INTRODUCTION TO LIE ALGEBRAS. LECTURE 2.

2. More examples. Ideals. Direct products.

2.1. **More examples.**

2.1.1. Let $k=\mathbb{R}$, $L=\mathbb{R}^3$. Define $[x,y]=x\times y$ — the cross-product. Recall that the latter is defined by the formulas

$$e_1\times e_2=e_3,\ e_2\times e_3=e_1,\ e_3\times e_1=e_2.$$

2.1.2. It is convenient to choose a basis of $\mathfrak{sl}_2$ as follows.

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix}, f=\begin{pmatrix}0&0\\1&0\end{pmatrix}, h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{1}$$

Than the bracket in $\mathfrak{sl}_2$ is given by the formulas

$$[ef]=h,\ [he]=2e,\ [hf]=-2f.$$

There are much more Lie subalgebras of $\mathfrak{gl}_n$. For instance,

2.1.3. $\mathfrak{b}_n=\{a\in\mathfrak{sl}_n|a_{ij}=0 \text{ for } i>j\}$ — upper-triangular matrices of trace zero.

This algebra has dimension $\frac{n(n+1)}{2}-1$.

2.1.4. $\mathfrak{n}_n=\{a\in\mathfrak{gl}_n|a_{ij}=0 \text{ for } i\geq j\}$ — strictly upper-triangular matrices.

This algebra has dimension $\frac{n(n-1)}{2}$.

2.2. **Direct product.** Let L and M be two Lie algebras. Define their direct product $L\times M$ as follows. As a set, this is the Cartesian product of L and M. The operations (multiplication by a scalar, sum and bracket) are defined componentwise. For instance,

$$[(x,y),(x',y')]=([x,x'],[y,y']).$$

2.2.1. Example. If L is commutative of dimension n and L' is commutative of dimension n' then $L\times L'$ is commutative of dimension $n+n'$.

2.2.2. Example. The Lie algebra $\mathfrak{gl}_n$ is isomorphic to the direct product $\mathfrak{sl}_n \times k$ (k is the one-dimensional algebra). The map from the direct product to $\mathfrak{gl}_n$ is given by the formula $(a, \lambda) \mapsto a + \lambda I$ where I is the identity matrix.

2.3. **Some calculations.**

2.3.1. Ideals in $\mathfrak{n}_3$. Quotients Choose a basis for $\mathfrak{n}_3$ as follows.

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Multiplication is given by

$$[x, y] = z, \ [x, z] = [y, z] = 0.$$

Let us describe all ideals in $\mathfrak{n}_3$. If I is a non-zero ideal, let $t = ax + by + cz \in I$ be non-zero. Then $[x, t] = bz, \ [y, t] = az, \ [z, t] = 0$. Thus, if $a \neq 0$ or $b \neq 0$ then $z \in I$. If $a = b = 0$ then once more $z \in I$. Therefore, z belongs to any non-zero ideal. Thus, the only one-dimensional ideal is $\langle z \rangle$; any two-dimensional ideal has form $\langle z, ax+by \rangle$. It is easy to see that all these formulas do define ideals.

The quotient algebra $\mathfrak{n}_3/\langle z \rangle$ has a basis $\overline{x}, \ \overline{y}$ with the bracket $[\overline{x}, \overline{y}] = \overline{z} = 0$. Thus, the quotient is a commutative two-dimensional algebra.

2.4. **Adjoint action.** Let L be a Lie algebra, $x \in L$. Define a linear transformation

$$\mathrm{ad}_x : L \longrightarrow L$$

by the formula $\mathrm{ad}_x(y) = [x, y]$.

Lemma 2.4.1. ad_x *is a linear transformation.*

In fact, this follows from the linearity of $[,]$ in the second argument.

Lemma 2.4.2. ad_x *is a derivation.*

In fact,

$$\mathrm{ad}_x[y, z] = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]$$

— this follows from the Jacobi identity.

Assembling together ad_x for all $x \in L$ we get therefore a map

$$\mathrm{ad} : L \longrightarrow \mathrm{Der}(L).$$

Lemma 2.4.3. *The map* $\mathrm{ad} : L \to \mathrm{Der}(L)$ *is a homomorphism of Lie algebras.*

One has to check that

$$\mathrm{ad}_{[x,y]} = \mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x.$$

This also follows from the Jacobi identity.

Definition 2.4.4. Center of a Lie algebra L is defined by the formula

$$Z(L) = \{x \in L | \forall y \in L \quad [x, y] = 0\}.$$

By definition of ad, one has $Z(L) = \mathrm{Ker}(\mathrm{ad})$.
For example, $Z(\mathfrak{n}_3) = \langle z \rangle$.

2.5. **Simplicity of $\mathfrak{sl}_2$.**

Definition 2.5.1. A Lie algebra L is *simple* if it is not one-dimensional and if it has no non-trivial ideals.

Our aim is to prove the following

Theorem 2.5.2. *$\mathfrak{sl}_2$ is simple.*

2.5.3. Some linear algebra

Let V be a finite dimensional vector space and $f \in \mathrm{End}(V)$.

Endomorphism f is called *diagonalizable* if V has a basis of eigenvectors.

If f is diagonalizable then $V = \oplus_{\lambda \in S} V_\lambda$ where $V_\lambda = \{x \in V | f(x) = \lambda x\}$ is the eigenspace corresponding to the eigenvalue λ and S is the set of eigenvalues of f (*spectrum of f*).

Lemma 2.5.4. *Let $f \in \mathrm{End}(V)$ be diagonalizable and let W be a f-invariant subspace of V (i.e., $f(W) \subseteq W$). Then*

$$W = \oplus_{\lambda \in S} W_\lambda \text{ where } W_\lambda = W \cap V_\lambda.$$

Proof. We have to prove that if $x \in W$ and if $x = \sum x_\lambda$ with $x_\lambda \in V_\lambda$ then $x_\lambda \in W$.

In fact, $W \ni f^k(x) = \sum f^k(x_\lambda) = \sum \lambda^k x_\lambda$ for each k.

Let $T = \{\lambda \in S | x_\lambda \neq 0\}$ and let $t = |T|$. This is the number of non-zero summands in the decomposition of x. The vectors $x, f(x), \ldots, f^{t-1}(x)$ can be expressed as linear combinations of t linearly independent vectors x_λ. The transition matrix has form

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_t \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{t-1} & \lambda_2^{t-1} & \ldots & \lambda_t^{t-1} \end{pmatrix}.$$

This is Vandermonde matrix. Its determinant is

$$\prod_{i<j}(\lambda_i - \lambda_j) \neq 0.$$

This proves that x_λ can be expressed through $f^k(x)$ and therefore belong to W. □

2.5.5. Proof of Theorem 2.5.2.

Consider endomorphism ad_h of $\mathfrak{sl}_2$. It is diagonalizable with eigenvalues $-2,\ 0,\ 2$. Any ideal I is invariant with respect to ad_h. Therefore, I should be spanned by a subset of f, h, g. It is easy to check that this is impossible for any nonempty proper subset of generators.

Problem assignment, 1

1. Derivations.

(a) Let $A = k[t]$ be the algebra of polynomials. Fix $f \in A$ and define $d : A \longrightarrow A$ by the formula

$$d(g) = fg'.$$

Prove d is a derivation.

(b) The same A, f and $d : A \longrightarrow A$ is given by the formula

$$d(g) = fg' + g.$$

is this a derivation?

(c) Prove that any derivation of A is of form described in (a). *Hint*: consider the value of d on $1,\ t \in A$.

2. Find all ideals and all quotient algebras of the algebra

$$L = \{a \in \mathfrak{gl}_2 | a_{21} = 0\}.$$

Prove that L is isomorphic to the direct product of k (one-dimensional algebra) and $\mathfrak{b}_2$.

3. (bonus). Let $L = \mathbb{R}^3$ with cross-product as a bracket. Prove that $L_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Here $L_{\mathbb{C}}$ denotes the Lie algebra over $\mathbb{C}$ having the base $e_1,\ e_2,\ e_3$ with the bracket given by the formulas

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2.$$