition $G \times H \overset{f\times \mathrm{id}}{\to} H \times H \overset{m_H}{\to} H$(in other words, $L_g(h) = f(g)h$. Then K is the stabilizer of $1 \in H$ under this action and its Lie algebra is the kernel of $\mathtt{Lie}(f)$ by 3.5.8, see also 7.7.2. □

7.7.4. *Exercise.* Define the direct product of Lie groups and of Lie algebras. Prove that $\mathtt{Lie}(G \times H)$ is isomorphic to the direct product $\mathtt{Lie}(G) \times \mathtt{Lie}(H)$.

8. Fundamental Lie theorems

8.1. In this section we study the precise connection between Lie groups and Lie algebras. If G, H are connected Lie groups, we denote by $\mathrm{Hom}_{\mathtt{LG}}(G, H)$ the set of Lie group homomorphisms from G to H. Similarly, if $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, we denote by $\mathrm{Hom}_{\mathtt{LA}}(\mathfrak{g}, \mathfrak{h})$ the set of Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{h}$. We already know that any Lie group homomorphism $f : G \to H$ gives rise to a Lie algebra homomorphism $\mathtt{Lie}(f) := T_1(f) : \mathtt{Lie}(G) \to \mathtt{Lie}(H)$. This defines a map

$$\mathrm{Hom}_{\mathtt{LG}}(G, H) \to \mathrm{Hom}_{\mathtt{LA}}(\mathtt{Lie}(G), \mathtt{Lie}(H)).$$

Here is the first fundamental theorem.

8.1.1. **Theorem.** *(Lie theorem 1) Let G be a Lie group with $\mathfrak{g} = \mathtt{Lie}(G)$. There is a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.*

8.1.2. **Remark.** Note that the theorem is talking about immersed (not necessarily closed) subgroups in G.

The second Lie theorem extends the first one to general Lie group or Lie algebra homomorphisms.

8.1.3. **Theorem.** *(Lie theorem 2) Let G, H be connected Lie groups, $\mathfrak{g} = \mathtt{Lie}(G)$, $\mathfrak{h} = \mathtt{Lie}(H)$. The map*

$$\mathrm{Hom}_{\mathtt{LG}}(G, H) \to \mathrm{Hom}_{\mathtt{LA}}(\mathfrak{g}, \mathfrak{h})$$

assigning to $f : G \to H$ the tangent map $\mathtt{Lie}(f) : \mathfrak{g} \to \mathfrak{h}$ is injective. It is bijective if G is simply connected.

Here is the third principal theorem.

8.1.4. **Theorem.** *(Lie theorem 3) For any finite dimensional Lie algebra $\mathfrak{g}$ there exists a Lie group G such that $\mathfrak{g} = \mathtt{Lie}(G)$.*

8.1.5. **Corollary.** *There is an equivalence of the category* LA *of finite dimensional Lie algebras and the full subcategory of* LG *spanned by the simply connected Lie groups.*

□

`In other words, for any finite dimensional Lie algebra` $\mathfrak{g}$ `there is a unique, up to isomorphism, simply connected Lie group and to any Lie algebra homomorphism` $\mathfrak{g} \to \mathfrak{h}$ `uniquely lifts to a Lie group homomorphism` $G \to H$`.`

There is a number of approaches to these results. Moreover, some of them can be easily deduced from he others.

8.1.6. For instance, here is an easy way to deduce Theorem 1 from Theorems 2 and 3. By Lie 3 theorem, $\mathfrak{h}$ admits a simply connected Lie group $\tilde{H}$. By Lie theorem 2 there is a Lie group homomorphism $f : \tilde{H} \to G$ such that $\mathtt{Lie}(f)$ is the embedding $\mathfrak{h} \to \mathfrak{g}$. This means that the kernel of f is a discrete subgroup K and the quotient $H = G/K$ is a Lie group locally isomorphic to $\tilde{H}$.

Obviously, H is a unique Lie subgroup of G whose Lie algebra identifies with $\mathfrak{h}$.

8.2. **Lie theorem 1.** Recall that we defined a smooth map $\exp : U \to G$ for an open neighborhood U of 0 in $\mathfrak{g}$. This makes the image $\exp(U) \subset G$ endowed with the inverse map $\log : \exp(U) \to \mathfrak{g}$ a chart for the group U at the neighborhood of $1 \in G$.

Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the image $\exp(\mathfrak{h}) \subset G$ is a submanifold. We cannot expect it to be a subgroup, but we hope it to be a small neighborhood of a subgroup. How can one prove this? At least this should mean that, if $h, h' \in \mathfrak{h}$ are small enough, the product $\exp(h)\exp(h')$ belongs to $\exp(\mathfrak{h})$.

There are different approaches. A more algebraic one is based on the beautiful BCH (Backer-Campbell-Hausdorff) formula that claims $\exp(x)\exp(y) = \exp(z)$

where z can be expressed through x and y using (multiple) commutators,

$$z = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \ldots.$$

A more geometric way of proving this theorem is based on a very important Frobenius theorem. Let us first indicate that Lie theorem 1 has already been proven in the case when $\mathfrak{h}$ has dimension 1: if $\mathfrak{h} = \mathbb{R} \cdot X$, the corresponding Lie subgroup of G is the one-parametric subgroup $t \mapsto \exp(tX)$ constructed above. The exponent map was constructed using the theorem on the existence and uniqueness of a solution of ODE. Our proof of Theorem 1 is based on a multi-dimensional generalization of this theorem. This is Frobenius theorem.

8.3. **Distribution, involutive distributions.** Let M be a smooth manifold of dimension n.

8.3.1. **Definition.**
1. A d-dimensional family D of tangent vectors in M is an assignment, for any $x \in M$, of a d-dimensional subspace $D_x \subset T_x(M)$.
2. A d-dimensional *distribution* D on M is a d-dimensional family locally generated by d linearly independent vector fields: for any $x \in M$ there exist an open neighborhood U of x and vector fields $X_1, \ldots, X_d \in \mathtt{Vect}(U)$ such that for any $y \in U$ $D_y = \mathrm{Span}((X_1)_y, \ldots (X_d)_y)$.

Thus, a one-dimensional distribution is just a direction field (a notion very close to a non-vanishing vector field).

8.3.2. **Definition.** A d-dimensional distribution D on M is *integrable* if at any $x \in M$ there are local coordinates $x_1, \ldots, x_n$ such that D is (locally) generated by $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, d$. In other words, this means that for any $x \in M$ there is a (small) d-dimensional submanifold $\tilde{D}$ of M containing x such that for any $y \in \tilde{D}$ one has $D_y = T_y\tilde{D} \subset T_yM$.

Any one-dimensional distribution is integrable, as the existence and uniqueness of a solution of ODE shows. This is not so in general: integral distributions have to be *involutive* in the sense of the definition below.

Let D be a distribution on M. We say that a vector field X on M belongs to D if for any $x \in M$ $X_x \in D_x$.

8.3.3. **Definition.** A distribution D on M is called involutive if for any two vector fields X and Y belonging to D their bracket $[X,Y]$ also belongs to D.

8.3.4. **Remark.** Let D be a d-dimensional distribution on U generated by the vector fields $X_1, \ldots, X_d$. Then, in order to verify that D is involutive, it is sufficient to verify that $[X_i, X_j]$ belongs to D. In effect, $X \in D$ decomposes $X = \sum f_i X_i$ for some functions X_i. One has

$$[X,Y] = [\sum f_i X_i, \sum g_j X_j] = \sum_{i,j} [f_i X_i, g_j X_j],$$

so it is sufficient to find an expression for $[fX, gY]$. Here is the general formula that can be easily verified in the local coordinates.

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$$

in particular, if X, Y and $[X, Y]$ belong to D, $[fX, gY]$ also belongs to D.

The above remark implies, in particular, that any one-dimensional distribution is automatically involutive: if D is (locally) generated by $X \in \mathtt{Vect}(U)$, $[X, X] = 0$ implies $[fX, gX] \in D$.

8.3.5. **Lemma.** *Let D be an integrable distribution on M. Then it is involutive.*

Proof. In a local chart where D is generated by $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, d$, one has $[\frac{\partial}{\partial x_i}$, so by the above remark D is involutive. $\square$

Frobenius theorem claims that the above condition is also sufficient. Moreover, the d-dimensional submanifold containing x is uniquely defined in a neighborhood of x. We will prove Frobenius theorem in 8.4. Meanwhile we will show how to deduce Fundamental Lie theorem 1 from it.

8.3.6. *Proof of Lie theorem 1.* Let G be a Lie group, $\mathfrak{g} = \mathtt{Lie}(G)$ identified with the Lie algebra of left-invariant vector fields, and let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. For any $g \in G$ we define $\mathcal{H}_g = T_1 L_g(\mathfrak{h}) \in T_g(G)$. This is a d-dimensional distribution on G that is invariant under left multiplication. It is obviously closed under the Lie bracket as $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Therefore, it satisfies the Frobenius theorem, so that there exists a d-dimensional ($d = \dim \mathfrak{h}$) submanifold H of G tangent to the distribution $\mathcal{H}$. Since $\mathcal{H}$ is left-invariant, the uniqueness implies that for any $g \in H$ the submanifolds H and $L_g(H)$ coincide at the intersection. This implies the existence of H, the maximal among those immersed d-dimensional submanifolds tangent to $\mathcal{H}$. Moreover, $H = L_g(H)$ for any $g \in H$. This implies that H is a subgroup.

8.4. **Proof of Frobenius theorem.** Let us formulate once more the existence and the uniqueness part.

8.4.1. **Theorem.** *(Frobenius, 1849–1917) A d-dimensional distribution D on M is integrable iff for any two vector fields $X, Y \in \mathtt{Vect}(U)$ belonging to D the bracket $[X, Y]$ also belongs to D. In this case a d-dimensional submanifold $\tilde{D}$ containing x and tangent to D is unique in a neighborhood of x.*

Here is a global version of the theorem.

8.4.2. **Corollary.** *If a d-dimensional distribution D on M is closed under the brackets as above, for any $x \in M$ there exists a unique maximal immersed submanifold containing x and tangent to D.*

Proof of Frobenius theorem. The claim of the theorem being local, we can think of M as an open subset U of $\mathbb{R}^n$. We have d vector fields $X_1, \ldots, X_d$ on U that generate D. The conditions give that $[X_i, X_j]$ is a linear combination of X_k with coefficients in $C^\infty(U)$. The idea of the proof is to find another basis $X'_1, \ldots, X'_d$ satisfying $[X'_u, X'_j] = 0$.

Step one. We choose the coordinates $x_1, \ldots, x_n$ on U so that $X_d = \frac{\partial}{\partial x_n}$ (this is possible by ODE (Picard-Lindelöf) as X_d is nonzero at x; we may need to make U smaller). We have

$$X_i = \sum_{k=1}^{n} h_{ik} \frac{\partial}{\partial x_k} \tag{6}$$

for $i < d$ and $X_d = \frac{\partial}{\partial x_n}$.

Step two. Replacing X_i with $X_i - h_{id} X_d$ for $i < d$ we can assume that $h_{id} = 0$ in (6).

Now,

$$[X_d, X_i] = \frac{\partial}{\partial x_n} \circ \sum_{k=1}^{n-1} h_{ik} \frac{\partial}{\partial x_k} - \sum_{k=1}^{n-1} h_{ik} \frac{\partial}{\partial x_k} \circ \frac{\partial}{\partial x_n} = \sum_{k=1}^{n-1} \frac{\partial h_{ik}}{\partial x_n} \frac{\partial}{\partial x_k}. \tag{7}$$

and, by the assumption,

$$[X_d, X_i] = \sum_{j=1}^{d} g_{ij} X_j = \sum_{j=1}^{d-1} \sum_{k=1}^{n-1} g_{ij} h_{jk} \frac{\partial}{\partial x_k} + g_{id} \frac{\partial}{\partial x_n} \tag{8}$$

which implies, in particular, that $g_{id} = 0$ so that $[X_d, X_i] = \sum_{j=1}^{d-1} g_{ij} X_j$ is a linear combination of X_j with $j < d$.

Step three. Our next step is to make a change of variables to ensure that $[X_d, X_i] = 0$ for all i. This is done as follows.

For any collection of $c_1, \ldots, c_{d-1}$ we can find the functions $f_1. \ldots, f_{d-1}$ satisfying the differential equations

$$\frac{\partial f_i}{\partial x_n} = - \sum_{j=1}^{d-1} f_j g_{ji},$$

with the initial condition $f_i(x_1, \ldots, x_{n-1}, 0) = c_i$. Then one has

$$[X_d, \sum_{i=1}^{d-1} f_i X_i] = \sum_{i=1}^{d-1} \frac{\partial f_i}{\partial x_n} X_i + \sum_{i,j=1}^{d-1} f_i g_{ij} X_j = 0.$$

Choose, for instance, $(c_1, \ldots, c_{d-1}) = (1, 0, \ldots, 0)$. The vector field $Y_1 := \sum f_i X_i$ coincides with X_1 on the hyperplace $x_n = 0$, so Y_1 can (locally) replace X_1 in the basis of vector fields. Once can similarly correct $X_2, \ldots, X_{d-1}$.

Step four. We are now ready to prove the theorem by induction in d. Going back to the formula (8), we deduce that $\frac{\partial h_{ik}}{\partial x_n} = 0$, so h_{ik} are independent of x_n. This implies that $X_1, \ldots, X_{d-1}$ can be seen as defining a $(d-1)$-dimensional integrable family of $\mathbb{R}^{n-1}$, and so by the inductive hypothesis it can be integrated to a $(d-1)$-dimensional manifold N_0. Then $N := N_0 \times \mathbb{R} \subset \mathbb{R}^n$ is a d-dimensional manifold tangent to D. $\square$

8.5. **Lie theorem 2.** The first claim of the theorem is very easy. In fact, given a Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$, any its lifting $\bar{f} : G \to H$ shooud carry, for $X \in \mathfrak{g}$, an element $\exp(tX) \in G$ to $\exp(tf(X)) \in H$. Since G is connected, it is generated by the $\exp(tX)$, so the homomorphism $\bar{f}$ is (at most) uniquely determined by f.

The second claim of the theorem will be deduced from Lie theorem 1. Let $\mathfrak{g} = \mathtt{Lie}(G)$ and $\mathfrak{h} = \mathtt{Lie}(H)$. Then one has $\mathfrak{g} \times \mathfrak{h} = \mathtt{Lie}(G \times H)$. Any Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ defines an injective homomorphism $F : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{h}$ given by the formula $F(X) = (X, f(X))$. By Lie theorem 1 the Lie group $G \times H$ admits a unique Lie subgroup $\tilde{G} \subset G \times H$ so that $\mathtt{Lie}(\tilde{G}) = \mathfrak{g}$. The composition

$$p_1 : \tilde{G} \to G \times H \to G$$

induces the identity $T_1 p_1 : \mathfrak{g} \to \mathfrak{g}$, so p_1 is a covering. Since G is simply connected, p_1 is an isomorphism. This implies that the second composition

$$p_2 \circ p_1^{-1} : G \to \tilde{G} \to G \times H \to H$$

induces the map $f : \mathfrak{g} \to \mathfrak{h}$.

8.6. **Lie theorem 3.** The last Lie theorem is the most difficult. We will allow ourselves to deduce it from a non-trivial result of Ado (1910–1983). An alternative proof (also based on some properties of Lie algebras that we will not prove) will be presented later.

8.6.1. **Theorem.** *(Igor Ado) Any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ admits an embedding into $M_n(\mathbb{R})$.*

Let us deduce from this that any finite dimensional Lie algebra is a Lie algebra of a Lie group. Choose an embedding $\mathfrak{g} \to M_n(\mathbb{R})$. By Lie theorem 1 there is a Lie subgroup G of $GL_n(\mathbb{R})$ whose Lie algebra is $\mathfrak{g}$.

8.6.2. **Remark.** The proof of Lie theorem 3 is a bit cheating as its deduction from Ado theorem is immediate, so that the main difficulty is in the proof of Ado theorem. The proof of Ado ttheorem is based on a structure theory of Lie algebras. One can prove Lie theorem 3 directly using the same structure theory; but in this case one will have to say something about it.

9. Lie groups vs Lie algebras

We will now start to exploit the connection between Lie groups and Lie algebras.

9.1. Commutative Lie groups.

9.1.1. **Definition.** A Lie algebra $\mathfrak{g}$ is called commutative if for $X, Y \in \mathfrak{g}$ one has $[X, Y] = 0$.

Obviously, commutative Lie algebras are just real vector spaces.

9.1.2. **Lemma.** *Let G be a commutative Lie group. Then $\mathtt{Lie}(G)$ is a commutative Lie algebra.*

Proof. For $X, Y \in \mathfrak{g} = \mathtt{Lie}(G)$ the bracket $[X, Y]$ is calculated differentiating the map $\phi : \mathbb{R}^2 \to G$ given by the formula

$$\phi(s, t) = \exp(sX)\exp(tY)\exp(-sX)\exp(-tY).$$

Since G is commutative, $\phi(s, t) = 1$, so $[X, Y] = 0$. $\square$

9.1.3. By the Lie theorems, there exists a unique simply connected Lie group having a commutative Lie algebra $\mathfrak{g}$ as a Lie algebra. Since $(\mathfrak{g}, +)$ obviously satisfies this property, there are no others.

9.1.4. By the general theory, any Lie group having a commutative Lie algebra $\mathfrak{g}$ is the quotient of $(\mathfrak{g}, +)$ by a discrete subgroup D. Discrete subgroups of a finite dimensional vector space can be easily classified. Here is the answer.

9.1.5. **Proposition.** *Let V be a finite dimensional real vector space and D be a discrete subgroup of V. Then there is a basis of V, say, $v_1, \ldots, v_n$, and a number $d \leq n$ such that D is the free abelian group spanned by $v_1, \ldots, v_d$.*

$\square$

Naturally, a discrete subgroup generated by $v_1, \ldots, v_d$ is called *a discrete subgroup of rank d*.

This result immediately implies the following classification of of collutative Lie groups.

9.1.6. **Proposition.** *Any commutative Lie group of dimension n is isomorphic to the direct product $C^d \times \mathbb{R}^{n-d}$, where $C = \mathbb{R}/\mathbb{Z}$ is the circle group.*

$\square$

9.1.7. **Remark.** Classification of complex-analytic commutative groups is much more interesting. Similarly to the real case, they all have form $\mathbb{C}^n/D$ where D is a discrete subgroup of $\mathbb{C}^n$. However, the maximal rank d of D is now $2n$ instead of n, and, for instance, different rank 2 discrete subgroups of $\mathbb{C}$ lead to non-isomorphic complex-analytic groups (they are of course isomorphic as groups but not as analytic manifolds). For instance, the case $n = 1$, $d = 2$ gives *elliptic curves* – and there is a continuous family of them.

9.2. **Center.** Let G be a Lie group. Its center Z is defined as the collection of elements of G commuting with all elements of G.

9.2.1. **Definition.** The center of a Lie algebra $\mathfrak{g}$ is defined as

$$\mathfrak{z} = \{X \in \mathfrak{g} | [X, Y] = 0 \text{ for all } Y\}.$$

9.2.2. **Proposition.** *Let G be a connected Lie group. Then Z is a closed Lie subgroup with* $\mathtt{Lie}(Z) = \mathfrak{z}$.

Proof. We know that any neighborhood of 1 in G generates G. Therefore, $g \in Z$ iff g commutes with all $\exp(X)$, $X \in \mathfrak{g}$. In other words, if $\mathrm{Ad}_g(X) = X$. This means that Z is the kernel of the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$. Therefore, the Lie algebra of Z is the kernel of $\mathrm{ad} = \mathtt{Lie}(\mathrm{Ad}) : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. This is precisely the center of $\mathfrak{g}$ as $\mathrm{ad}(x)(y) = [x, y]$. □

9.3. **Normal Lie subgroups versus Lie ideals.** A normal subgroup of a Lie group is, by definition, a *closed* Lie subgroup that is normal as an abstract group. (The meaning of a normal subgroup is that it can serve as the kernel of a homomorphism. Since the kernel of a Lie group homomorphism is automatically a closed Lie subgroup, we require closedness.)

We know that if H is a normal Lie subgroup of a Lie group G, the quotient group G/H acquires a structure of a Lie group. We will now show that, in terms of the correspondence between Lie subgroups and Lie subalgebras, normal Lie subgroups corresponds to Lie ideals in the sense of the following definition.

9.3.1. **Definition.** Let $\mathfrak{g}$ be a Lie algebra. An ideal $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra satisfying the property $[x, y] \in \mathfrak{h}$ for any $x \in \mathfrak{g}$, $y \in \mathfrak{h}$.

9.3.2. **Proposition.** *Let H be a closed connected subgroup of connected Lie group G with* $\mathfrak{h} = \mathtt{Lie}(H) \subset \mathfrak{g} = \mathtt{Lie}(G)$. *Then H is a normal Lie subgroup of G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.*

Proof. If H is normap subgroup, $\mathrm{Ad}_g(H) = H$ for all $g \in G$, so $\mathrm{Ad}_g(\mathfrak{h}) = \mathfrak{h}$. So, Ad defines a representation of G in $\mathfrak{h}$, so ad defines a representation of $\mathfrak{g}$ on $\mathfrak{h}$. This precisely means that $[X, Y] \in \mathfrak{h}$ for $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

In the opposite direction, assume $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, so that $\mathrm{ad}_X(Y) \in \mathfrak{h}$ for any $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$. This means that $\mathfrak{h}$ is an ad_X-invariant subspace of $\mathfrak{g}$, and, therefore, it is also an $\mathrm{Ad}_{\exp(sX)}$-invariant subspace of $\mathfrak{g}$. In other words, $\mathrm{Ad}_{\exp(sX)}(\mathfrak{h}) = \mathfrak{h}$. By Lie theorem 1 this implies that H and $\mathrm{Ad}_{\exp(sX)}(H)$ coincide as they have the same Lie algebra. Since G is connected, it is generated by $\exp(sX)$, so this implies that H is normal. □

9.3.3. *Exercise.* Show that both connectedness of H and connectedness of G are important: find examples where one of G, H is not connected, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ but H is not normal in G.

9.3.4. **Definition.**
- A connected Lie group G is called simple if it is not abelian and has no nontrivial normal connected Lie subgroups.
- Similarly, a noncommutative Lie algebra $\mathfrak{g}$ is called simple if it has no nontrivial ideals.

9.3.5. *Example.* The group $G = SL(2, \mathbb{R})$ is simple even though its center $\{\pm 1\}$ is nontrivial. Of course, the center of a simple Lie group is always discrete.

9.3.6. **Corollary.** *A connected Lie group G is simple iff its Lie algebra is simple.*

9.4. **Direct product. Semidirect product.** It is easy to find out that $\texttt{Lie}(G \times H) = \texttt{Lie}(G) \times \texttt{Lie}(H)$. We will now describe a more general operation (both for Lie groups and for Lie algebras) that assigns a new Lie group (algebra) to a pair of groups (algebras) and an action of one on the other.

We start with the groups.

9.4.1. *Semidirect product of Lie groups.* We have two Lie groups G and H and a smooth action $\phi : G \times H \to H$ of G on H such that, for any $g \in G$ the action $\phi_g : H \to H$ is a group automorphism. The semidirect product $G \ltimes H$ is defined as follows.

- As a manifold, $G \ltimes H$ is the product $G \times H$.
- The multiplication is given by the formula $(g, h)(g', h') = (gg', \phi_{g'^{-1}}(h)h')$.

Here is a simplest example.

The group $G = \mathbb{R}^*$ acts on $H = \mathbb{R}$ by multiplication. The semidirect product is isomorphic to the group of matrices in $GL(2, \mathbb{R})$ having the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$.

9.4.2. *Exercise.* Present the group of matrices $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ with $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$ as a semidirect product of $\mathbb{R}^*$ and $\mathbb{R}$, with respect to a certain action of $\mathbb{R}^*$ on $\mathbb{R}$.

9.4.3. *Exercise.* Let A be a finite dimensional algebra. Define $\mathrm{Aut}(A)$ as the subgroup of $GL(A)$ consisting of the algebra automorphisms. Prove that $\mathrm{Aut}(A)$ is a Lie subgroup of $GL(A)$. Prove that the Lie algebra of $\mathrm{Aut}(A)$ is $\mathrm{Der}(A)$, the Lie algebra of derivations of A.

9.4.4. An action of a Lie algebra $\mathfrak{g}$ on a Lie algebra $\mathfrak{h}$ is, by definition, a Lie algebra homomorphism

$$\psi : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h}).$$

Thus, an action is described by a bilinear map

$$\psi : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$$

such that for each $X \in \mathfrak{g}$ the map $\psi_X : \mathfrak{h} \to \mathfrak{h}$ is a derivation.

Given an action of a Lie group G on H

$$\phi : G \times H \to H,$$

Defines a representation of G on $\mathfrak{h} = T_1(H)$ so that each $g \in G$ defines a Lie algebra automorphism of $\mathfrak{h}$. Deriving, we get an action of $\mathfrak{g}$ on the Lie algebra $\mathfrak{h}$.

9.4.5. *Semidirect product of Lie algebras.* Let $\psi : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ be an action of $\mathfrak{g}$ on $\mathfrak{h}$. Then the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ (it depends on ψ) is defined as the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of vector spaces, with the bracket defined by the formula

$$[X + Y, X' + Y'] = [X, X'] + (\psi(X)(Y') - \psi(X')(Y) + [Y, Y']).$$

9.4.6. **Proposition.** *The Lie algebra of the semidirect product $G \ltimes H$ is the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$.*

Conversely, let $\psi : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ be a Lie algebra homomorphism. Let, furthermore, G and H be simply connected Lie groups such that $\mathfrak{g} = \mathtt{Lie}(G)$ and $\mathfrak{h} = \mathtt{Lie}(H)$. We know that $\mathrm{Der}(\mathfrak{h})$ is the Lie algebra of the Lie group $\mathrm{Aut}(\mathfrak{h})$. By Lie theorem 2, ψ lifts to a Lie group homomorphism $G \to \mathrm{Aut}(\mathfrak{h})$. Once more, Lie theorem 2 implies that $\mathrm{Aut}(\mathfrak{h}) = \mathrm{Aut}(H)$. Therefore, the Lie algebra action ψ lifts to a unique Lie group action $\phi : G \to \mathrm{Aut}(H)$. To be completely honest, one has to verify that the action of G on H leads to a smooth map $G \times H \to H$. We will leave this fact without the proof.

This allows one to construct a semidirect product $G \ltimes H$ whose Lie algebra is $\mathfrak{g} \ltimes \mathfrak{h}$.

9.4.7. Ado theorem is based on the following structural result in the theory of Lie algebras.

Theorem. *Any finite dimensional Lie algebra can be recursively constructed, using the operation of semidirect product, from simple Lie algebras and commutative Lie algebras.*

This result follows from the Levi theorem in Lie algebra theory, presentation of a semisimple Lie algebra as a product of simple Lie algebras and presentation of a solvable Lie algebra as a consecutive semidirect product of commutative Lie algebras.

Based on this result, we can give an alternative proof of Lie theorem 3.

Case 1. $\mathfrak{g}$ is a simple Lie algebra. In this case $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is injective, so we do not need Ado theorem to deduce the existence of a Lie group with the Lie algebra $\mathfrak{g}$.

Case 2. $\mathfrak{g}$ is a commutative Lie algebra. Then we know the answer: the additive group $(\mathfrak{g}, +)$ is a Lie group with the Lie algebra $\mathfrak{g}$.

We can now proceed, using the mentioned above stucture theorem, using the semidirect product construction. If G and H are simply connected Lie groups and an action $\mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is given, it lifts to an action of G on H by automorphisms, so we can take the semidirect product $G \ltimes H$ as the Lie group with the LIe algebra $\mathfrak{g} \ltimes \mathfrak{h}$. This completes the proof of Lie theorem 3.

10. Compact groups (digest)

In this section we sketch the argument that allows to prove the following important property of compact Lie groups.

10.1. **Theorem.** *Any representation of a compact Lie group G is a direct sum of irreducible representations.*

For simplicity, we will be talking about finite dimensional representations only.

Let G be a compact group and $\rho : G \to GL(V)$ be a finite dimensional complex representation.

10.2. Unitary representations.

10.2.1. **Definition.** Let V be a complex vector space endowed with an inner product. A representation $\rho : G \to GL(V)$ is called unitary if $\rho(G)$ lies in the group of unitary transformations of V.

10.2.2. **Lemma.** *Let V be a unitary representation and $W \subset V$ be a subrepresentation. Then $W^{\perp}$ is also a subrepresentation so that $V = W \oplus W^{\perp}$.*

Proof. The claim is quite obvious. □

10.2.3. *Unitarizability.*

Theorem. *Let $\rho : G \to GL(V)$ be any finite dimensional complex representation of a compact Lie group. Then V admits an inner product making ρ a unitary representation.*

We will discuss later the proof of this important result.

10.2.4. *The case G is a finite group.* Finite group are compact. So, let us think the simpler case of representation of finite groups. An inner product on V will satiisfy the requirement of the theorem if it is invariant, that is if

$$(g(v), g(w) = (v, w)$$

for any $v, w \in V$. Choose an arbitrary inner product $(-,-)$ on V. We can "average" it defining a new inner product by the formula

$$\langle v, w\rangle = \frac{1}{|G|}\sum_{g\in G}(g(v), g(w)). \tag{9}$$

The formula defines a symmetric bilinear form. It is an inner product as $\langle v, v\rangle = \frac{1}{|G|}\sum_{g\in G}(g(v), g(v)) > 0$.

10.2.5. *Integration.* The reasoning presented above cannot be generalized to infinite groups as the formula (9) makes no sense in general. However, if G is compact, one can replace the summation with an integration. This is what we will study now.

The easiest way to define intergation of functions on a manifold is the axiomatic way. We define a *distribution* δ on M a linear functional $\delta : C_c^\infty(M) \to \mathbb{R}$ on the space of compactly supported smooth functions. The functional is supposed to be continuous in a certain canonical topology. For example, any $x \in M$ defines the distribution δ_x carrying f to $f(x)$. This is what is called *the delta-function at* x. Another (more reasonable) type of distributions is given by a differential n-form on an n-dimensional oriented manifold M.

A distribution δ on a group G is called left-invariant iff $\delta(f) = \delta(L_g^*(f))$ form any $g \in G$.

It turns out, there is a unique, up to scalar, invariant measure on a Lie group (a much more general result is called Haar's theorem).

In particular, for a finite group G the distribution carrying any function $f : G \to \mathbb{R}$ to $\frac{1}{|G|}\sum_{g\in G} f(g)$, is invariant. For the group $G = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, an invariant distribution is given by the formula

$$\delta(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx.$$

10.2.6. *Unitarizability.* We can now prove that any finite dimensional representation of a compact Lie group is unitarizable. Given a representation $\rho : G \to GL(V)$, we have to construct a G-invarian inner product on V.

Choose any inner product $(-,-)$ on V and define a new one by the formula

$$\langle v, w\rangle = \frac{1}{\mathtt{vol}(G)}\delta((g(v), g(w))),$$

where δ is a left-invariant distribution on G and $\mathtt{vol}(G) = \delta(1)$ is the volume of G with respect to δ.

Similarly to the case of finite groups, the new Hermitian form is positive and invariant.

10.2.7. *Exercise.* Using the fact that $G := \mathbb{R}/\mathbb{Z}$ is commutative, prove that its any irreducible representation is one-dimensional.

Hint. Choose an element $g \in G$ and find an eigenvalue λ and a corresponding eigenvector $v \in V$. Prove that $\{v \in V | g(v) = \lambda v\}$ is a G-subrepresentation.

The exercise, together with Theorem 10.1, allows one to describe all irreducible representations of $\mathbb{R}/\mathbb{Z}$, see 6.8.1.

10.3. One can reformulate the unitarizability result as follows. Let G be a compact Lie group. Then any Lie group homomorphism $f : G \to GL(n, \mathbb{C})$ factors, after conjugation, through the embedding $U(n) \to GL(n, \mathbb{C})$. Simply saying, this means that $U(n) \subset GL(n, \mathbb{C})$ is a maximal compact subgroup.

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