INTRODUCTION TO LIE ALGEBRAS.
LECTURE 12.

12. Levi theorem. Reductive algebras

12.1. **Levi theorem.** Recall that a Lie algebra $\mathfrak{g}$ is semisimple if it has no solvable ideals. Recall as well that any Lie algebra has a maximal solvable ideal R (called the radical). Thus, the quotient $\mathfrak{g}/R$ is semisimple (prove this!).

We will prove today that (to some extent) $\mathfrak{g}$ is defined by R and $\mathfrak{g}/R$. More precisely, we will prove the following Levy theorem.

Theorem 12.1.1. *Let $f : \mathfrak{g} \to L$ be a surjective Lie algebra homomorphism with L semisimple. Then there exists a Lie algebra homomorphism $s : L \to \mathfrak{g}$ splitting f, that is satisfying $f \circ s = \mathrm{id}_L$.*

Let $I = \mathrm{Ker}(f)$. One has obviously $\mathfrak{g} = I \oplus s(L)$ as vector spaces; I is an ideal in $\mathfrak{g}$ and $s(L)$ is a Lie subalgebra. Since I is an ideal, $[s(x), y] \in I$ for all $x \in L$ and $y \in I$, so the formula

$$x(y) := [s(x), y]$$

define as action of L on I. Moreover, $\mathrm{ad}_x|_I$ is obviously a derivation of I for all $x \in L$. Vice versa, given a Lie algebra homomorphism $a : L \to \mathrm{Der}(I)$, one reconstructs the Lie algebra structure on $g := I \oplus I$ by the formula

$$[(x, y), (x', y')] = ([x, x'], [y, y'] + a(x)(y') - a(x')(y)).$$

The above construction is called *semidirect product* of L and I. In case $a = 0$ this is just a direct product of Lie algebras.

Corollary 12.1.2. *Any Lie algebra is a semidirect product of its radical with a semisimple Lie algebra.*

An image $s(L)$ of the quotient L of a Lie algebra by the radical is called *a Levi factor*. Note that Levi factor is not unique as there is a freedom in the choice of the section s.

12.1.3. Proof of Levy theorem

Step 1. Reduce the problem to the case I is a simple $\mathfrak{g}$-module. This is done by induction in dimension of I.

If I is not commutative, I contains an ideal J of smaller (nonzero) dimension. By induction, the projection $\mathfrak{g}/J \to L$ splits, so that there exists $t : L \to \mathfrak{g}/J$ with $\mathfrak{g}/J = t(L) \oplus I/J$. Let $\mathfrak{h}$ be the preimage of $t(L)$ in $\mathfrak{g}$. This is a subalgebra in $\mathfrak{g}$ having a projection $\mathfrak{h} \to L$ with the kernel J. Once more by induction there is a splitting of this projection.

Step 2. Let R be the radical of $\mathfrak{g}$. Since $f(R)$ is solvable ideal in L (f is surjective so the image of an ideal is ideal), $f(R) = 0$ so $R \subset I$. Since I has no submodules, $R = 0$ or $R = I$. If $R = 0$, $\mathfrak{g}$ is semisimple and any ideal in it is a direct factor. Thus, we can assume $R = I$. Therefore $[I, I] = 0$ since $I \neq [I, I]$ and I has no submodules. Therefore, I is commutative.

Step 3. Consider the case $\mathfrak{g}$ acts trivially on I. Then I is the center of $\mathfrak{g}$ and belongs to the kerner of the adjoint action (of $\mathfrak{g}$ on $\mathfrak{g}$). This means that $\mathfrak{g}$ is $\mathfrak{g}/I = L$-module and therefore by complete reducibility I has a direct complement which is automatically an ideal isomorphic to L.

Step 4. From now on we asume I is simple nontrivial $\mathfrak{g}$-module and $[I, I] = 0$.

We have to find a Lie subalgebra K in $\mathfrak{g}$ isomorphic to L and complement to I: $I \oplus K = \mathfrak{g}$. The idea of doing so is the following. We will present a $\mathfrak{g}$-module W and an element $w \in W$ such that K will appear as the stabilizer of w, that is $K = \{x \in \mathfrak{g} | xw = 0\}$.

Then, in order to have $K \oplus I = \mathfrak{g}$, we will require that the linear map

$$a : \mathfrak{g} \to W, \quad a(x) = xw$$

satisfy the following properties.

- $a|_I$ is injective.
- $a(\mathfrak{g}) = a(I)$.

Stabilizer of any element is always a Lie subalgebra; the isomorphism $K \to L$ will immediately follow from the isomorphism theorem.

Let us construct W and w. Put $W = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ with the action induced by the adjoint action on $\mathfrak{g}$. Define the following subspaces in W.

$$P = \{\mathrm{ad}_{\mathfrak{g}}(a),\ a \in I\}.$$

$$Q = \{\phi \in W | \phi(\mathfrak{g}) \subset I \& \phi(I) = 0\}.$$

$$R = \{\phi \in W | \phi(\mathfrak{g}) \subset I \ \&\ \phi|_I = \lambda \cdot \mathrm{id}_I, \lambda \in k\}.$$

The subspaces P, Q, R are $\mathfrak{g}$-submodules of W.

For instance, the equality $x \cdot \mathrm{ad}_a = \mathrm{ad}_{[x,a]}$ proves that P is a $\mathfrak{g}$-submodule. The fact that Q and R are $\mathfrak{g}$-submodules can be also checked by a calculation. Otherwise, one can identify the subspace Q with $\mathrm{Hom}(\mathfrak{g}/I, I)$ and R with the fiber product

$$R = \mathrm{Hom}(\mathfrak{g}, I) \times_{\mathrm{Hom}(I,I)} k,$$

which proves Q and R are $\mathfrak{g}$-submodules.

One has a short exact sequence of $\mathfrak{g}$-modules

$$0 \to Q \to R \to k \to 0$$

where k is the trivial representation.

Furthermore, $P \subset Q$ and one deduces the following short exact sequence

$$0 \to Q/P \to R/P \to k \to 0.$$

Note that I acts trivially on the quotients, so this is a short exact sequence of L-modules. Therefore, it splits, so that there exists an element $\bar{w}$ in R/P such that $R/P = Q/P \oplus k\bar{w}$. We choose $w \in R$ to be the preimage of $\bar{w}$. The element w so chosen is a linear map $\mathfrak{g} \to I$ spliting whose restriction on I is identity and such that any $x \in g$ carries it to an element of form ad_a with $a \in I$. Note that $x(w) = [ad_x, w]$.

Let us check this is precisely what we wanted to find. If $a \in I$ and $aw = 0$, we have $[ad_a, w] = 0$ that is $\mathrm{ad}_a \circ w(x) = w \circ \mathrm{ad}_a(x)$ that is

$$w([a, x]) = [a, w(x)] = 0$$

for all x. Since $[a, x] \in I$ and $w|_I = \mathrm{id}$, this gives that $[x, a] = 0$ for all x, which is possible only if $a = 0$ since I does not contain trivial subrepresentation.

It remains to check the second property. It says that for any $x \in \mathfrak{g}$ there exists $a \in I$ such that $xw = aw$. We know that $xw = \mathrm{ad}_a$ for some $a \in I$. But $aw(x) = [ad_a, w](x) = [a, w(x)] - w([a, x]) = -w([a, x]) = -[a, x]$ since $w|_I = \mathrm{id}$. Thus, $xw = -aw$.

Theorem is proven.

12.2. **Reductive Lie algebras.** A finite dimensional algebra $\mathfrak{g}$ is called *reductive* if $\mathfrak{g}$ considered as the $\mathfrak{g}$-module via the adjoint action, is completely reducible.

A semisimple algebra is reductive. A commutative algebra is reductive.

Lemma 12.2.1. *A product of two reductive algebras is reductive.*

Theorem 12.2.2. *Any reductive Lie algebra is a direct product of a semisimple and of a commutative algebra.*

Proof. Submodules of $\mathfrak{g}$ are just ideals. If I and J are two ideals such that $I \cap J = 0$, then $[I, J] = 0$ since $[I, J] \subset I \cap J$. This implies that

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