INTRODUCTION TO LIE ALGEBRAS.
LECTURE 6.

6. Representations of $\mathfrak{sl}_2$. Complete reducibility

6.1. **Verma modules.** Let V be a $\mathfrak{sl}_2$-module (not necessarily finite dimensional!) and let $v_0 \in V$ be a primitive weight vector of weight λ. In the previous lecture we defined the elements $v_n = F^n v_0$ and proved the formulas

$$(1) \qquad Fv_n = v_{n+1},\ Hv_n = (\lambda - 2n)v_n,\ Ev_n = n(\lambda - n + 1)v_{n-1}.$$

This suggests the following definition.

Definition 6.1.1. Let $\lambda \in \mathbb{C}$. Define a module $M(\lambda)$ as follows. The collection $\{v_k | k \in \mathbb{N}\}$ is a linear basis of $M(\lambda)$. The action of $e, f, h \in \mathfrak{sl}_2$ on $M(\lambda)$ is given by the formulas (1) above.

Module $M(\lambda)$ is called *Verma module of highest weight* λ.

Strictly speaking, we have not yet proven Verma module exists. To prove this, one has to check the formulas $[E, F] = H,\ [H, E] = 2E,\ [H, F] = -2F$. This is an easy exercise.

Now, if a module V has a primitive vector x of weight λ, a module homomorphism $f : M(\lambda \longrightarrow V$ sending v_0 to $x \in V$ is uniquely defined. One has $f(v_k) = F^k x$ and since the formulas (1) are satisfied both by v_k and by $F^k x$, this is an $\mathfrak{sl}_2$-homomorphism.

Let us study submodules of $M(\lambda)$.

Lemma 6.1.2. *If $\lambda \neq \mathbb{N}$ then $M(\lambda)$ is simple. If λ is a nonnegative integer, $M(\lambda)$ admits a unique non-trivial submodule N isomorphic to $M(-\lambda - 2)$. The quotient $M(n)/N$ of the Verma module of highest weight $n \in \mathbb{N}$ modulo its nontrivial submodule is isomorphic to $V(n)$.*

Proof. Let N be a submodule of $M(\lambda)$. In particular, this is a H-submodule. Therefore, N is a sum of weight subspaces which are all one-dimensional in $M(\lambda)$. Therefore, if $N \neq 0$, $v_k \in N$ for some k.

Now, if $\lambda \ni \mathbb{N}$ $Ev_i \neq 0$ for all $i \neq 0$ and this implies that $v_k \in N$ for all k. This proves the first claim of the lemma.

If $\lambda \in \mathbb{N}$, $Ev_i = 0$ for $i = 0, \lambda + 1$ only. This implies that submodule generated by v_k is $M(\lambda)$ if $k \leq \lambda$ and $N = \langle v_i | i \geq \lambda{+}1 \rangle$ otherwise. This proves the second claim of the lemma. The quotient module $M(n)/N$

has basis $\overline{v}_0, \dots, \overline{v}_n$ with the action of $\mathfrak{sl}_2$ given by the formulas identical to that for $V(n)$. This proves the third claim. $\square$

Corollary 6.1.3. *Let $n \in \mathbb{N}$ so that the Verma module $M(n)$ is not simple. The module $M(n)$ cannot be presented as a direct sum.*

Proof. If $M(n) = N' \oplus N''$ then $M(n)$ admits at least two different nontrivial submodules, N' and N''. We saw in 6.1.2 this is not the case. $\square$

We intend to prove later that finite dimensional $\mathfrak{sl}_2$-modules are completely reducible. However, before that we need to understand better complete reducibility.

6.2. **Complete reducibility: generalities.** All modules in this subsection are supposed to be modules over a fixed Lie algebra (or over a fixed associative algebra).

Let V be a module.

Theorem 6.2.1. *The following conditions are equivalent.*

- CR1. *V is a sum of its simple submodules.*
- CR2. *V is a direct sum of a family of its simple submodules.*
- CR3. *Any submodule V' of V is a direct summand, i.e. there exists a submodule V'' of V such that $V = V' \oplus V''$.*

The modules satisfying the equivalent properties (CR1)–(CR3) are called *semisimple modules* or *completely reducible representations.*

We start with two lemmas.

Lemma 6.2.2. *Let a module V satisfy the property (CR3). Then any submodule of V satisfies (CR3) as well.*

Proof. If $V \supseteq W \supseteq W'$ there exists a submodule V'' in V such that $V = W' \oplus V''$. Let $W'' = W \cap V''$. Then $W'' \cap W' = 0$ and the only thing to check is that $W = W' + W''$. Any element $w \in W \subseteq V$ has a decomposition $w = w_1 + v_2$ such that $w_1 \in W'$ and $v_2 \in V''$. Then $v_2 = w - w_1$ belongs to W automatically which implies that $v_2 \in W''$. $\square$

Lemma 6.2.3. *Let $V \neq 0$ satisfy the condition (CR3). Then V has a simple submodule.*

Proof. Choose $v \in V$, $v \neq 0$. By Zorn Lemma, the collection of submodules of V which do not contain v, admits a maximal element, say, W. By (CR3) $V = W \oplus W'$ for some W'. We claim that W' is simple. In fact, suppose W' has a proper nontrivial submodule. By Lemma 6.2.2 this implies that W' admits a non-trivial decomposition

$W' = W_1 \oplus W_2$. Then $V = W \oplus W_1 \oplus W_2$. Consider the corresponding decomposition of the element v chosen at the beginning of the proof. We have

$$v = w + w_1 + w_2 \text{ where } w \in W,\ w_1 \in W_1,\ w_2 \in W_2.$$

Since $v \notin W$, $w_1 \neq 0$ or $w_2 \neq 0$. Suppose $w_2 \neq 0$. Then $v \notin W \oplus W_1$ and we have a contradiction. $\square$

Now we are ready to prove the theorem.

Proof. Obviously, (CR2) implies (CR1). In the other direction, we claim that if $V = \sum_{i \in I} V_i$ with V_I simple, there exists a subset $J \subset I$ such that $V = \oplus_{i \in J} V_i$. The proof uses Zorn Lemma. Choose J to be a maximal subset of I such that $\sum_{i \in J} V_i$ is a direct sum. Let us prove the sum equals V. In fact, if this were not so, one would have

$$V_j \not\subseteq \sum_{i \in J} V_i.$$

Then the intersection $V_j \cap \sum_{i \in J} V_i$ is zero (it is a submodule of V_j). This proves that the sum $\sum_{i \in J \cup \{j\}} V_i$ is direct. Contradiction.

Thus, (CR1) implies (CR2). The same reasoning proves that (CR1) implies (CR3): if V' is a submodule of V, let J be a maximal subset of I such that $V' + \sum_{i \in J} V_i$ is a direct sum. Similarly to the above this sum is equal to V.

Let us prove now that (CR3) implies (CR1). Let $V' \subseteq V$ be the sum of all simple submodules of V, According to (CR3), $V = V' \oplus V''$ for some submodule V''. According to Lemma 6.2.3, if V'' is non-zero, it has a simple submodule. This gives a contradiction. $\square$

6.3. **Complete reducibility of finite dimensional $\mathfrak{sl}_2$-modules.** In this subsection we prove the following

Theorem 6.3.1. *Any finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is completely reducible.*

The proof will take a while.

6.3.2. The Casimir operator. Let V be a finite dimensional representation. The operator $Q = H^2 + 2EF + 2FE$ is a $\mathfrak{sl}_2$-module endomorphism of V. It is called *Casimir operator*. Recall that $Q : V \longrightarrow V$ gives rise to a decomposition

$$V = \bigoplus V_\theta \tag{2}$$

where V_θ is the generalized eigenspace corresponding to the eigenvalue θ of Q.

By definition $V_\theta = \{v \in V | \exists n : (Q - \theta \cdot \mathrm{id})^n v = 0\}$. Thus $V_\theta = \cup_n \mathrm{Ker}(Q - \theta \cdot \mathrm{id})^n$. Since Q is $\mathfrak{sl}_2$-invariant, V_θ is an $\mathfrak{sl}_2$-submodule of V. Thus, (2) is a direct decomposition of modules.

Thus, we reduced the claim of our theorem to the case Q has only one (generalized) eigenvalue θ.

6.3.3. Let now V be any finite dimensional representation and let $P(V) = \{v \in V | Ev = 0\}$ denote the set of primitive elements of V. We claim that $P(V)$ is an H-invariant vector subspace of V.

In fact, if $Ev = 0$ then $EHv = HEv - 2Ev = 0$.

Proposition 6.3.4. *The operator H is semisimple on $P(V)$.*

To prove Proposition 6.3.4 we need the following identity connecting E, F and H.

Lemma 6.3.5. *One has for $k > 0$*

$$EF^k = F^k E + kF^{k-1}(H - k + 1).$$

Proof. Induction on k. For $k = 1$ the claim is obvious. Suppose it has already been proven for k. We have

$$\begin{aligned} EF^{k+1} = (EF^k)F = F^k EF + kF^{k-1}(H - k + 1)F = \\ F^k FE + F^k H + kF^{k-1}F(H - k + 1) - 2kF^{k-1}F = \\ F^{k+1}E + F^k(H + k(H - k + 1) - 2k) = \\ F^{k+1}E + (k + 1)F^k(H - k). \end{aligned}$$

□

6.3.6. Proof of Proposition 6.3.4.

Let $v \in V$ be primitive and let $k > 0$ be a natural number. Let us check that

$$E^k F^k v = k! H(H - 1) \cdots (H - (k - 1))v. \tag{3}$$

In fact, the claim is obvious for $k = 1$ since v is primitive. Suppose we have already checked it for a given $k \in \mathbb{N}$. Then

$$\begin{aligned} E^{k+1}F^{k+1}v = E^k(EF^{k+1})v = E^k F^{k+1}Ev + (k + 1)E^k F^k(H - k)v = \\ (k + 1)! H(H - 1) \cdots (H - k)v. \end{aligned}$$

Now, since V is finite dimensional, there exists $k \in \mathbb{N}$ big enough, so that $F^k v = 0$ for each primitive element $v \in V$.

This implies that the restriction H_P of the operator H on $P(V)$ satisfies the identity

$$H_P(H_P-1)\cdots(H_P-k+1)=0.$$

This means semisimplicity.

$\square$

6.3.7. Suppose now that $V=V_\theta$. If $v\in P(V)$ is a weight primitive vector of weight n, V admits a simple submodule isomorphic to $V(n)$. By Schur lemma Q acts on $V(n)$ as a multiplication by a number. This number is obviously equal to θ.

On the other hand, one has

Lemma 6.3.8. *The Casimir operator Q acts on $V(n)$ as multiplication by $n(n+2)$.*

Proof. The highest weight vector v_0 of $V(n)$ has weight n. Thus, $Qv_0=H^2(v_0)+2FE+2EF=(n^2+2n)v_0$. The rest follows from Schur Lemma. $\square$

As an immediate corollary we deduce that all weight vectors of $P(V)$ have the same weight: this is the natural number n such that $\theta=n^2+2n$. This implies that all primitive vectors of V are weight vectors.

Denote $V_{(\lambda)}$ the generalized weight space of V corresponding to weight $\lambda\in k$. Recall that

$$V_{(\lambda)}=\{x\in V|\exists n:(H-\lambda)^n x=0\}.$$

We know that

$$V=\bigoplus_\lambda V_{(\lambda)}.$$

Lemma 6.3.9. *Let $x\in V_{(\lambda)}$. Then $Ex\in V_{(\lambda+2)}$ and $Fx\in V_{(\lambda-2)}$.*

Proof. We will prove only the first claim. Induction in n such that $(H-\lambda)^n x=0$. If $n=1$, there is nothing to prove (we have already checked this). If $n=k+1$, then $(H-\lambda)^k$ annihilates $(H-\lambda)x$. Therefore by the inductive assumption $E(H-\lambda)x$ has generalized weight $\lambda+2$.

$$E(H-\lambda)x=(H-\lambda-2)Ex,$$

we get the required claim. $\square$

6.3.10. Proof of Theorem 6.3.1.

Choose a basis $\{v^1,\dots,v^k\}$ of $P(V)$. Each vector v^i is a primitive weight vector. It, therefore, defines a simple submodule $V^i=\langle v^i,Fv^i,\dots F^n v^i\rangle$. We claim that $V=\oplus V^i$. Here is the proof.

Consider the natural map $f : V(n)^k \longrightarrow V$ sending the i-th component of $V(n)$ to V^i. The map f is an isomorphism in the weight n part by the choice of v^i. If $\mathrm{Ker}(f) \neq 0$, this is a non-trivial submodule on which Q acts by multiplicatin on $\theta = n^2 + 2n$. Therefore, its primitive vectors have weight n which contradicts bijectivity of f in weight n.

The proof of surjectivity of f is similar. Consider the quotient module $V/\mathrm{Im}(f)$. The Casimir acts on it with the same generalized eigenvalue θ since if V is annihilated by a power of $Q - \theta$ the same is true for $V/\mathrm{Im}(f)$. Therefore, the primitive vectors of the quotient have teh same weight n as the primitive vectors of V. If $\bar{v}$ is a primitive vector of the quotient and v is its (generalized) weight n representative in V, there exists a nonnegative integer k such that $E^k v$ is primitive in V. Since all primitive vectors in V have weight n, $k = 0$, v is primitive and therefore belongs to the image of f (recall: f is bijective on the primitive part). This proves that $\bar{v} = 0$.

Theorem is proven.

Corollary 6.3.11. *A finite dimensional representation V of $\mathfrak{sl}_2$ decomposes as*

$$V = \bigoplus_n V(n)^{d_n}$$

where $d_n = \dim P(V)_n$ is the dimension of the space of primitive vectors of V of weight n.

Problem assignment, 4

1. Let V be a finite dimensional $\mathfrak{sl}_2$-module and let V_k be the subspace of weight k vectors. Let $P(V)_k$ be the space of primitive weight k vectors. Prove that for $k \geq 0$
$$\dim P(V)_k = \dim V_k - \dim V_{k+2}.$$
2. Define $i : \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}_n$ by the formulas
$$\begin{array}{rcl} i(e) & = & \sum_{i=1}^{n-1} E_{i,i+1} \\ i(h) & = & \sum_{i=1}^{n} a_i E_{i,i} \\ i(f) & = & \sum_{i=1}^{n-1} b_i E_{i+1,i} \end{array}$$
Find a_i and b_i so that i is a Lie algebra homomorphism. Consider $\mathfrak{gl}_n$ as a $\mathfrak{sl}_2$-module with respect to the restriction of the adjoint action along i. Using Problem 2, find the multiplicity of each irreducible module $E(k)$ in $\mathfrak{gl}_n$.

   Note. Whoever prefers working in a more concrete setting is allowed to put $n = 3$.
3. We say that a module M satisfies condition (CR4) if for any surjective homomorphism $f : M \longrightarrow N$ there exists $g : N \longrightarrow M$ such that $fg = \mathrm{id}_N$.

   Prove that condition (CR4) is equivalent to (CR3).