INTRODUCTION TO LIE ALGEBRAS.
LECTURE 3.

2.1. Simplicity of \((\mathbb{R}^3, \times)\). The proof of the simplicity of this Lie algebra is very geometric. Let \(I\) be a non-zero ideal in it and let \(0 \neq v \in I\). We can normalize \(v\) so that \(||v|| = 1\). There exists a pair of vectors \(v_2, v_3\) so that the triple \(v, v_2, v_3\) forms an orthonormal base. Then \(v_2 = v \times v_3\) and \(v_3 = v \times v_2\) up to sign, therefore, all three vectors belong to \(I\). This proves the assertion.

3. Modules

The notion of module over a Lie algebra is of extreme importance.

3.1. Two definitions and their equivalence. Let \(L\) be a Lie algebra over a field \(k\).

Definition 3.1.1. An \(L\)-module is a \(k\)-vector space \(M\) together with a bilinear map

\[ r : L \times M \longrightarrow M \]

satisfying the following property

\[ r([x, y], m) = r(x, r(y, m)) - r(y, r(x, m)). \]

Usually one writes simply \(xm\) instead of \(r(x, m)\). Then our axiom reads

\[ [x, y]m = xym - yxm. \]

To give another definition of \(L\)-module recall that for every vector space \(M\) the collection of endomorphisms \(\text{End}(V)\) admits an associative composition. The operation

\[ f, g \in \text{End}(M) \mapsto [f, g] = fg - gf \in \text{End}(M) \]

defines a Lie algebra structure on \(\text{End}(M)\). The Lie algebra of endomorphisms so obtained is denoted \(\mathfrak{gl}(M)\).

Definition 3.1.2. An \(L\)-module is a vector space \(M\) endowed with a Lie algebra homomorphism

\[ \rho : L \longrightarrow \mathfrak{gl}(M). \]
The proof of the equivalence of the above definitions is fairly standard.

Another name for an $L$-module is representation of $L$.

If $M$ is finite dimensional, we are talking about a finite dimensional representation.

3.2. Examples.

3.2.1. $L = k$. If $L$ is one-dimensional, say, $L = ke$, a module structure

$$\rho : L \longrightarrow \mathfrak{gl}(M)$$

is given by an endomorphism of $M$ (the image $\rho(e)$).

3.2.2. $L$ is commutative. A representation of $L$ is a Lie algebra homomorphism. If $L = \langle e_1, \ldots, e_n \rangle$, a homomorphism $r : L \rightarrow \mathfrak{gl}(M)$ is given by the images $r(e_i)$. Since $r$ is a homomorphism, $r(e_i)$ commute. Vice versa, any collection of $n$ commuting endomorphisms of $M$ define on $M$ a structure of $L$-module.

3.2.3. An $\mathfrak{sl}_2$-module is a vector space $M$ with three endomorphisms $E, F, H$ of $M$ satisfying the conditions

$$EF - FE = H; \ HE - EH = 2E; \ HF - FH = -2F.$$  

This means that an $\mathfrak{sl}_2$-module defines a representation of $\mathfrak{sl}_2$ in matrices. This is the explanation of the term representation.

3.2.4. Natural representation. By definition, Lie algebra $\mathfrak{gl}_n$ admits an $n$-dimensional representation. It is given by the identity map

$$\text{id} : \mathfrak{gl}_n \longrightarrow \mathfrak{gl}(k^n).$$

It is called the natural representation. Similarly, if $g \subseteq \mathfrak{gl}_n$ is a Lie subalgebra, we have a natural $n$-dimensional representation of $g$.

Examples include $g = \mathfrak{sl}_n, \ b_n, n_n$ and some other algebras.

3.3. Category of $L$-modules. Fix a Lie algebra $L$.

A linear map $f : M \longrightarrow N$ is an $L$-module homomorphism if

$$f(ax) = af(x)$$

for each $a \in L, \ x \in M$. Clearly, composition of homomorphisms is a homomorphism.

Lemma 3.3.1. Let $f : M \longrightarrow N$ be a bijective homomorphism of $L$-modules. Then $f^{-1} : N \longrightarrow M$ is also a homomorphism.
Proof. Straightforward. □

The notion of submodule and quotient module are defined in a standard way.

3.3.2. Direct sum. Given two \( L \)-modules \( M \) and \( N \), one defines an \( L \)-module structure on \( M \oplus N \) by the formula

\[
a(m, n) = (am, an).
\]

3.4. Representation theory. Representation theory of Lie algebras studies the category of modules over a Lie algebra. Here are the typical questions and the typical notions studied.

3.4.1. Classification. Description of all isomorphism classes of \( L \)-modules. Sometimes only modules satisfying special properties are considered (e.g., finite dimensional modules).

Today we will see that in the case \( L \) is one-dimensional we already know the answer from Linear Algebra.

3.4.2. Simple modules. A module is called simple if it does not admit non-trivial submodules. (A synonym: irreducible representation).

3.4.3. Semisimple modules. A module is called semisimple if it is isomorphic to a direct sum of simple modules (there are other equivalent definitions). Synonym: a completely reducible representation.

We will study soon the following result.

**Theorem 3.4.4.** All finite dimensional representations of \( \mathfrak{sl}_2 \) are completely reducible.

3.5. Representations of a one-dimensional Lie algebra.

3.5.1. Isomorphism classes. We are looking for isomorphism classes of \( n \)-dimensional representations. A map \( f : M \to N \) is a homomorphism of representations if \( f\alpha_M = \alpha_N f \). Since \( M = N = k^n \) as vector spaces, we deduce that endomorphisms \( \alpha_1 \) and \( \alpha_2 \) define isomorphic representations iff there exists an automorphism \( f \) such that

\[
\alpha_2 = f\alpha_1 f^{-1}.
\]

Thus the problem of classifications of \( n \)-dimensional representations is equivalent to that of classification of square matrices up to conjugation.

Theory of Jordan normal form answers this question in the case \( k \) is algebraically closed.
Let us recall the most important steps in this theory.

3.5.2. Recollections from Linear Algebra.

Let $f : V \rightarrow V$ be an endomorphism of a finite dimensional vector space over an algebraically closed field $k$. Recall that $\lambda \in k$ is an eigenvalue of $f$ iff $f - \lambda I$ is not invertible. The collection of eigenvalues of $f$ is therefore the set of roots of the characteristic polynomial of $f$ defined as

$$P_f(t) = \det(f - tI).$$

In what follows $S(f)$ will denote the set of eigenvalues of $f$ (the spectrum of $f$).

Let $\lambda \in S(f)$. A vector $v \in V$ is called an eigenvector corresponding to $\lambda$ if $f(v) = \lambda v$. Each eigenvalue admits a non-zero eigenvector. Furthermore, $v \in V$ is called generalized eigenvector if there exists $n \in \mathbb{N}$ such that $(f - \lambda I)^n v = 0$.

Fix $\lambda \in S(f)$. Let $V^\lambda$ denote the set of eigenvectors and $V_\lambda$ the set of generalized eigenvectors corresponding to $\lambda$. These are vector subspaces of $V$ and

$$V_\lambda \supseteq V^\lambda \neq 0.$$

The following are the main results of this study.

- $V = \bigoplus_{\lambda \in S(f)} V_\lambda$.
- $\dim V_\lambda$ equals the multiplicity of $\lambda$ in the characteristic polynomial of $f$.
- Each $V_\lambda$ is isomorphic to a direct sum of Jordan blocks having eigenvalue $\lambda$ (definition of J. b. see below).

Jordan block having eigenvalue $\lambda$ is the matrix $A = (a_{ij})$ defined by the formulas

$$a_{ij} = \begin{cases} 
\lambda & \text{if } i = j, \\
1 & \text{if } i = j - 1, \\
0 & \text{otherwise.}
\end{cases}$$

3.5.3. Simple modules. Direct consequence of the above: Any simple module has dimension 1; it is defined up to isomorphism by its (only) eigenvalue $\lambda$.

This module will be sometimes denoted by $k_\lambda$.

3.5.4. Semisimple modules.

Semisimple module is a direct sum of simple modules. Thus (for $k$ algebraically closed) $(V, f)$ is semisimple iff $f$ is diagonalizable.
3.5.5. Example of non-semisimple modules. It is given by the matrix
\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

3.6. Examples of representations.

3.6.1. Adjoint representation Let \( L \) be any Lie algebra. the map \( \text{ad} : L \rightarrow \mathfrak{gl}(L) \) defines a representation of \( L \) called adjoint representation.

Note that submodules of the adjoint representation are the ideals. Therefore, \( L \) is simple iff \( \dim L > 1 \) and the adjoint representation is irreducible.

3.6.2. ... and its restrictions If \( M \subseteq L \) is a Lie subalgebra, one can consider \( L \) as a \( M \)-module restricting the adjoint representation of \( L \) on \( M \).

Consider, for example, \( L = \mathfrak{sl}_2 \) and \( M = \langle h \rangle \). Algebra \( M \) is one-dimensional and \( L \) is an \( M \)-module. It is semisimple with eigenvalues \(-2, 0, 2\).

If we take another \( M \), say, \( \langle e \rangle \), the picture changes: all eigenvalues are zero and the \( M \)-module \( L \) is not semisimple.

3.7. One-dimensional representations. Let \( L \) be a Lie algebra and let \( \rho : L \rightarrow \mathfrak{gl}(V) \) is a one-dimensional representation. The algebra \( \mathfrak{gl}(V) \) is one-dimensional and therefore commutative in this case. Thus,
\[
\rho[x, y] = [\rho(x), \rho(y)] = 0.
\]
In particular, for \( L = \mathfrak{sl}_2 \) one gets \( \rho = 0 \). This proves \( \mathfrak{sl}_2 \) does not admit non-trivial one-dimensional representations.
Problem assignment, 2

1. Let $\mathfrak{h} \subseteq \mathfrak{gl}_n$ be the set of diagonal matrices. Check that $\mathfrak{h}$ is a commutative Lie subalgebra. Check that $\mathfrak{gl}_n$ as the $\mathfrak{h}$-module (with respect to adjoint action) is a sum of one-dimensional representations.

2. Prove that the adjoint representation of $\mathfrak{gl}_2$ is isomorphic to a direct sum of a three-dimensional and one-dimensional representations.

3. Prove that the adjoint representation of $\mathfrak{sl}_2$ is not isomorphic to the sum of the natural representation with the trivial representation.