INTRODUCTION TO LIE ALGEBRAS. LECTURE 9.

8.3. **Remarks.**

8.3.1. The Lie theorem immediately implies that if V is a finite dimensional irreducible representation of a solvable Lie algebra then it is one-dimensional.

8.3.2. Even though Lie theorem looks very similar to Engel theorem, they are very different. Lie theorem provides an information about the structure of representations of a solvable Lie algebra. Engel theorem says nothing about representations of a nilpotent Lie algebra; it is applicable only if the image of the Lie algebra consists of nilpotent endomorphisms. For example, if L is one-dimensional (therefore nilpotent), it has representations corresponding to any (not necessarily nilpotent) endomorphisms. However, existence of Jordan normal form shows that any (single) representation can be written by as an upper-triangular matrix, provided the base field is algebraically closed.

Corollary 8.3.3. *Let k be algebraically closed field of characteristic zero. Let L be a solvable Lie algebra. Then there exists a sequence of ideals*

$$0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = L$$

where $\dim\ I_k = k$.

Proof. Apply Lie theorem to the adjoint representation. □

Corollary 8.3.4. *Let k, L be as above. Then $[L, L]$ is nilpotent Lie algebra.*

Proof. Let $x \in [L, L]$. Then 8.3.3 implies that ad_x is nilpotent. Then by Engel theorem $[L, L]$ is nilpotent. □

Corollary 8.3.5. *Let k, L be as above. Let $B : L \times L \longrightarrow k$ be the Killing form. Then $B(x, y) = 0$ if $x \in L, \quad y \in [L, L]$.*

Proof. According to 8.3.3, L admits a base in which ad_x has an apper triangular form. Then ad_y for $y \in [L, L]$ has in this base a strictly upper triangular form. This implies the claim. □

It turn out the converse is also true. It is called Cartan criterion (of solvability).

Theorem 8.3.6. *Let k be a field of characteristic zero and let V be a finite dimensional k-vector space. Let $L \subseteq \mathfrak{gl}(V)$. The following conditions are equivalent.*

- *The Lie algebra L is solvable.*
- *For each $x \in L$ and $y \in DL = [L, L]$ one has $\mathrm{Tr}_V(xy) = 0$.*

The proof will be given (hopefully, if we have enough time) later on.

9. Semisimplicity

9.1. **Radical.** The following result is an easy consequence of the last home assignments.

Proposition 9.1.1. *Let L be a finite dimensional Lie algebra. Then there exists an ideal $R \subseteq L$ such that*

- *R is solvable.*
- *R contains any solvable ideal of L.*

Proof. It is enough to check that if I, J are solvable ideals of L then $I + J$ is also solvable. One has

- $\subseteq I + J$ is a solvable ideal.
- The quotient $(I + J)/I = J/(I \cap J)$ is solvable.

Therefore, $I + J$ is solvable. □

The maximal solvable ideal $R \subseteq L$ is called *the radical of L*.

9.2. **Semisimple Lie algebras.**

Definition 9.2.1. A Lie algebra L is called semisimple if its radical is trivial.

It is clear that any simple algebra is semisimple (consider separately the one-dimensional case!)

Here is an equivalent definition.

Definition 9.2.2. A Lie algebra L is semisimple if it has no nontrivial abelian ideals.

In fact, any abelian ideal is solvable, so if L has nontrivial abelian ideals, it is not semisimple. In the other direction, if L is not semisimple, i.e. if its radical R is non-zero, consider the sequence of derived ideals $D^k(R)$; the last non-zero ideal $D^n(R))$ will be abelian.

There are some more very nice criteria of semisimplicity.

The first one uses the Killing form

$$K(x, y) = \mathrm{Tr}((\mathrm{ad}_x \mathrm{ad}_y).$$

The proof of the following lemma is based on Cartan criterion.

Lemma 9.2.3. *Let L be a finite dimensional Lie algebra over a field of characteristic zero and let I be the kernel of the Killing form:*

$$I = \{x \in L | \forall y \in L K(x,y) = 0\}. \tag{1}$$

The I is a solvable ideal in L.

Proof. I is an ideal since K is invariant. In fact, if $x \in I$ and $z \in L$ then

$$K([x,z],y) = K(x,[z,y]) = 0 \text{ for all } y \in L.$$

Solvability of I immediately follows from Cartan criterion. □

Theorem 9.2.4. *A Lie algebra L is semisimple iff its Killing form is non-degenerate.*

Proof. If L is semisimple, the kernel of the Killing form should be trivial. This means that the Killing form is non-degenerate.

In the other direction, suppose that L is not semisimple. Then L admits a nontrivial abelian ideal I. Then for $x \in I,\ y \in L$ one has

- $\mathrm{ad}_x \mathrm{ad}_y(L) \subset I$;
- $\mathrm{ad}_x \mathrm{ad}_y(I) = 0$.

This implies that $\mathrm{ad}_x \mathrm{ad}_y$ is nilpotent, therefore, $K(x,y) = 0$. □

The above criterion is very powerful.

Corollary 9.2.5. *Let L be a semisimple Lie algebra and let I be an ideal in L. Put $J = I^\perp$ (the orthogonal complement to I with respect to K). Then J is an ideal in L and $L = I \times J$.*

Proof. J is ideal by invariance of K: $y \in J$ iff $\forall x \in I\ K(x,y) = 0$. Then for $z \in L$ and for all $x \in I$ one has

$$K([z,y],x) = -K([y,z],x) = -K(y,[z,x]) = 0.$$

The intersenction $I \cap J$ is solvable by Cartan criterion. Therefore, $I \cap J = 0$ and this implies $L = I \oplus J$ as vector spaces. The rest is obvious. □

Lemma 9.2.6. *The direct product $L \times M$ is semisimple iff both L and M are semisimple.*

Proof. Killing form on $L \times M$ is the orthogonal sum of the Killing forms on L and on M. Orthogonal sum is non-degenerate iff the summands are non-degenerate. □

Taking into account everything said above we deduce

Theorem 9.2.7. *A Lie algebra L is semisimple iff it is a direct product of simple algebras.*

9.3. **The algebra $\mathfrak{sl}_n$.** We will prove here that $\mathfrak{sl}_n$ is a simple Lie algebra. The idea is similar to the case $n = 2$.

It is more convenient to work with the algebra $L = \mathfrak{gl}_n$. Our aim is the following

Theorem 9.3.1. *The only ideals of $\mathfrak{gl}_n$ are $\mathfrak{sl}_n$ and kI where I is the unit matrix.*

In particular, since $\mathfrak{gl}_n = \mathfrak{sl}_n \times kI$, this implies that $\mathfrak{sl}_n$ is a simple algebra.

Let us choose a convenient basis.

Let e_{ij} be the matrix having 1 at (i, j)-th place and 0 elsewhere.

Denote $H = \langle e_{11}, \ldots, e_{nn} \rangle$. This is an n-dimensional commutative Lie subalgebra. We can easily dscribe L as a H-module with respect to the adjoint action. The result is given by the formula

$$L = H \oplus \bigoplus_{i \neq j} \langle e_{ij} \rangle. \tag{2}$$

Where H is a sum of trivial H-modules and each $\langle e_{ij} \rangle$ is a one-dimensional H-module defined by the character $\chi_{ij} : H \longrightarrow k$ given by the formula

$$\chi_{ij}(h) = h_{ii} - h_{jj}.$$

Let I be an ideal of L.

We are in the situation of Theorem 6.2.1. L is a completely reducible representatin of H, I is a subrepresentation. Therefore, I is also completely reducible. Every simple H-submodule of L is either $\langle e_{ij} \rangle$ for $i \neq j$ or a one-dimensional subspace of H.

Suppose there exists $i \neq j$ so that $e_{ij} \in I$. Then $[e_{ji}, e_{ij}] = e_{jj} - e_{ii} \in I$. Then $[e_{jj} - e_{ii}, e_{kl}]$ which is e_{kl} up to a non-zero constant whenever $k \neq l$ and $\{k, l\} \cap \{i, j\} \neq \emptyset$.

Then it is easy to see that $e_{ij} \in I$ for all $i \neq j$ and finally $I \supseteq \mathfrak{sl}_n$.

Suppose now that none of the e_{ij}, $i \neq j$, belongs to I. Then $I \subseteq H$. Since $e_{ij} \in I$, for each $h \in I$ one should have

$$[h, e_{ij}] = (h_{ii} - h_{jj}) e_{ij} = 0$$

which implies that $I \subseteq kI$.

Theorem is proven.

Problem assignment, 7

1. The algebra $\mathfrak{sp}_4$ consists of 4×4-matrices of form

$$\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$$

where M, N, P, Q are two-by-two matrices satisfying the conditions

$$N, P \text{ are symmetric; } Q = -N^t.$$

Prove that $\mathfrak{sp}_4$ is simple.

Hint. The reasoning similar to that of Theorem 9.3.1, where H is once more the subalgebra of diagonal matrices.