INTRODUCTION TO LIE ALGEBRAS. LECTURE 3.

2.1. **Simplicity of** $(\mathbb{R}^3, \times)$**.** The proof of the simplicity of this Lie algebra is very geometric. Let I be a non-zero ideal in it and let $0 \neq v \in I$. We can normalize v so that $||v|| = 1$. There exists a pair of vectors v_2, v_3 so that the triple v, v_2, v_3 forms an orthonormal base. Then $v_2 = v \times v_3$ and $v_3 = v \times v_2$ up to sign, therefore, all three vectors belong to I. This proves the assertion.

3. Modules

The notion of module over a Lie algebra is of extreme importance.

3.1. **Two definitions and their equivalence.** Let L be a Lie algebra over a field k.

Definition 3.1.1. An L-module is a k-vector space M together with a bilinear map

$$r : L \times M \longrightarrow M$$

satisfying the following property

$$r([x, y], m) = r(x, r(y, m)) - r(y, r(x, m)).$$

Usually one writes simply xm instead of $r(x, m)$. Then our axiom reads

$$[x, y]m = xym - yxm.$$

To give another definition of L-module recall that for every vector space M the collection of endomorphisms $\mathrm{End}(V)$ admits an associative composition. The operation

$$f, g \in \mathrm{End}(M) \mapsto [f, g] = fg - gf \in \mathrm{End}(M)$$

defines a Lie algebra structure on $\mathrm{End}(M)$. The Lie algebra of endomorphisms so obtained is denoted $\mathfrak{gl}(M)$.

Definition 3.1.2. An L-module is a vector space M endowed with a Lie algebra homomorphism

$$\rho : L \longrightarrow \mathfrak{gl}(M).$$

The proof of the equivalence of the above definitions is fairly standard.

Another name for an L-module is *representation of* L.

If M is finite dimensional, we are talking about a finite dimensional representation.

3.2. **Examples.**

3.2.1. $L = k$**.** If L is one-dimensional, say, $L = ke$, a module structure

$$\rho : L \longrightarrow \mathfrak{gl}(M)$$

is given by an endomorphism of M (the image $\rho(e)$).

3.2.2. L **is commutative.** A representation of L is a Lie algebra homomorphism. If $L = \langle e_1, \ldots, e_n \rangle$, a homomorphism $r : L \to \mathfrak{gl}(M)$ is given by the images $r(e_i)$. Since r is a homomorphism, $r(e_i)$ commute. Vice versa, any collection of n commuting endomorphisms of M define on M a structure of L-module.

3.2.3. An $\mathfrak{sl}_2$-module is a vector space M with three endomorphisms E, F, H of M satisfying the conditions

$$EF - FE = H;\ HE - EH = 2E;\ HF - FH = -2F.$$

This means that an $\mathfrak{sl}_2$-module defines a *representation of* $\mathfrak{sl}_2$ *in matrices.* This is the explanation of the term *representation.*

3.2.4. Natural representation. By definition, Lie algebra $\mathfrak{gl}_n$ admits an n-dimensional representation. It is given by the identity map

$$\mathrm{id} : \mathfrak{gl}_n \longrightarrow \mathfrak{gl}(k^n).$$

It is called *the natural representation.* Similarly, if $\mathfrak{g} \subseteq \mathfrak{gl}_n$ is a Lie subalgebra, we have a natural n-dimensional representation of $\mathfrak{g}$.

Examples include $\mathfrak{g} = \mathfrak{sl}_n,\ \mathfrak{b}_n, \mathfrak{n}_n$ and some other algebras.

3.3. **Category of** L**-modules.** Fix a Lie algebra L.

A linear map $f : M \longrightarrow N$ is an L-module homomorphism if

$$f(ax) = af(x)$$

for each $a \in L,\ x \in M$. Clearly, composition of homomorphisms is a homomorphism.

Lemma 3.3.1. *Let* $f : M \longrightarrow N$ *be a bijective homomorphism of* L*-modules. Then* $f^{-1} : N \longrightarrow M$ *is also a homomorphism.*

Proof. Straighforward. □

The notion of submodule and quotient module are defined in a standard way.

3.3.2. Direct sum. Given two L-modules M and N, one defines an L-module structure on $M \oplus N$ by the formula

$$a(m, n) = (am, an).$$

3.4. **Representation theory.** Representation theory of Lie algebras studies the category of modules over a Lie algebra. Here are the typical questions and the typical notions studied.

3.4.1. Classification. Description of all isomorphism classes of L-modules. Sometimes only modules satisfying special properties are considered (e.g., finite dimensional modules).

Today we will see that in the case L is one-dimensional we already know the answer from Linear Algebra.

3.4.2. Simple modules. A module is called simple if it does not admit non-trivial submodules. (A synonym: irreducible representation).

3.4.3. Semisimple modules A module is called semisimple if it is isomorphic to a direct sum of simple modules (there are other equivalent definitions). Synonym: a completely reducible representation.

We will study soon the following result.

Theorem 3.4.4. *All finite dimensional representations of $\mathfrak{sl}_2$ are completely reducible.*

3.5. **Representations of a one-dimensional Lie algebra.** .

3.5.1. Isomorphism classes. We are looking for isomorphism classes of n-dimensional representations. A map $f : M \longrightarrow N$ is a homomorphism of representations if $f\alpha_M = \alpha_N f$. Since $M = N = k^n$ as vector spaces, we deduce that endomorphisms α_1 and α_2 define isomorphic representations iff there exists an automorphism f such that $\alpha_2 = f\alpha_1 f^{-1}$.

Thus the problem of classifications of n-dimensional representations is equivalent to that of classification of square matrices up to conjugation.

Theory of Jordan normal form answers this question in the case k is algebraically closed.

Let us recall the most important steps in this theory.

3.5.2. Recollections from Linear Algebra.

Let $f : V \longrightarrow V$ be an endomorphism of a finite dimensional vector space over an algebraically closed field k. Recall that $\lambda \in k$ is an eigenvalue of f iff $f - \lambda I$ is not invertible. The collection of eigenvalues of f is therefore the set of roots of the characteristic polynomial of f defined as

$$P_f(t) = \det(f - tI).$$

In what follows $S(f)$ will denote the set of eigenvalues of f (the spectrum of f).

Let $\lambda \in S(f)$. A vector $v \in V$ is called an eigenvector corresponding to λ if $f(v) = \lambda v$. Each eigenvalue admits a non-zero eigenvector. Furthermore, $v \in V$ is called generalized eigenvector if there exists $n \in \mathbb{N}$ such that $(f - \lambda I)^n v = 0$.

Fix $\lambda \in S(f)$. Let V^λ denote the set of eigenvectors and V_λ the set of generalized eigenvectors corresponding to λ. These are vector subspaces of V and

$$V_\lambda \supseteq V^\lambda \neq 0.$$

The following are the main results of this study.

- $V = \bigoplus_{\lambda \in S(f)} V_\lambda$.
- $\dim V_\lambda$ equals the multiplicity of λ in the characteristic polynomial of f.
- Each V_λ is isomorphic to a direct sum of Jordan blocks having eigenvalue λ (definition of J. b. see below).

Jordan block having eigenvalue λ is the matrix $A = (a_{ij})$ defined by the formulas

$$a_{ij} = \begin{cases} \lambda & \text{if} \quad i = j, \\ 1 & \text{if} \quad i = j - 1, \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

3.5.3. Simple modules. Direct consequence of the above: Any simple module has dimension 1; it is defined up to isomorphism by its (only) eigenvalue λ.

This module will be sometimes denoted by k_λ.

3.5.4. Semisimple modules.

Semisimple module is a direct sum of simple modules. Thus (for k algebraically closed) (V, f) is semisimple iff f is diagonalizable.

3.5.5. Example of non-semisimple modules. It is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

3.6. **Examples of representations.**

3.6.1. Adjoint representation Let L be any Lie algebra. the map $\mathrm{ad} : L \longrightarrow \mathfrak{gl}(L)$ defines a repersentation of L called adjoint representation.

Note that submodules of the adjoint representation are the ideals. Therefore, L is simple iff $\dim\ L > 1$ and the adjoint representation is irreducible.

3.6.2. ... and its restrictions If $M \subseteq L$ is a Lie subalgebra, one can consider L as a M-module restricting the adjoint representation of L on M.

Consider, for example, $L = \mathfrak{sl}_2$ and $M = \langle h \rangle$. Algebra M is one-dimensional and L is an M-module. It is semisimple with eigenvalues $-2,\ 0,\ 2$.

If we take another M, say, $\langle e \rangle$, the picture changes: all eigenvalues are zero and the M-module L is not semisimple.

3.7. **One-dimensional representations.** Let L be a Lie algebra and let $\rho : L \longrightarrow \mathfrak{gl}(V)$ is a one-dimensional representation. The algebra $\mathfrak{gl}(V)$ is one-dimensional and therefore commutative in this case. Thus,

$$\rho[x, y] = [\rho(x), \rho(y)] = 0.$$

In particular, for $L = \mathfrak{sl}_2$ one gets $\rho = 0$. This proves $\mathfrak{sl}_2$ does not admit non-trivial one-dimensional representations.

Problem assignment, 2

1. Let $\mathfrak{h} \subseteq \mathfrak{gl}_n$ be the set of diagonal matrices. Check that $\mathfrak{h}$ is a commutative Lie subalgebra. Check that $\mathfrak{gl}_n$ as the $\mathfrak{h}$-module (with respect to adjoint action) is a sum of one-dimensional representations.
2. Prove that the adjoint representation of $\mathfrak{gl}_2$ is isomorphic to a direct sum of a three-dimensional and one-dimensional representations.
3. Prove that the adjoint representation of $\mathfrak{sl}_2$ is not isomorphic to the sum of the natural representation with the trivial representation.