INTRODUCTION TO LIE ALGEBRAS.
LECTURE 13.

13. Study of semisimple Lie algebras

We assume here that the base field is algebraically closed.

Let L be a semisimple Lie algebra.

13.1. **Maximal toral subalgebras and roots.** If L does not contain ad-semisimple elements, all its elements are ad-nilpotent by Jordan-Chevalley theorem and L is nilpotent by Engel theorem.

Since this is not the case, L contains semisimple elements.

A Lie subalgebra of L is called *toral* if it consists of semisimple elements. The above reasoning implies that any semisimple Lie algebra contains nontrivial toral subalgebras.

One has

Lemma 13.1.1. *(For k algebraically closed) Any toral subalgebra is commutative.*

Proof. Let T be a toral subalgebra and $x \in T$. Observe that $(\mathrm{ad}x)T \subset T$ and denote the restriction of $\mathrm{ad}x$ to T by $\mathrm{ad}_T x$. We need to show that $\mathrm{ad}_T x = 0$. Obviously $\mathrm{ad}_T x$ is semisimple.

Let $y_0 \in T$ be an eigenvector of $\mathrm{ad}_T x$ that is $(\mathrm{ad}x)y_0 = cy_0$ for some $c \in k$. We have to prove $c = 0$.

Since $\mathrm{ad}_T y_0$ is semisimple, T admits a basis $y_0, y_1, \ldots, y_r$ of eigenvectors of $\mathrm{ad}_T y$. One has $(\mathrm{ad}y_0)T \subset \{y_1, \ldots, y_r\}$ because $(\mathrm{ad}y_0)y_0 = 0$. Now the equality $(\mathrm{ad}x)y_0 = -(\mathrm{ad}y_0)x$ gives $c = 0$. □

Example 13.1.2. In the case $k = \mathbb{R}$, a linear operator $\psi \in \mathrm{End}(V)$ is called semisimple if it complexification $\psi_{\mathbb{C}} \in \mathrm{End}(V \otimes_{\mathbb{R}} \mathbb{C})$ is diagonalizable.

Let $L = \mathrm{Vect}$ be a three dimensional Lie algebra over $\mathbb{R}$ with respect to vector product: $[v, w] := v \times w$. This algebra is simple (see Lecture 2) since an ideal containing a given non-zero element v contains also all vectors which are orthogonal to v and so coincides with Vect. Any element $x \in L$ is ad-semisimple since all eigenvalues of $\mathrm{ad}x$ are distinct $(0, i, -i)$. However L is not commutative. Thus, the condition $k = \overline{k}$ is very important.

13.1.3. Fix a maximal toral subalgebra H in L. Since H consists of commuting semisimple elements, L has a basis for which all matrices $\mathrm{ad}x : x \in H$ are diagonal. Let v be a non-zero common eigenvector of the linear operators $\mathrm{ad}x : x \in H$. An element μ of the dual space H^* is called *the weight of* v if $(\mathrm{ad}x)(v) = \mu(x)v$ for all $x \in H$. One has

$$L = \oplus_{\mu\in\Delta} L_\mu \oplus L_0$$

where

$$L_\mu := \{x \in L |\, (\mathrm{ad}h)(x) = \mu(h)x,\ \forall h \in H\}, \quad \Delta := \{\mu \in H^*\backslash\{0\}|\, L_\mu \neq 0\}.$$

The elements of Δ are called *the roots* of L.

Denote by K the Killing form of L.

Lemma 13.1.4. *i)* $[L_\mu, L_\nu] \subset L_{\mu+\nu}$,
ii) for $\alpha \in \Delta$ *all elements of* L_α *are nilpotent,*
iii) if $\alpha + \beta \neq 0$ *then* $\forall x \in L_\alpha, y \in L_\beta$ *one has* $K(x, y) = 0$,
iv) the restriction of K *to* L_0 *is non-degenerate.*

Proof. (i) follows from the Jacobi identity. (ii) follows from (i) and the fact that the set of weights of L with respect to H is a finite set (it is equal to $\Delta \cup \{0\}$). (iii) follows from ad-invariance of K. Finally, (iv) is an immediate consequence of non-degeneracy of K and (iii). □

Corollary 13.1.5. *For any* $\alpha \in \Delta$ *one has* $\dim L_\alpha = \dim L_{-\alpha}$.

Proof. Combining the non-degeneracy of K and (iii), we conclude that for any non-zero $x \in L_\alpha$ there exists $y \in L_{-\alpha}$ such that $K(x, y) \neq 0$. Therefore the formula $x \mapsto f_x :\ f_x(y) := K(x, y),\ \ \forall y \in L_{-\alpha}$ defines an embedding $L_\alpha \to L^*_{-\alpha}$. In particular, $\dim L_\alpha \leq \dim L_{-\alpha}$. Applying the last inequality for $\alpha' := -\alpha$, one concludes $\dim L_\alpha = \dim L_{-\alpha}$. □

Since any toral subalgebra is commutative, $L_0 \supseteq H$.

Proposition 13.1.6. *One has* $L_0 = H$ *that is a maximal toral subalgebra coincides with its centralizer.*

Proof. Note that by definition $L_0 = \{x \in L|[h, x] = 0\ \forall\ h \in H\}$ is the centralizer of H.

Step 1. Let $x = s + n$ be the Jordan-Chevalley decomposition of an element $x \in L_0$. Then s and n are in L_0.

In effect, $L_0 = \{x \in L|\mathrm{ad}_x(H) = 0\}$. Thus, by the property of Jordan decomposition, both ad_s and ad_n satisfy the same property.

Step 2. If $x \in L_0$ is semisimple then $x \in H$. This follows from maximality of H: x commutes with H, therefore $H \oplus k \cdot x$ consists of semisimple elements.

Step 3. The restriction of K to H is nondegenerate. Let $h \in H$ and let $K(h, H) = 0$. We have to prove that $h = 0$. We will first check that $K(h, L_0) = 0$ and then we deduce $h = 0$ from the nondegeneracy of $K|_{L_0}$. For a general $x \in L_0$ one has $x = s + n$ where $s \in H$ by Step 2. Obviously, $\mathrm{Tr}(\mathrm{ad}_h \cdot \mathrm{ad}_n) = 0$ since h and n commute and n is nilpotent. Thus, $K(h, x) = K(h, s) + K(h, n) = 0$ for all x.

Step 4. L_0 is nilpotent. By Engel theorem it suffices to check that ad_x is nilpotent for all $x \in L_0$. This is true for ad_s since $s \in H$ so $\mathrm{ad}_s = 0$ and this is true for ad_n. Therefore, this is true for ad_x.

Step 5. $H \cap [L_0, L_0] = 0$. In effect, $K(H, [L_0, L_0]) = 0$ since K is invariant. If $h \in H \cap [L_0, L_0]$, $K(h, H)$ would vanish, therefore, h would vanish since $K|_H$ is nondegenerate.

Step 6. L_0 is commutative. Otherwise there would exist $x \in [L_0, L_0] \cap Z(L_0)$. Let $x = n + s$ be the Jordan decomposition. One has $n \neq 0$ since otherwise x would be semisimple, which is impossible by Steps 2 and 5. The nilpotent element n belongs to L_0 and therefore to the center of L_0 by the properties of the Jordan decomposition. Then $K(n, x) = \mathrm{Tr}(ad_n \cdot \mathrm{ad}_x) = 0$ for all $x \in L_0$ which contradicts to the nondegeneracy of $K|_{L_0}$.

Step 7. Finally, assume $L_0 \neq H$. Then there exists a nonzero nilpotent element $x \in L_0$ by Steps 1,2. Then $K(x, y) = \mathrm{Tr}(\mathrm{ad}_x \cdot \mathrm{ad}_y) = 0$ for all $y \in L_0$ since ad_x is nilpotent and commutes with ad_y. This contradicts nondegeneracy of $K|_{L_0}$.

□

13.2. **Root space decomposition.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Recall that the Killing form

$$K(x, y) := \mathrm{Tr}(\mathrm{ad}x \cdot \mathrm{ad}y)$$

is a non-degenerate invariant bilinear form on $\mathfrak{g}$.

13.2.1. Recall that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called toral if it consists of semisimple elements:

$$\forall x \in \mathfrak{h} \quad \mathrm{ad}x : \mathfrak{g} \to \mathfrak{g} \text{ is a semsimple linear operator.}$$

We have shown that any toral subalgebra is commutative and that a maximal toral subalgebra coincides with its centralizer. Moreover the restriction of the Killing form K to a maximal toral subalgebra is non-degenerate.

Note the following fact without proof: *All maximal toral subalgebras are conjugate: if $\mathfrak{h}, \mathfrak{h}'$ are maximal toral subalgebras of $\mathfrak{g}$ then there exists an automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi(\mathfrak{h}) = \mathfrak{h}'$.*

13.2.2. Fix a maximal toral subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and denote by Δ the set of roots:

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha).$$

Recall that the restriction of K to $\mathfrak{h}$ is nondegenerate. This means that for each $\alpha \in \mathfrak{h}^*$ there exists a unique element $t_\alpha \in \mathfrak{h}$ such that $K(t_\alpha, h) = \alpha(h)$ for $h \in \mathfrak{h}$.

The set of roots Δ satisfies the following properties.

Proposition 13.2.3. 1. *The set $\Delta \subset \mathfrak{h}^*$ spans $\mathfrak{h}^*$.*
2. *$\alpha \in \Delta$ iff $-\alpha \in \Delta$.*
3. *If $\alpha \in \Delta$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional spanned by t_α.*
4. *$\alpha(t_\alpha) = K(t_\alpha, t_\alpha) \neq 0$ for $\alpha \in \Delta$.*
5. *Let $\alpha \in \Delta$, $0 \neq x \in \mathfrak{g}_\alpha$. There exists an element $y \in \mathfrak{g}_{-\alpha}$ such the triple $(x, y, h = [x, y])$ generate a subalgebra isomorphic to $\mathfrak{sl}_2$ and $h = \frac{2t_\alpha}{K(t_\alpha,t_\alpha)}$.*

Proof. 1. If Δ does not span $\mathfrak{g}^*$, there exists a non-zero element $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta$. This implies that h commutes with elements of $\mathfrak{g}_\alpha$ for all $\alpha \in \Delta$. Then h is central. Since the center of $\mathfrak{g}$ is trivial, this leads to a contradiction.

2. Since $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\beta \neq -\alpha$ and since K is nondegenerate, Δ is symmetric with respect to $\alpha \mapsto -\alpha$.

3. Let $x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha}$ and let $h \in \mathfrak{h}$. One has

$$K(h, [x, y]) = K([h, x], y) = \alpha(h)K(x, y) = K(t_\alpha, h)K(x, y) = K(K(x, y)t_\alpha, h).$$

This implies that $\mathfrak{h}$ is orthogonal to $[x, y] - K(x, y)t_\alpha$ which in turn yields

$$[x, y] = K(x, y)t_\alpha.$$

4. Assume $\alpha(t_\alpha) = K(t_\alpha, t_\alpha) = 0$. Choose $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) = 1$ so that $[x, y] = t_\alpha$ and $[t_\alpha, x] = [t_\alpha, y] = 0$. One can consider $\mathrm{Span}\{x, y, t_\alpha\}$ as a solvable subalgebra of $\mathfrak{gl}(\mathfrak{g})$; thus, its commutator containing t_α is nilpotent; since it is in $\mathfrak{h}$, it is as well semisimple, that is $\mathrm{ad}_{t_\alpha} = 0$ or t_α is in the center of $\mathfrak{g}$.

5. If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ so that $K(x, y) = c$, we have

$$[x, y] = ct_\alpha,\ [ct_\alpha, x] = c\alpha(t_\alpha)x,\ [ct_\alpha, y] = -c\alpha(t_\alpha)y.$$

Thus, if we set $c = \frac{2}{K(t_\alpha,t_\alpha)}$, we get the required $\mathfrak{sl}_2$-triple. □

13.2.4.

Let S_α be the Lie subalgebra of $\mathfrak{g}$ spanned by an element $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ such that $[x, y] = h_\alpha := \frac{2t_\alpha}{K(t_\alpha,t_\alpha)}$ and h_α. Note that we cannot, at the moment, claim that S_α so defined is unique.

The map $S_\alpha \longrightarrow \mathfrak{g}$ is a map of Lie algebras and S_α acts on $\mathfrak{g}$ via the adjoint action. The elements of $\mathfrak{g}_\beta$ have weight $\beta(h_\alpha) = 2\frac{K(t_\alpha,t_\beta)}{K(t_\alpha,t_\alpha)}$. In particular, $\alpha(h_\alpha) = 2$.

We know that $\mathfrak{g}$ decomposes, as S_α-module, into a direct sum of irreducible modules whose structure we can read off the occuring weights.

Consider the vector subspace M of $\mathfrak{g}$ of the form

$$M = \mathfrak{h} \oplus_{c\in k^*} \mathfrak{g}_{c\alpha}.$$

This is a S_α-submodule since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. Since the weights of M with respect to S_α are even ($h_\alpha(k\alpha) = 2k!$), M is a sum of simple S_α-modules having even highest weight. Each such simple module has precisely one-dimensional zero weight space, so the number of such components is $\dim \mathfrak{h}$. But $\mathfrak{h} = k \cdot h_\alpha \oplus \mathfrak{h}'$ where $\mathfrak{h}' = \{h \in \mathfrak{h} | \alpha(h) = 0\}$, and each element of $\mathfrak{h}'$ generates a one-dimensional S_α-submodule. This allows us to completely determine the decomposition of M as a S_α-module: $M = S_\alpha \oplus \mathfrak{h}'$.

This immediately implies the following property of the root systems and corresponding Lie algebras:

Proposition.

1. *Let α and $c\alpha$ belong to a root system Δ. Then $c = \pm 1$.*
2. *For each $\alpha \in \Delta$ one has* $\dim\ \mathfrak{g}_\alpha = 1$.

13.2.5. Let us now fix $\beta \in \Delta$ and put

$$M = \oplus_{c\in\mathbb{Z}}\ \mathfrak{g}_{\beta+c\alpha}.$$

Once more, M is an S_α-submodule of $\mathfrak{g}$. For each $c \in \mathbb{Z}$ such that $\beta + c\alpha \in \Delta$ the space $\mathfrak{g}_{\beta+c\alpha}$ is a one-dimensional subspace of M of weight $(\beta + c\alpha)(h_\alpha) = \beta(h_\alpha) + 2c$. These numbers have to be integers since the module M is finite dimensional. Thus, $\beta(h_\alpha) \in \mathbb{Z}$ for all α and β.

Since there are no multiplicities, M is simple. This has a far-reaching implication.

Proposition 13.2.6. *Let $I = \{c \in \mathbb{Z} | \beta + c\alpha \in \Delta\}$ be nonempty. Then I has form $[-m, n]$ where $m, n \in \mathbb{Z}$ and $n - m = -\beta(h_\alpha)$. Moreover, if α, β and $\alpha + \beta$ are in Δ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

The segment I defined above is called *the α-string through β*.

We are now ready to prove the main property of root systems.

For $\alpha, \beta \in \Delta$ define $(\alpha, \beta) = K(t_\alpha, t_\beta)$. This is the scalar product on $\mathfrak{h}^*$ obtained from K via the isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ induced (once more) by K.

For $\alpha \in \Delta$ denote by s_α the reflection of $\mathfrak{h}^*$ with respect to the hyperplane orthogonal to α. The formula for s_α reads

$$s_\alpha(x) = x - x(h_\alpha)\alpha = x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha.$$

In particular, $s_\alpha(\alpha) = -\alpha$ and for $\beta \in \Delta$ one has $s_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha$. Note tat $\beta(h_\alpha)$ is integer and always belongs to the segment I describing the α-string passing through β. In effect, if $I = [-m, n]$, one has $n - m = -\beta(h_\alpha)$ so $-m \leq -\beta(h_\alpha) \leq n$ as required.

13.2.7. The habe proven that the set $\Delta \in \mathfrak{h}^*$ spans the whole $\mathfrak{h}^*$ and is symmetric with respect to any hyperplane orthogonal to $\alpha \in \Delta$.

It is easy to check that the base field k plays no role in the story as the vectors Δ lie in a certain real (or even rational) vector space of the same dimension. This observation is important since it allows usage of positively definite forms and this turn out to help in classification.

Definition 13.2.8. A finite subset Δ of a real vector space V is called *a root system* if it spans V and if for any $\alpha \in \Delta$ the reflection with respect to a hyperplane orthogonal to α, carries Δ to Δ.

13.3. **Example: $\mathfrak{sl}(n)$.** To describe a root system of $\mathfrak{sl}(n)$, it is convinient to start from $\mathfrak{gl}(n)$. The latter is reductive: $\mathfrak{gl}(n) = \mathfrak{sl}(n) \times \mathbb{C}z$ and we can define a maximal toral subalgebra in the same manner; it is easy to check that such a subalgebra is of the form $\mathfrak{h}' = \mathfrak{h} \times \mathbb{C}z$ where $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{sl}(n)$.

The natural choice for $\mathfrak{h}'$ is the set of diagonal matrices

$$\mathfrak{h}' := \{\sum a_i E_{i,i},\ a_i \in \mathbb{C}\}.$$

The natural choice for $\mathfrak{h}$ is the set of traceless diagonal matrices

$$\mathfrak{h} := \{\sum a_i E_{i,i} |\ \sum a_i = 0\}.$$

Let $\{\varepsilon_i\}_{i=1}^n$ be a basis of $(\mathfrak{h}')^*$ which is dual to the basis $E_{i,i}$. Since $\mathfrak{h}$ is a subspace of $\mathfrak{h}'$, the dual space $\mathfrak{h}^*$ may be naturally viewed as a factor space

$$\mathfrak{h}^* = span\{\varepsilon_i\}_{i=1}^n / \sum_{i=1}^n \varepsilon_i.$$

In this notation, one has

$$\Delta := \{\varepsilon_i - \varepsilon_j\}_{i \neq j}.$$

The $\mathfrak{sl}(2)$ triple corresponing to $(\varepsilon_i - \varepsilon_j)$ is

$$E_{i,j},\ h_{\varepsilon_i-\varepsilon_j} := E_{i,i} - E_{j,j},\ E_{j,i}.$$

The $(\varepsilon_i - \varepsilon_j)$-strings take form

$$\varepsilon_k - \varepsilon_i; \varepsilon_k - \varepsilon_j.$$

Therefore the integers $\beta(h_\alpha)$ are

$$(\varepsilon_k - \varepsilon_i)(h_{\varepsilon_i-\varepsilon_j}) = -1$$
$$(\varepsilon_i - \varepsilon_j)(h_{\varepsilon_i-\varepsilon_j}) = 2$$
$$(\varepsilon_k - \varepsilon_j)(h_{\varepsilon_i-\varepsilon_j}) = 1$$

where $k \neq i, j$ and zero for all remaining cases.

13.4. **Example: $\mathfrak{sp}(n)$ $(n = 2l)$.** This is a Lie subalgebra of $\mathfrak{gl}(n)$ which consists of all matrices T satisfying $TA + AT^t = 0$ where

$$A = \begin{pmatrix} 0 & | & I_l \\ -- & - & -- \\ -I_l & | & 0 \end{pmatrix}$$

and I_l stands for the identity $l \times l$ matrix.

The matrices in $\mathfrak{sp}(n)$ are of the form

$$T_{x,y,z} := \begin{pmatrix} x & | & y \\ -- & - & -- \\ z & | & -x^t \end{pmatrix}$$

where x, y, x are $l \times l$ matrices and y, z are symmetric: $y^t = y, z^t = z$. We have a natural embedding $\mathfrak{gl}(l) \subset \mathfrak{sp}(2l)$ $(x \mapsto T_{x,0,0})$.

Let $\mathfrak{h}$ be the set of diagonal matrices

$$\mathfrak{h} := \{\sum_{i=1}^{l} a_i(E_{i,i} - E_{l+i,l+i})\}$$

(it corresponds to $\mathfrak{h}'$ in the previous example). Retain notation for the dual basis.

Obviously this is a commutative Lie subalgebra. To check that $\mathfrak{h}$ is a maximal toral subalgebra, let us show that it consists of ad-semisimple elements and coincides with the own centralizer (so it is a maximal commutative subalgebra).

Indeed, if $x = E_{i,j}, y = z = 0$ then $T_{x,0,0}$ has weight $\varepsilon_i - \varepsilon_j$ (in this example i, j are assumed to be distinct integeres from 1 to l).

If $x = 0, y = E_{i,i}$ then $T_{0,y,0} = E_{i,l+i}$ and has weight $2\varepsilon_i$.

Similarly, if $x = 0, z = E_{i,i}$ then $T_{0,0,z} = E_{l+i,i}$ and has weight $-2\varepsilon_i$.

If $x = 0, y = E_{i,j} + E_{j,i}$ then $T_{0,y,0}$ has weight $\varepsilon_i + \varepsilon_j$.

If $x = 0, z = E_{i,j} + E_{j,i}$ then $T_{0,0,z}$ has weight $-(\varepsilon_i + \varepsilon_j)$.

Thus $\mathfrak{h}$ is a maximal toral subalgebra and

$$\Delta := \{\varepsilon_i - \varepsilon_j; \pm(\varepsilon_i + \varepsilon_j); \pm 2\varepsilon_i\}.$$

We have the following $\mathfrak{sl}(2)$-triples. The $\mathfrak{sl}(2)$ triple corresponing to $(\varepsilon_i - \varepsilon_j)$ comes from $\mathfrak{gl}(l) \subset \mathfrak{sp}(2l)$ and takes form

$$E_{i,j} - E_{l+j,l+i},\ h_{\varepsilon_i - \varepsilon_j} := (E_{i,i} - E_{j,j}) - (E_{l+i,l+i} - E_{l+j,l+j}),\ E_{j,i} - E_{l+i,l+j}.$$

The $\mathfrak{sl}(2)$ triple corresponing to $2\varepsilon_i$ is

$$E_{i,l+i}, h_{\varepsilon_i + \varepsilon_j} := E_{i,i} - E_{l+i,l+i}, E_{l+i,i}$$

Finally, the $\mathfrak{sl}(2)$ triple corresponing to $(\varepsilon_i + \varepsilon_j)$ is

$$E_{i,l+j} + E_{j,l+i},\ h_{\varepsilon_i + \varepsilon_j} := (E_{i,i} + E_{j,j}) - (E_{l+i,l+i} + E_{l+j,l+j}),\ E_{l+j,i} + E_{l+i,j}.$$

Examples of strings:

$$2\varepsilon_2 - \text{string}: \quad \varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2;$$
$$(\varepsilon_1 - \varepsilon_2) - \text{string}: \quad 2\varepsilon_2, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1;$$
$$(\varepsilon_1 + \varepsilon_2) - \text{string}: \quad -2\varepsilon_2, \varepsilon_1 - \varepsilon_2, 2\varepsilon_1.$$

The numbers $< \alpha, \beta >= \alpha(h_\beta)$:

$$< \varepsilon_1 - \varepsilon_2, 2\varepsilon_2 >= -1, \quad < \varepsilon_1 + \varepsilon_2, 2\varepsilon_2 >= 1$$
$$< 2\varepsilon_2, \varepsilon_1 - \varepsilon_2 >= -2, \quad < 2\varepsilon_2, \varepsilon_1 + \varepsilon_2 >= 2$$