INTRODUCTION TO LIE ALGEBRAS.
LECTURE 8.

8. Solvable Lie algebras

Recall that if V, W are vector subspaces of a Lie algebra L then $[V, W]$ denotes the vector subspace of L generated by all elements $[v, w]$ where $v \in V,\ w \in W$.

Recall also that if I, J are ideals in L then $[I, J]$ is also an ideal of L (Jacobi identity).

8.1. **Definition and first properties.** Define a sequence of ideals of L (the derived series) by

$$D^0(L) := L,\ D^1(L) := [L, L],\ D^{i+1}(L) := [D^i(L), D^i(L)].$$

Definition 8.1.1. A Lie algebra L is called *solvable* if $D^n(L) = 0$ for some n.

8.1.2. Examples. Recall that the sequence of ideals $C^n(L)$ was defined before the Pesach vacation by the formulas

$$C^1(L) = L; \quad C^{n+1}(L) = [L, C^n(L)].$$

A Lie algebra is nilpotent if $C^n(L) = 0$ for some n.

Therefore, any nilpotent Lie algebra is solvable since $D^n(L) \subset C^{n+1}(L)$.

Define $\mathfrak{b}_n(k) \subset \mathfrak{gl}_n(k)$ as the Lie subalgebra consisting of upper triangular matrices. *Please take care: this contradicts our notation at the beginning of the course where* $\mathfrak{b}$ *denoted traceless upper triangular matrices. Sorry.*

The algebra $\mathfrak{b}_n(k)$ is solvable since $D^1(\mathfrak{b}) = \mathfrak{m}$ and so $D^i(\mathfrak{b}) \subset C^i(\mathfrak{m})$. However $\mathfrak{b}$ is not nilpotent since $C^2(\mathfrak{b}) = D^1(\mathfrak{b}) = \mathfrak{m}$ and $C^3(\mathfrak{b}) = C^2(\mathfrak{b}) = \ldots$. (Check this!)

Proposition 8.1.3. *The following conditions are equivalent*

(i) $D^n(L) = 0$.

(ii) *There exists a chain of ideals*

$$L = I_0 \supset I_1 \supset \ldots \supset I_n = 0$$

such that I_k/I_{k+1} *is a commutative Lie algebra that is* $[I_k, I_k] \subset I_{k+1}$ *for all* k.

Proof. (i)$\Longrightarrow$ (ii) since the chain $I_k := D^k(L)$ satisfies the condition of (ii). Moreover, this is the minimal chain which satisfies the condition: if $I_0, \ldots, I_n$ is such that $[I_k, I_k] \subset I_{k+1}$ for all k, then $I_k \supset D^k(L)$. Hence (ii)$\Longrightarrow$ (i). □

8.2. **Lie theorem.**

Theorem 8.2.1. *Assume that the base field k is algebraically closed and has characteristic zero. Let L be a solvable finite dimensional Lie algebra and let $\rho : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of L. Then one can choose a basis of V so that the image $\rho(L)$ is a subalgebra of $\mathfrak{b}(m)$, $m := \dim V$.*

The proof is similar to the proof of Engel theorem.

First of all, note that the theorem is an easy consequence (by induction on the dimension of V) of the following assertion.

Theorem 8.2.2. *Let L and ρ be as above. Suppose that* $\dim\ V > 0$. *Then all endomorphisms $\rho(g), g \in L$ have a common eigenvector.*

In fact, suppose 8.2.2 have been proven. Then Theorem 8.2 immediately follows by induction on $\dim\ V$. In fact, by 8.2.2 there exists a common eigenvector $v_0 \in V$ for all operators $\rho(x),\ x \in L$. This means that v_0 generates a one-dimensional L-submodule of V. By the induction hypothesis, the quotient module V/kv_0 admits a basis $\overline{v_1}, \ldots, \overline{v_{n-1}}$ satisfying the requirements of the theorem. Then choose representatives $v_i \in V$ of the classes $\overline{v_i}$ and the basis $v_1, v_2, \ldots, v_{n-1}, v_0$ is the one we were looking for.

From now on we will concentrate on proving Theorem 8.2.2. The proof will go by induction on $\dim\ L$.

Here are the steps of the proof.

- Find in L an ideal I of codimension 1.
- By the induction hypothesis, all endomorphisms $\rho(g), g \in I$ have a common eigenvector v_0. Let W be the space of all common eigenvectors of $\rho(g), g \in I$ with the same eigenvalues. We prove that W is L-submodule.
- Choose $x \in L$ such that $L = I + kx$. The element $\rho(x)$ has an eigenvector in W. It will be automatically a common eigenvector to all $\rho(g), g \in L$.

8.2.3. Proof of 8.2.2

We are proving the theorem by induction on $\dim\ L$.

Basis of the induction.

For dim $L = 1$ one has $L = kx$. Since the field k is algebraically closed and $\dim V < \infty$, $\rho(x)$ has an eigenvector v. Obviously v is an eigenvector of all endomorphisms $\rho(g), g \in L$.

The step of the induction

1. First of all, we will prove there exists an ideal I in L having codimesion one.

Since L is solvable, $[L, L] \neq L$.

Observe that any subspace of L containing the commutant $[L, L]$ is an ideal of L.

Hence L has an ideal I of codimension 1: $L = I \oplus kx$.

2. By the induction hypothesis, all endomorphisms $\rho(g), g \in I$ have a common eigenvector $v_0 \neq 0$. For any $g \in I$ denote by $\chi(g)$ the scalar satisfying $\rho(g)v_0 = \chi(g)v_0$. This uniquely defines a map $\chi : L \longrightarrow k$ satisfying two properties

- χ is linear;
- $\chi([x, y]) = 0$ for all $x, y \in I$.

(Explanation: χ is the homomorphism $I \longrightarrow \mathfrak{gl}_1$ corresponding to the one-dimensional representation generated by v_0).

3. The space $W := \{v \in V \mid \rho(g)v = \chi(g)v, \ \forall g \in I\}$ contains v_0. Any vector of W is a common eigenvector for all endomorphisms $\rho(g), g \in I$. To finish the proof, it is enough to verify that $\rho(x)W \subset W$ since this inclusion implies the existence of an eigenvector of $\rho(x)$ in W.

4. Hence we need to show that $\rho(x)W \subset W$ or, equivalently, that $\rho(g)\rho(x)v = \chi(g)\rho(x)v$ for all $v \in W, g \in I$.

One has

$$\rho(g)\rho(x)v = \rho(x)\rho(g)v + \rho([x, g])v = \chi(g)\rho(x)v + \chi([x, g])v$$

since $[x, g] \in I$. Therefore, our aim is to prove that $\chi([x, g]) = 0$ for each $g \in I$.

5. Fix $v \in W \setminus \{0\}$ and for each $k \in \mathbb{N}$ denote by V_k the span of $v = \rho(x)^0 v, \rho(x)v, \ldots, \rho(x)^k v$.

Let us show that for any $g \in I, k \in \mathbb{N}$ $\rho(g)V_k \subset V_k$ and, moreover, $\rho(g)\rho(x)^k v = \chi(g)\rho(x)^k v$ modulo V_{k-1} (set $V_{-1} = 0$). We show this by induction on k. For $k = 0$ the assertion follows from the definition of W. For the induction step, observe that

$$\rho(g)\rho(x)^{k+1}v = \rho(x)\rho(g)\rho(x)^k v + \rho([g, x])\rho(x)^k v.$$

Since I is an ideal, $[g, x] \in I$. Then, by the induction hypothesis, $\rho([g, x])\rho(x)^k v \in V_k$ and $\rho(g)\rho(x)^k v = \chi(g)\rho(x)^k v$ modulo V_{k-1}. Therefore $\rho(g)\rho(x)^{k+1}v = \chi(g)\rho(x)^{k+1}v$ modulo V_k as required.

Let $k \in \mathbb{N}$ be maximal such that the vectors $v, \rho(x)v, \dots, \rho(x)^k v$ are linearly independent. Then $V_m = V_k$ for all $m \geq k$. As we showed, V_k is $\rho(I)$-stable and relative to the basis $v, \rho(x)v, \dots, \rho(x)^k v$ each matrix $\rho(g)$ $(g \in I)$ is upper triangular with the diagonal entries equal to $\chi(g)$. Obviously $\rho(x)V_k = V_k$ and so for any $g \in I$ $\rho([x, g]) = \rho(x)\rho(g) - \rho(g)\rho(x)$ is a traceless endomorphism. Thus $\chi([x, g]) = 0$ as required.

Thus, W is $\rho(x)$-invariant, therefore, W contains a $\rho(x)$-invariant vector which is invariant with respect to the whole of L.

The theorem is proven.

Problem assignment, 6

1. Prove that any subalgebra of a solvable Lie algebra is solvable.
2. Prove that any quotient algebra of a solvable Lie algebra is solvable.
3. Let L be a Lie algebra and I be an ideal in L. Prove that if L and L/I are solvable then L is solvable.