INTRODUCTION TO LIE ALGEBRAS.
LECTURE 11.

11. Complete reducibility

11.1. The aim of this section is proving the following important theorem.

Theorem 11.1.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then each finite dimensional $\mathfrak{g}$-module is completely reducible.*

Note that the theorem has already been proven for $\mathfrak{g} = \mathfrak{sl}_2$. We have as well seen that the complete reducibility fails for non-finitely dimensional modules — nonsimple Verma modules are indecomposable into a sum of irreduclible modules. Therefore, this theorem is as general as possible. We will see later on that the converse of the theorem is also correct: if $\mathfrak{g}$ is a finite dimansional Lie algebra such that all finite dimensional $\mathfrak{g}$-modules are completely reduclible, then $\mathfrak{g}$ is semisimple.

The proof will result from a sequence of steps. One of the steps is a long digression about tensor algebra of $\mathfrak{g}$-modules.

11.2. Step 1: Killing. Let $\mathfrak{g}$ be semisimple and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be injective. Then the Killing form B_ρ given by the formula $B_\rho(x,y) = \mathrm{Tr}_V(\rho(x)\rho(y))$, is nondegenerate.

In effect, the $I = \mathrm{Ker}(B_\rho)$ is an ideal. It is solvable by Cartan criterion. Therefore, it vanishes.

11.3. Step 2: Casimir. Let B be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ (here $\mathfrak{g}$ is not required to be semisimple!). Let $x_1, \ldots, x_n$ be a basis in $\mathfrak{g}$ and let $y_1, \ldots, y_n$ be the dual basis: $B(x_i, y_j) = \delta_{ij}$. We claim that the endomorphism

$$Q = \sum x_i y_i : V \longrightarrow V$$

does not depend on the choice of the basis $x_1, \ldots, x_n$, commutes with any $\mathfrak{g}$-homomorphism of representations and with the action of $\mathfrak{g}$.

A proof via a direct calculation.

Let $x'_i = \sum a_{ij} x_j$. Then $y_i = \sum a_{ji} y'_j$ and

$$\sum x_i y_i = \sum a_{ji} x_i y'_j = \sum x'_i y'_i.$$

This means that the resulting operator does not depend on the choice of a basis in $\mathfrak{g}$. The homomorphism Q commutes with any $\mathfrak{g}$-homomorphism since this is true for any expression of this kind. Let us check it commutes with the action of $\mathfrak{g}$. We have to prove that for any $z \in \mathfrak{g}$ one has

$$\sum_i zx_iy_i = \sum_i x_iy_iz : V \to V,$$

or, in other words, that one has the following identity

$$[z, \sum x_iy_i] = 0$$

in endomorphisms of any $\mathfrak{g}$-module V. Note that

$$[z, \sum x_iy_i] = \sum_i [z, x_i]y_i + \sum_i x_i[z, y_i].$$

Since the form is invariant,

$$B([z, x_i], y_j) + B(x_i, [z, y_j]) = 0,$$

which implies that

$$[z, x_i] = \sum B([z, x_i], y_j)x_j = -\sum B(x_i, [z, y_j])x_j$$

and similarly

$$[z, y_i] = \sum B([z, y_i], x_j)y_j.$$

Therefore,

$$\sum_i [z, x_i]y_i + \sum_i x_i[z, y_i] = -\sum B(x_i, [z, y_j])x_jy_i + \sum B([z, y_i], x_j)x_iy_j = 0.$$

We will present later on a more "scientific" explanation of this fact.

11.4. Step 3: Calculation. Let $B = B_\rho$ as in Step 1 and Q be the corresponding Casimir endomorphism of V. Then $\mathrm{Tr}_V(Q) = \dim \mathfrak{g}$.

11.5. Step 4: V simple. Assume now that V is simple. Then by Schur lemma Q is an isomorphism since it is nonzero.

11.6. Step 5: A special case. Assume there is a pair $V \subset W$ of $\mathfrak{g}$-modules so that W/V is one-dimensional. We claim that in this case W is isomorphic to the direct sum $V \oplus k$ where k is the trivial one-dimensional nmodule.

First of all, any one-dimensional module M is defined by a Lie algebra homomorphism $\mathfrak{g} \longrightarrow k$ which sends $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ to zero. Thus, M is trivial.

We will prove by induction on the dimension of V that the embedding $V \subset W$ splits, that is W is isomorphic to $V \oplus k$. The splitting of the embedding is equivalent to esistence of element $w \in W$ such that $xw = 0$ for each $x \in \mathfrak{g}$ and $W = V + kw$.

First of all, we will show that V can be assumed to be irreducible. In effect, if V' is a nontrivial submodule of V, we can factor both V and W by V' and get an embedding $\bar{V} = V/V' \subset \bar{W} = W/V'$ of representations of codimension one, having smaller dimension of the submodule $\bar{V}$. Then by the inductive assumption there exists an element $\bar{w} \in \bar{W}$ with $x\bar{w} = 0$ for all $x \in \mathfrak{g}$ such that $\bar{W} = \bar{V} + k\bar{w}$. Look at the natural projection $\pi : W \to \bar{W}$ of $\mathfrak{g}$-modules. The submodule $k\bar{w}$ is a trivial subrepresentation of $\bar{W}$, so $W' := \pi^{-1}(k\bar{w})$ is a subrepresentation of W containing $V' = \mathrm{Ker}(\pi)$. Since $W'/V' = k\bar{w}$ has dimension one, we can use induction to lift $\bar{w}$ to W. The above reasoning shows that we can assume V is irreducible.

Now we will show we can assume that the map $\mathfrak{g} \to \mathfrak{gl}(V)$ is injective. In effect, let I be the kernel of this map. This means that any $x \in I$ annihilates V. Also any $x \in \mathfrak{g}$ carries W to V since the quotient is the trivial representation. Therefore, $[I, I]$ annihilates W. Since an ideal of a semisimple Lie algebra is semisimple, $[I, I] = I$ and therefore both V and W are $\mathfrak{g}/I$-modules. The latter is also semisimple, so we can reduce the problem to the case $I = 0$.

After all reductions made, we have the following. A simple $\mathfrak{g}$ module V is a submodule of W such that the quotient is the trivial representation. The map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is injective. Look now at the Casimir $Q_\rho : W \to W$. Its restriction to V is multiplication by a nonzero constant, and its action on W/V is zero. Therefore, Q_ρ is degenerate and $\mathrm{Ker}Q_\rho$ is one-dimensional. This is a $\mathfrak{g}$-submodule, so this is precisely what we were looking for.

11.7. The general case. We wish to prove that any $\mathfrak{g}$-submodule V of a finite dimensional $\mathfrak{g}$-module W is a direct summand. In other words, that there is another submodule U of W so that $W = V \oplus U$. This will imply complete reducibility.

We already know this in the special case V has codimension one in W. We will prove the general claim deducing it from the special case.

The idea of the proof goes as follows. In order to find the direct summand U, we have to find a $\mathfrak{g}$-homomorphism $\pi : W \to V$ whose restriction on V is identity. Then $U := \mathrm{Ker}(\pi)$ would solve the problem.

This can be encoded as follows. Denote $\mathrm{Hom}(W, V)$ the set of linear transformations from W to V. Let $H \subset \mathrm{Hom}(W, V)$ be the collection of transformations $\phi : W \to V$ such that their restriction to V is a multiplication by a scalar. The map $p : H \to k$ assigns to any $\phi \in H$ this scalar.

The kernel of this map is the collection of $\phi : W \to V$ such that $\phi(V) = 0$. It identifies with the set of linear maps $\mathrm{Hom}(W/V, V)$.

Thus, the vector space $\mathrm{Hom}(W/V, V)$ is a subspace of H having codimension 1. We will now show that H has a natural structure of $\mathfrak{g}$-module and the map $\pi : H \to k$ is a morphism of $\mathfrak{g}$-modules (where k has the trivial $\mathfrak{g}$-module structure). Then by the previous step there is an element $\pi \in H$ such that $x\pi = 0$ for all $x \in \mathfrak{g}$ and $p(\pi) = 1$. This is the map $\pi : W \to V$ we were looking for.

11.8. Digression: Operations with $\mathfrak{g}$-modules. Let V and W be $\mathfrak{g}$-modules. In this subsection we will define the tensor product $V \otimes W$ and the module $\mathrm{Hom}(V, W)$. Then we will study various connections with the objects defined.

11.8.1. Hom

Let V, W be $\mathfrak{g}$-modules. Define on the vector space $\mathrm{Hom}(V, W)$ of the linear maps from V to W a structure of $\mathfrak{g}$-module as follows.

Given $f :\in \mathrm{Hom}(V, W)$ and $x \in \mathfrak{g}$ we define $xf : V \to W$ by the formula

$$(xf)(v) = x(f(v)) - f(x(v)).$$

The construction being obviously linear in x and in f, we have only to check that $[x, y](f) = x(y(f)) - y(x(f))$. We leave this as an exercise.

An obvious special case: if $V = k$ is the trivial representation, the assignment

$$f \in \mathrm{Hom}(k, W) \mapsto f(1) \in W$$

defines a bijection between $\mathrm{Hom}(k, W)$ and W; this bijection is an isomorphism of $\mathfrak{g}$-modules.

Another special case — the $\mathfrak{g}$-module structure on the dual space $V^* = \mathrm{Hom}(V, k)$. Here the action of $\mathfrak{g}$ on the second argument k is assumed to be trivial. We have here

$$(xf)v = -f(xv).$$

Lemma 11.8.2. *The invariant elements of $Hom(V, W)$ are precisely the $\mathfrak{g}$-maps $f : V \to W$ of modules.*

□

11.8.3. Tensor product Reall first of all what is the tensor product of vector spaces. Given V, W two vector spaces. Let U be a third vector space. A bilinear map $\phi : V \times W \to U$ is a map which is linear in each argument. Given a bilinear map ϕ as above, and a linear map

$f : U \to U'$, the composition $f \circ \phi : V \times W \to U'$ is also bilinear. Thus, a map

$$f_* : \mathrm{Bil}(V, W : U) \to \mathrm{Bil}(V, W : U')$$

is induced where $\mathrm{Bil}(V, W : U)$ denotes the set of bilinear maps $V \times W \to U$.

A bilinear map $\Phi : V \times W \to Z$ is called universal if the map

$$\mathrm{Hom}(Z, U) \to \mathrm{Bil}(U, V : U)$$

sending f to $f \circ \Phi$, is a bijection.

Lemma 11.8.4. *If $\Phi : V \times W \to Z$ and $\Phi' : V \times W \to Z'$ are both universal, there exists a* unique *isomorphism $\theta : Z \to Z'$ such that*

$$\Phi' = \theta \circ \Phi.$$

Proof. By universality of Φ there exists a unique θ satisfying the above property. By the uniqueness of Φ', there exists a unique $\theta' : Z' \to Z$ such that $\Phi = \theta' \circ \Phi'$. This implies that $\Phi' = \theta \circ \theta' \circ \Phi$. Once more by the universality of Φ we deduce that $\theta \circ \theta'$ is identity. □

We have proven that a universal bilinear map is essentially unique, if it exists. We will now prove the existence.

Chooce a basis $v_1, \ldots, v_n$ in V and a basis $w_1, \ldots, w_m$ in W. Define a vector space Z as the one spanned by a basis $\{z_{ij}\}$ where i runs from 1 to n and j from 1 to m. Define the bilinear map $\Phi : V \times W \to Z$ by the formula

$$\Phi(v_i, w_j) = z_{i,j}.$$

In a more detail, if $v = \sum c_i v_i$ and $w = \sum d_j w_j$, we have

$$\Phi(v, w) = \sum c_i d_j z_{i,j}.$$

Now any bilinear map $\phi : V \times W \to U$ is uniquely defined by the linear map $f : Z \to U$ given by $f(z_{i,j}) = \phi(v_i, w_j)$.

Notation. Let $\Phi : V \times W \to Z$ be a universal bilinear map. The veector space Z is denoted $V \otimes W$: the element $\Phi(v, w)$ is denoted $v \otimes w$. Note that a general element of $V \otimes W$ does not necessarily have form $v \otimes w$; it is a linear combination of such elements.

11.8.5. Tensor product of $\mathfrak{g}$-modules

Let now V, W be $\mathfrak{g}$-modules. For each $x \in \mathfrak{g}$ we define a map $x : V \otimes W \to V \otimes W$ as the one defined uniquely by the bilinear map

$$V \times W \to V \otimes W$$

given by the formula $x(v, w) = xv \otimes w + v \otimes xw$.

11.8.6. Some identities A bilinear map $\phi : V \times W \to U$ can be equally interpreted as a linear map from $V \otimes W \to U$ or as a linear map from V to $\mathrm{Hom}(W, U)$. This defined a canonical isomorphism

$$\mathrm{Hom}(V \otimes W, U) = \mathrm{Hom}(V, \mathrm{Hom}(W, U)).$$

If the vector spaces are $\mathfrak{g}$-modules, the above isomorphism is an isomorphism of $\mathfrak{g}$-modules.

A linear transformation $V^* \otimes W \to \mathrm{Hom}(V, W)$ is defined by the formula

$$f \otimes w \mapsto u, \quad u(v) = f(v)w.$$

One can easily see that it is an isomorphism provided V or W is finite dimensional. In case V and W are $\mathfrak{g}$-modules, this isomorphism is an isomorphism of $\mathfrak{g}$-modules.

Finally, it is worthwhile to mention that the $\mathfrak{g}$-invarirant part of $\mathrm{Hom}(V, W)$ is precisely the set of $\mathfrak{g}$-homomorphisms from V to W. In effect, let $f \in \mathrm{Hom}(V, W)$, $x \in \mathfrak{g}$. Then $(xf)(v) = x(f(v)) - f(xv)$, so $xf = 0$ iff f commutes with the action of x. Thus, $xf = 0$ for all x iff f commutes with the action of $\mathfrak{g}$.