INTRODUCTION TO LIE ALGEBRAS.
LECTURE 10.

10. JORDAN DECOMPOSITION: THEME WITH VARIATIONS

10.1. Recall that \( f \in \text{End}(V) \) is semisimple if \( f \) is diagonalizable (over the algebraic closure of the base field). Equivalently, this means that \( V \) admits a basis of eigenvectors. Equivalently, this means that the minimal polynomial for \( f \) has distinct roots. This formulation is convenient to prove that if \( W \) is an invariant subspace of \( V \) and \( f \) is semisimple, then \( f|_W \) is semisimple as well.

Proposition 10.1.1. Let \( x \in \text{End}(V) \).

1. There exist unique elements \( s, n \in \text{End}(V) \) such that \( x = s + n \), \( s \) is semisimple, \( n \) is nilpotent and \([s, n] = 0\).
2. There exist polynomials \( p, q \in k[t] \) with no constant term, such that \( s = p(x) \), \( n = q(x) \). Thus, \( s \) and \( n \) commute with every endomorphism commuting with \( x \).
3. If \( A \subseteq B \subseteq V \) and \( x(B) \subseteq A \) then \( s(B) \subseteq A \) and \( n(B) \subseteq A \).

Proof. Existence in 1. follows from the standard linear algebra theorem (Jordan decomposition).

Let \( a_i, i = 1, \ldots, k \) be the eigenvalues of \( x \) with multiplicities \( m_i \). Then \( V = \sum V_i \) and the characteristic polynomial of \( x|_{V_i} \) is \((t - a_i)^{m_i}\). If we define \( s \) by the condition \( s|_{V_i} = a_i \) and \( n = x - s \), we get a required decomposition.

The claim 2 seems to be ugly. It will, however, help us to prove the rest of the claims (including the uniqueness part of claim 1).

We claim there is a polynomial \( p \in k[t] \) such that \( p \equiv a_i \mod (t - a_i)^{m_i} \) and \( p \equiv 0 \mod t \) (the proof see below; this claim is called Chinese remainder theorem).

Then for each \( i \) \( p(x) = a_i + (x - a_i)^{m_i} = a_i \). This proves that \( p(x) = s \). If we put \( q = t - p \), we get \( q(x) = n \).

Thus, we have proven claim 2 for the specific decomposition \( x = s + n \). Let us now prove uniqueness of the decomposition. Let \( x = s + n = s' + n' \). Since \( s' \) and \( n' \) commute with \( x \) and \( s = p(x), n = q(x) \), \( s' \) commutes with \( s \) and \( n' \) commutes with \( n \). Then one has

\[ s - s' = n' - n. \]
The sum of two commuting nilpotent elements is nilpotent. The sum of two commuting semisimple elements is semisimple. A nilpotent semisimple element is zero. This together gives the uniqueness claim.

Finally, since \( p, q \) have no constant term, the claim 3 follows. □

Here is Chinese Remainder theorem.

**Lemma 10.1.2.** Let \( f_1, \ldots, f_k \) be pairwise coprime polynomials in \( k[t] \). Let \( a_1, \ldots, a_k \in k[t] \). Then there exists a polynomial \( p \in k[t] \) satisfying the equality

\[
p \equiv a_i \mod (f_i).
\]

**Proof.** For each \( i \) we have a canonical homomorphism

\[
\pi_i : k[t] \to k[t]/(f_i).
\]

Together they yield a homomorphism

\[
\pi : k[t] \to \prod_i k[t]/(f_i).
\]

The kernel of \( \pi \) consists of polynomials divisible by all \( f_i \). Since all \( f_i \) are pairwise coprime, the kernel of \( \pi \) is the ideal generated by the product \( f = \prod f_i \).

The map \( \bar{\pi} : k[t]/(f) \to \prod_i k[t]/(f_i) \) induced by \( \pi \) is injective. Since the dimensions of the source and the target as vector spaces over \( k \) are the same, \( \bar{\pi} \) is surjective. This implies there is a polynomial whose image under \( \pi_i \) is \( a_i \). □

**Lemma 10.1.3.** Let \( x \in \text{End}(V), \; x = s + n \). Then \( \text{ad}_x = \text{ad}_s + \text{ad}_n \) is the Jordan-Chevalley decomposition of \( \text{ad}_x \).

**Proof.** \( \text{ad}_s \) is semisimple and \( \text{ad}_n \) is nilpotent (see Lemma in Engel theorem). They commute:

\[
\text{ad}_s \circ \text{ad}_n (f) = snf - sf n - nfs + fns = \text{ad}_n \circ \text{ad}_s (f),
\]

since \( s \) and \( n \) commute. □

**10.2. Replicas.** The following trick is difficult to grasp.

Let \( \phi : k \to k \) be a \( \mathbb{Q} \)-linear map.

Let \( s : V \to V \) be a semisimple endomorphism. This means that \( V \) is uniquely decomposed as \( V = \oplus V_i \) where \( V_i \) is the eigenspace for \( s \) with an eigenvalue \( \lambda_i \). Then we define \( \phi(s) \) as the semisimple endomorphism of \( V \) acting on \( V_i \) as \( \phi(\lambda_i) \cdot \text{id} \).

We will call \( \phi(s) \) a replica of \( s \). Choose a polynomial \( P \in k[t] \) such that \( P(0) = 0, \; P(\lambda_i) = \phi(\lambda_i) \). This is always possible since \( \phi(0) = 0 \). Then obviously \( \phi(s) = P(s) \).
Lemma 10.2.1. Let $s \in \text{End}(V)$ be semisimple. Then for each $\phi$ one has
\[ \phi(\text{ad}_s) = \text{ad}_{\phi(s)}. \]

Proof. Choose a basis $x_1, \ldots, x_n$ of eigenvectors in $V$. Then $\text{End}(V)$ has a basis $e_{ij}$ defined by the formula $e_{ij}(x_k) = \delta_{jk}x_i$ (Kronecker’s delta). If $s(x_i) = \lambda_i x_i$, then
\[ \text{ad}_s(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}. \]

Since $\phi(\lambda_i - \lambda_j) = \phi(\lambda_i) - \phi(\lambda_j)$, the result follows. \hfill \square

Corollary 10.2.2. Let $u = s + n$ be the canonical decomposition of an endomorphism of $V$. Let $A \subset B \subset \text{End}(V)$ be subspaces satisfying the condition $\text{ad}_u(B) \subset A$. Then for each $\phi : k \rightarrow k$ over $\mathbb{Q}$ one has $\text{ad}_{\phi(s)}(B) \subset A$.

Proof. We already know that $\text{ad}_s$ is the semisimple part of $\text{ad}_u$, so $\text{ad}_s(B) \subset A$. Since $\phi(\text{ad}_s) = \text{ad}_{\phi(s)}$ is a polynomial without constant term of $\text{ad}_s$, we are done. \hfill \square

Lemma 10.2.3. Let $u = s + n$ be a decomposition of an endomorphism of $V$ as above. If $\text{Tr}_V(u\phi(s)) = 0$ for all $\phi : k \rightarrow k$ over $\mathbb{Q}$, then $s = 0$ that is $u$ is nilpotent.

Proof. Choose a basis of $V$ so that $s$ is diagonal and $n$ is upper-triangular. The trace will be
\[ \text{Tr}(u\phi(s)) = \sum_i m_i \lambda_i \phi(\lambda_i), \]
where $m_i$ are the multiplicities of the respective eigenvalues. Choose $\phi$ whose image belongs to $\mathbb{Q}$. Then
\[ 0 = \phi(\text{Tr}(u\phi(s))) = \sum_i m_i \phi(\lambda_i)^2 \]
which is possible only when $\phi(\lambda_i) = 0$. If this is valid for all $\phi : k \rightarrow \mathbb{Q}$, all eigenvalues are equal to zero. \hfill \square

We are ready now to prove Cartan criterion.

Theorem 10.2.4. Let $L \subset \mathfrak{gl}(V)$ be a Lie subalgebra. The following conditions are equivalent:

- $L$ is solvable.
- $\text{Tr}_V(xy) = 0$ for $x \in L$, $y \in [L, L]$.

Proof. First of all, we can assume that $k$ is algebraically closed. If $L$ is solvable, $V$ admits a basis for which $L$ consists of upper-triangular matrices. Then $[L, L]$ consists of strictly upper-triangular matrices, and the trace vanishes.
Let us now prove the converse. It suffices to check that \([L, L]\) consists of nilpotent elements: then by Engel theorem \([L, L]\) is a nilpotent Lie algebra and then \(L\) is solvable.

Let \(u \in [L, L]\). Write \(u = s + n\) as above. According to the lemma, it suffices to check that \(\text{Tr}_V(u\phi(s)) = 0\) for all \(\phi\).

Let \(u = \sum c_i[x_i, y_i]\). We have

\[
\text{Tr}(u\phi(s)) = \sum c_i \text{Tr}([x_i, y_i]\phi(s)) = \sum c_i\text{Tr}(y_i[\phi(s), x_i]).
\]

It remains now to prove that the brackets \([\phi(s), x_i]\) belong to \([L, L]\).

This follows from Proposition 10.1.1(3) applied to \(A = [L, L]\) and \(B = L\).

\[\square\]

10.3. Let \(L\) be a semisimple Lie algebra. Each element \(x \in L\) defines an endomorphism \(\text{ad}_x \in \text{End}(L)\) which has a unique semisimple and nilpotent part

\[
\text{ad}_x = s + n.
\]

We will see later that the elements \(s\) and \(t\) can be also expressed (in a unique way) as

\[
s = \text{ad}_{x_s}; \quad n = \text{ad}_{x_n}.
\]

The presentation \(x = x_s + x_n\) is called the abstract Jordan decomposition.

The existence of such decomposition in a semisimple Lie algebra is a first step in the classification of semisimple Lie algebras.

**Lemma 10.3.1.** Let \(V\) be a finite dimensional algebra and \(D = \text{Der}(V)\). If \(x = s + n \in D\) then \(s \in D, \ n \in D\).

Proof. Let \(V = \bigoplus a V_a\) be the decomposition of \(V\) into generalized eigenspaces with respect to the eigenvalues of \(x\). We claim that \(V_a \cdot V_b \subseteq V_{a+b}\). In fact, if \(v \in V_a, \ w \in V_b\), so that

\[(x - a)^i v = 0, \quad (x - b)^j w = 0,
\]

then

\[(1) \quad (x - a - b)^{i+j}(vw) = \sum_k \binom{i+j}{k} (x - a)^{i-k}(v)(x - b)^k(w) = 0.
\]

In formula (1) we used the identity

\[(2) \quad (x - a - b)^n(vw) = \sum_k \binom{n}{k} (x - a)^{n-k}(v)(x - b)^k(w)
\]
which can be easily proven by induction. In fact, for \( n = 1 \) we have
an obvious claim
\[
(x - a - b)(vw) = (x - a)(v) \cdot w + v \cdot (x - b)(w).
\]
Assuming it for \( n \), we can apply it to \((x - a - b)(vw)\) and, using the
standard binomial identities, get the required formula.

The endomorphism \( s \) has value \( a \) on \( V_a \). Therefore, Leibniz identity
is obvious for \( s \).

Finally, since \( x, s \in D, n = x - s \) is in \( D \) as well.

Proposition 10.3.2. Let \( L \) be semisimple. Then the map
\[
ad : L \longrightarrow \text{Der}(L)
\]
is a Lie algebra isomorphism.

Proof. \( L \) is semisimple, therefore, has no center. Thus, \( \text{ad} : L \longrightarrow D = \text{Der}(L) \) is injective. Identify \( L \) with \( \text{ad}(L) \). We claim \( L \) is an ideal in
\( D \). In fact, if \( x \in L \) and \( d \in D \) then \([d, \text{ad}_x] = \text{ad}_{d(x)}\].

Let us check that the Killing form of \( D \) restricted to \( L \), gives the
Killing form of \( L \). Choose a base in \( L \) and complete it to a base in \( D \).
Then one sees that for \( x, y \in L \) one has
\[
\text{Tr}_D(\text{ad}_x \text{ad}_y) = \text{Tr}_L(\text{ad}_x \text{ad}_y)
\]
since \( \text{ad}_y(D) \subseteq L \) and the trace depends on diagonal elements only.

Now, use that the Killing form of \( L \) is non-degenerate. This means
that \( L^\perp \cap L = 0 \) which implies \( D = L \oplus L^\perp \). By invariantness of the
Killing form, we deduce that \( L^\perp \) is an ideal. Therefore, \([L, L^\perp] = 0 \)
that is \( D = L \times L^\perp \).

Finally, if \( d \in L^\perp \) and \( x \in L \) then \([d, \text{ad}_x] = \text{ad}_{d(x)}\] which implies
that \( d(x) = 0 \). Thus, \( d = 0 \) and we are done.

\( \square \)

10.4. We are now ready to deduce the main result.

Proposition 10.4.1. Let \( L \) be a semisimple Lie algebra, \( x \in L \). Then
there exist unique elements \( x_s, x_n \in L \) such that

- \( x = x_s + x_n \), and the three elements commute with each other.
- \( \text{ad}_{x_s} \) is semisimple and \( \text{ad}_{x_n} \) is nilpotent.

Proof. \( \text{ad}_x \) is a derivation, therefore its semisimple and nilpotent parts
are derivations by Lemma 10.3.1. Then by Proposition 10.3.2 the
semisimple and nilpotent parts of \( \text{ad}_x \) come also from \( L \).

\( \square \)
Problem assignment, 8

1. Compute the basis in $\mathfrak{sl}_2$ dual to the standard basis $e, f, h$ with respect to the Killing form.

2. Let $L = L_1 \times L_2$ is a product of semisimple Lie algebras. Let $x \in L$ be presented $x = x^1 + x^2$ with $x^i \in L_i$. Prove that $x_s = x^1_s + x^2_s$.

3. Calculate the Killing form for the two-dimensional non-abelian Lie algebra.