INTRODUCTION TO LIE ALGEBRAS. LECTURES 4-5.

4. Schur Lemma

4.1. **Space of homomorphisms.** Let M, N be two L-modules. The collection of homomorphism of modules is denoted $\mathrm{Hom}_L(M, N)$. This is a vector space over k.

Thus,

$$\begin{aligned}\mathrm{Hom}_L(M,N) : &= \{\phi \in \mathrm{Hom}_k(M,N)|\ \forall v \in M, \forall x \in L\ \phi(xv) = x\phi(v)\}\\ &= \{\phi \in \mathrm{Hom}_k(M,N)|\ \forall x \in L\ \phi\rho(x) = \rho(x)\phi\}.\end{aligned}$$

4.2. **Schur's lemma.**

Theorem 4.2.1. *Suppose the base field k is algebraically closed. If V is a simple finite dimensional module over a Lie algebra L then* $\mathrm{Hom}_L(V, V) = k \cdot \mathrm{id}$.

Proof. Take $\phi \in \mathrm{Hom}_L(V, V)$. For any $c \in k$ the linear operator $(\phi - c \cdot \mathrm{id})$ is a L-homomorphism and so it is either an isomorphism or zero. Let c be an eigenvalue of ϕ; then the operator $(\phi - c \cdot \mathrm{id})$ has a non-zero kernel and so it is not an isomorphism. Hence $\phi - c \cdot \mathrm{id} = 0$ as required. □

4.3. **Application: Casimir operator for $\mathfrak{sl}_2(\mathbb{C})$.** Take $k := \mathbb{C}$. Fix the standard basis h, e, f of $\mathfrak{sl}_2$. Recall that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Let $\rho : \mathfrak{sl}(2) \to \mathfrak{gl}(V)$ be a representation. Denote

$$E = \rho(e),\ F = \rho(f), H = \rho(h).$$

Then the above relations imply

$$\begin{aligned} HE - EH &= 2E,\\ HF - FH &= -2F,\\ EF - FE &= H. \end{aligned} \tag{1}$$

Consider the endomorphism

$$Q := H^2 + 2FE + 2EF.$$

This is a linear endomorphism of V. We will check now that Q is an $\mathfrak{sl}_2$-endomorphism. To check this, it is enough to prove

$$QE = EQ,\ QF = FQ,\ QH = HQ.$$

The following easy lemma is useful in calculations.

Lemma 4.3.1. *Let $f, g, h \in \mathrm{End}(V)$. Then*

$$[f, gh] = [f, g]h + g[f, h].$$

Here, as usual, the bracket is defined by the formula $[f, g] = fg - gf$.

Proof.

$$[f, g]h + g[f, h] = fgh - gfh + gfh - ghf = fgh - ghf = [f, gh].$$

□

Now one can easily get

Lemma 4.3.2. *The operator Q commutes with E, F, H.*

Proof. Recall that all calculations are done in $\mathrm{End}(V)$.
One has

$$[E, Q] = [E, H^2 + 2EF + 2FE] = [E, H]H + H[E, H]+ \\ 2E[E, F] + 2[E, F]E = -2EH - 2HE + 2EH + 2HE = 0.$$

Similarly,

$$[F, Q] = [F, H^2 + 2EF + 2FE] = [F, H]H + H[F, H]+ \\ 2[F, E]F + 2F[F, E] = 2FH + 2HF - 2HF - 2FH = 0$$

and

$$[H, Q] = [H, H^2 + 2EF + 2FE] = 2[H, E]F + 2E[H, F]+ \\ 2[H, F]E + 2F[H, E] = 4EF - 4EF - 4FE + 4FE = 0.$$

□

Corollary 4.3.3. *Let V be finite dimensional and simple $\mathfrak{sl}_2$-module. Then $Q = c \cdot \mathrm{id}$ for some $c \in \mathbb{C}$.*

5. Finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$

Our next goal is to describe all finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

5.1. **Representations** $V(n)$**.** As a first step, we will describe a collection of irreducible representations which will turn out to be the collection of all irreducible representations.

We denote by $\mathbb{N}$ the set of non-negative integers.

5.1.1. Recall that $\mathfrak{sl}_2 \subseteq \mathfrak{gl}_2 = \langle E_{11}, E_{12}, E_{21}, E_{22}\rangle$ where E_{ij} denotes the matrix whose only non-zero entry is 1 in the (ij) position.

Note that in this notation $E = E_{12},\ F = E_{21},\ H = E_{11} - E_{22}$.

Consider the polynomial algebra $\mathbb{C}[x, y]$ and define the action of $\mathfrak{gl}_2$ on it by the formulas

$$E_{11}(p) = xp'_x,\quad E_{22}(p) = yp'_y,\quad E_{12}(p) = xp'_y,\quad E_{21}(p) = yp'_x. \tag{2}$$

Lemma 5.1.2. *The formulas (2) define a* $\mathfrak{gl}_2$*-module structure on* $\mathbb{C}[x, y]$*.*

Proof. One can check this claim directly.

Here is another way which allows to almost avoid calculations. Note that the formulas (2) assign to E_{ij} *derivations* of $\mathbb{C}[x, y]$ (compare to Problem assignment,1, # 1).

Any derivation of $\mathbb{C}[x, y]$ is uniquely defined by its value on the degree one polynomials x and y: if $d(x) = p,\ d(y) = q$ then $d(f) = pf'_x + qf'_y$ (once more, compare to Problem assignment, 1).

Then, in order to prove the formulas (2) are compatible with the brackets it is enough to check them on x and on y. One can see that the formulas (2) restricted on $\langle x, y\rangle$ give just the natural representation of $\mathfrak{gl}_2$. □

The set of homogeneous polynomials of a degree n is, obviously, a $\mathfrak{gl}_2$-submodule and an $\mathfrak{sl}_2$-submodule. Denote this $\mathfrak{sl}_2$-submodule by $V(n)$. Let us show that $V(n)$ is a simple $\mathfrak{sl}_2$-module.

5.1.3. Module V(n). Fix n. Consider the following basis of $V(n)$:

$$v_0 := x^n, v_1 := nx^{n-1}y, v_2 := n(n-1)x^{n-2}y^2, \ldots,$$
$$v_k := n!/(n-k)!x^{n-k}y^k, \ldots, v_n := n!y^n.$$

One has

$$Fv_k = E_{2,1}v_k = v_{k+1},\quad Ev_k = E_{1,2}v_k = k(n+1-k)v_{k-1},$$
$$Hv_k = (E_{1,1} - E_{2,2})v_k = (n-2k)v_k. \tag{3}$$

We see that H acts diagonally on the basis and all eigenvalues are distinct. By a lemma proven in Lecture 2, any submodule $W \subseteq V(n)$ is spanned by the elements of our basis belonging to W. In particular, any non-zero submodule contains v_k for some k; the relations (3) imply that such a submodule contains all basis elements. Hence $V(n)$ is simple.

5.2. **We have got all of them...** Now we will prove there are no finite dimensional irreducible representations of $\mathfrak{sl}_2$ except for the $V(n)$ described above.

5.2.1. Definitions

A vector v of $\mathfrak{sl}_2$-module is called *a weight vector* if $Hv \in \mathbb{C}v$.
A vector v of $\mathfrak{sl}_2$-module is called *of weight* c $(c \in \mathbb{C})$ if $Hv = cv$.
A vector v of $\mathfrak{sl}_2$-module is called *primitive* if $Ev = 0$.

5.2.2. The set of vectors of weight λ in V is denoted V^λ.

Let $v \in V^\lambda$. We claim that $Ev \in V^{\lambda+2}$ and $Fv \in V^{\lambda-2}$. In fact,

$$HEv = EHv + [H,E]v = \lambda Ev + 2Ev = (\lambda+2)Ev$$

and similarly for Fv.

5.2.3. Let V be a finite dimensional $\mathfrak{sl}_2$-module. We claim that V has a primitive weight vector.

In fact, $H : V \longrightarrow V$ is an endomorphism of a finite dimensional vector space. Therefore, H admits an eigenvector $v \in V$. Let $v \in V^\lambda$. Then $E^k v \in V^{\lambda+2k}$. Since V is finite dimensional, this proves that $E^k v = 0$ for k big enough. Thus, if $n = \max\{k | E^k v \neq 0\}$, the element $E^n v$ is a primitive weight vector.

5.2.4. Let V be a finite dimensional $\mathfrak{sl}_2$-module and let v_0 be a primitive vector of weight λ.

Put $v_n = F^n v_0$. One has $Fv_n = v_{n+1}$ and $Hv_n = (\lambda - 2n)v_n$. It turns out there is an very nice formula for Ev_n.

In Lemma 5.2.5 below we will prove the following identity.

$$EF^k = F^k E + kF^{k-1}(H - (k-1)). \tag{4}$$

The formula (4) implies that

$$Ev_n = EF^n v_0 = F^n E v_0 + nF^{n-1}(H - n + 1)v_0 = n(\lambda - n + 1)v_{n-1}.$$

Let us rewrite once more these formulas

$$Fv_n = v_{n+1},\ Hv_n = (\lambda - 2n)v_n,\ Ev_n = n(\lambda - n + 1)v_{n-1}. \tag{5}$$

Lemma 5.2.5. *The identity (4) is valid for any $n \geq 1$ for any representation of $\mathfrak{sl}_2$.*

Proof. Induction on k. For $k = 1$ it says that $EF = FE + H$ which is obvious. Suppose it has already been proven for $k = n$ and let $k = n+1$. We have

$$EF^{n+1} = EF^nF = (F^nE+nF^{n-1}(H-n+1))F = F^nEF+nF^{n-1}(H-n+1)F =$$
$$F^n(FE+H)+nF^{n-1}F(H-n+1)-nF^{n-1}(2F) = F^{n+1}E+(n+1)F^n(H-n).$$

□

5.2.6. We have made a substantial progress. In fact, we already know that any finite dimensional $\mathfrak{sl}_2$ module V contains a primitive weight vector v_0; The collection of $v_n = F^n v_0$ is a submodule. This implies that only finite number of v_i is nonzero.

This has very unexpected consequences. In fact, suppose $n = \max\{i | v_i \neq 0\}$. Then

$$0 = Ev_{n+1} = (n+1)(\lambda - n)v_n$$

and this implies that $\lambda = n$.

We have (easily!) proven the following

Theorem 5.2.7. *Let V be a finite dimensional representation and let v_0 be a weight primitive vector of weight λ. Then $\lambda \in \mathbb{N}$. The submodule of V generated by v_0 is $\langle v_0, v_1, \ldots, v_\lambda \rangle$. It has dimension $\lambda + 1$ and its module structure is given by the formulas (5).*

Problem assignment, 3

1. Let M, N be two non-isomorphic irreducible representations of a Lie algebra L. Prove that $\mathrm{Hom}_L(M,N)=0$.
2. Let L be a Lie algebra over an algebraically closed field k. Let $M_1,\dots M_n$ be non-isomorphic irreducible L-modules and let
$$M=\bigoplus_{i=1}^{n} M_i^{d_i}.$$
Calculate $\dim \mathrm{Hom}_L(M,M)$.
3. Let $V(n)$ be the standard $n+1$-dimensional representation of $\mathfrak{sl}_2$. Write down the matrices of the action of $E,\ F,H$ in the standard basis $v_0,\dots,v_n$ of $V(n)$.
4. Let V be the natural n-dimensional representation of the Lie algebra $\mathfrak{sl}_n$. Consider the map of Lie algebras
$$f:\mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_n$$
sending each matrix $M\in\mathfrak{sl}_2$ to the matrix $f(M)$ defined by the formula
$$f(M)_{ij}=\begin{cases} M_{ij} & \text{if} \qquad i,j\in\{1,2\}\\ 0 & \text{otherwise}\end{cases}$$
This defines an action ρ of $\mathfrak{sl}_2$ on V: $\rho(x)(v)=f(x)v$. For which n the resulting representation is irreducible? Write down the matrices of the operators $E,\ F,H$ acting on V.