INTRODUCTION TO LIE ALGEBRAS.
LECTURE 7.

7. Killing form. Nilpotent Lie algebras

7.1. Killing form.

7.1.1. Let L be a Lie algebra over a field k and let $\rho : L \longrightarrow \mathfrak{gl}(V)$ be a finite dimensional L-module. Define a map

$$B_\rho : L \times L \longrightarrow k$$

by the formula

$$B_\rho(x, y) = \mathrm{Tr}(\rho(x) \circ \rho(y)),$$

where Tr denotes the trace of endomorphism.

Lemma 7.1.2. *The map B_ρ is bilinear and symmetric.*

Proof. Bilinearity is obvious. Symmetricity follows from the well-known property of trace we have already used:

$$\mathrm{Tr}(fg) = \mathrm{Tr}(gf).$$

□

The map B_ρ satisfies another property called *invariance.*

Definition 7.1.3. Let V, W, K be three L-modules. A map

$$f : V \times W \longrightarrow K$$

is called L-invariant if for all $x \in L,\ v \in V,\ w \in W$ one has

$$xf(v, w) = f(xv, w) + f(v, xw).$$

Lemma 7.1.4. *Let $\rho : L \longrightarrow \mathfrak{gl}(V)$ be a finite dimensional representation. The bilinear form $B_\rho : V \times V \longrightarrow k$ is invariant (k is the trivial representation). This means that $B_\rho([x, y], z) + B_\rho(y, [x, z]) = 0$.*

Proof. Let $X = \rho(x)$, and similarly Y and Z. One has

$$B_\rho([x, y], z) = \mathrm{Tr}([X, Y]Z) = \mathrm{Tr}(XYZ - YXZ) = \mathrm{Tr}(XYZ) - \mathrm{Tr}(YXZ).$$

Similarly,

$$B_\rho(y, [x, z]) = \mathrm{Tr}(Y[X, Z]) = \mathrm{Tr}(YXZ) - \mathrm{Tr}(YZX).$$

Finally, $\mathrm{Tr}(X(YZ)) = \mathrm{Tr}((YZ)X)$ and the lemma is proven. □

Definition 7.1.5. Killing form on a Lie algebra L is the bilinear form

$$B : L \times L \longrightarrow k$$

defined by the adjoint representation.

Example 7.1.6. Let L be commutative. Then the Killing form on L is zero.

Example 7.1.7. Let $L = \mathfrak{sl}_2$. Then the Killing form is non-degenerate, i.e. for any nonzero $x \in L$ there exists $y \in L$ such that $B(x, y) \neq 0$. (see Problem assignment, 1).

7.1.8. Let $B : V \times V \longrightarrow k$ be a symmetric bilinear form. The kernel of B is defined by the formula

$$\mathrm{Ker}(B) = \{x \in V | \forall y \in V \quad B(x, y) = 0\}.$$

The form is *non-degenerate* if its kernel is zero. In this case the linear transformation

$$B' : V \longrightarrow V^*$$

from V to the dual vector space V^* given by the formula

$$B'(x)(y) = B(x, y),$$

is injective. If $\dim V < \infty$ this implies that B' is an isomorphism.

Proposition 7.1.9. *Let $\rho : L \longrightarrow \mathfrak{gl}(V)$ be a finite dimensional representation. Then the kernel of B_ρ is an ideal in L.*

Proof. Suppose $x \in \mathrm{Ker} B_\rho$. This means that $\mathrm{Tr}(\rho(x)\rho(y)) = 0$ for all y. Then for all $z \in L$

$$B_\rho([z, x], y) = -B_\rho(x, [z, y]) = 0$$

by the invariantness of B_ρ. □

Today we will study a large class of algebras having vanishing Killing form.

7.2. **Nilpotent Lie algebras.** Let $V, W \subseteq L$ be two vector subspaces of a Lie algebra L. We define $[V, W]$ as the vector subspace of L spanned by the brackets $[v, w], \ v \in V, \ w \in W$.

Jacobi identity implies that if V, W are ideals in L then $[V, W]$ is also an ideal in L. Define a sequence of ideals $C^k(L)$ by the formulas

$$C^1(L) = L; \quad C^{n+1}(L) = [L, C^n(L)].$$

Lemma 7.2.1. *One has $[C^r(L), C^s(L)] \subseteq C^{r+s}(L)$.*

Proof. Induction in r. □

Example 7.2.2. Recall that

$$\mathfrak{n}_n = \{A = (a_{ij}) \in \mathfrak{gl}_n | a_{ij} = 0 \text{ for } j < i + 1\}.$$

If $L = \mathfrak{n}_n$ then

$$C^k(L) = \{A = (a_{ij}) \in \mathfrak{gl}_n | a_{ij} = 0 \text{ for } j < i + k\}.$$

Check this!

Definition 7.2.3. A Lie algebra L is called nilpotent if $C^n(L) = 0$ for $n \in \mathbb{N}$ big enough.

Thus, commutative Lie algebras as well as the algebras $\mathfrak{n}_n$ are nilpotent.

7.3. **Engel theorem.**

Lemma 7.3.1. *Let $L, R \in \operatorname{End}(V)$ be two commuting nilpotent operators. Then $L + R$ is also nilpotent.*

Proof. Since L and R commute, one has a usual Newton binomial formula for $(L + R)^n$. This implies that if $L^n = R^n = 0$ then $(L + R)^{2n} = 0$. □

Theorem 7.3.2. *Let $L \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra. Suppose that all $x \in L$ considered as the operators on V, are nilpotent. Then there exists a non-zero vector $v \in V$ such that $xv = 0$ for all $x \in L$.*

Proof.

Step 1. Let us check that for each $x \in L$ the endomorphism ad_x of L is nilpotent. In fact, let $L_x : \operatorname{End}(V) \to \operatorname{End}(V)$ be the left multiplication by x and R_x be the right multiplication. Then $\operatorname{ad}_x = L_x - R_x$. The operators L_x and R_x commute. Both of them are nilpotent since x is a nilpotent endomorphism of V. Therefore, ad_x is nilpotent by 7.3.1.

Step 2. By induction on $\dim L$ we assume the theorem has been already proven for Lie algebras K of dimension $\dim K < \dim L$.

Step 3. Consider a maximal Lie subalgebra K of L strictly contained in L. We will check now that K is a codimension one ideal of L.

Let us prove first K is an ideal in L. Let

$$I = \{x \in L | \forall y \in K \quad [x, y] \in K\}.$$

This is a Lie subalgebra of L (Jacobi identity). Obviously $I \supseteq K$. We claim $I \neq K$. By maximality of K this will imply that $I = L$ which precisely means that K is an ideal in L.

In fact, consider the (restriction of the) adjoint action of K on L. K is a K-submodule of L. Consider the action of K on L/K. By the induction hypothesis (here we are using ad_x is nilpotent!), there exists a non-zero element $\overline{x} \in L/K$ invariant with respect to K. This means $[a, \overline{x}] = 0$ for all $a \in K$ or, in other words, choosing a representative $x \in L$ of $\overline{x}$, we get $[a, x] \in K$ for all $a \in K$. Thus, $x \in I \setminus K$ and we are done.

Now we know that K is an ideal. Let us check dim $L/K = 1$. In fact, If $x \in L \setminus K$, the vector space $K + kx \subseteq L$ is a subalgebra of L. Since K was chosen to be maximal, $K + kx = L$.

Step 4. Put

$$W = \{v \in V | Kv = 0\}.$$

Check that W is an L-submodule of V. If $x \in K,\ y \in L$ and $w \in W$, one has

$$x(yw) = y(xw) + [x, y]w = 0.$$

By the induction hypothesis, $W \neq 0$. Choose $x \in L \setminus K$. This is a nilpotent endomorphism of W. Therefore, there exists $0 \neq w \in W$ such that $xw = 0$. Clearly, w annihilates the whole of L.

Theorem is proven. □

Corollary 7.3.3. *Let $\rho : L \longrightarrow \mathfrak{gl}(V)$ be a representation. Suppose that for each $x \in L$ the operator $\rho(x)$ is nilpotent. Then one can choose a basis $v_1, \ldots, v_n$ of V so that*

$$\rho(L) \subseteq \mathfrak{n}_n \subseteq \mathfrak{gl}_n = \mathfrak{gl}(V).$$

Note that the choice of a basis allows one to identify $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n$.

Proof. Induction on $n = \dim V$.

One can substitute L by $\rho(L) \subseteq \mathfrak{gl}(V)$. By Theorem 7.3.2 there exists a nonzero element $v_n \in V$ satisfying

$$xv_n = 0 \text{ for all } x \in L.$$

Now consider L-module $W = V/\langle v_n \rangle$ and apply the inductive hypothesis. □

Corollary 7.3.4. *(Engel theorem) A Lie algebra L is nilpotent iff the endomorphism ad_x is nilpotent for all $x \in L$.*

Proof. Suppose L is nilpotent, $x \in L$. By definition $ad_x(C^k(L)) \subseteq C^{k+1}(L)$. This implies that ad_x is nilpotent.

In the other direction, suppose ad_x is nilpotent for all $x \in L$.

By 7.3.3 there exists a basis $y_1, \ldots, y_n$ of L such that

$$\mathrm{ad}_x(y_i) \in \langle y_{i+1}, \ldots, y_n \rangle \text{ for all } x \in L, i.$$

This implies by induction that

$$C^i(L) \subseteq \langle y_i, \ldots, y_n \rangle.$$

Therefore, $C^{n+1}(L) = 0$. □

Problem assignment, 5

1. Calculate the Killing form for $\mathfrak{sl}_2$ in the standard basis.
2. Let $\rho : \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V_n)$ be the irreducible representation of $\mathfrak{sl}_2$ of highest weight $n > 0$. Prove that B_ρ is non-degenerate.

   Hint. Check that $B_\rho(h, h) \neq 0$.
3. Prove that the Killing form of a nilpotent Lie algebra vanishes.
4. Prove that subalgebra and quotient algebra of a nilpotent Lie algebra is nilpotent.

   Let K be a nilpotent ideal of L and let L/K be nilpotent. Does this imply that L is nilpotent?
5. Prove that a nilpotent three-dimensional Lie algebra L is either abelian or isomorphic to $\mathfrak{n}_3$.