# INTRODUCTION TO LIE ALGEBRAS.
# LECTURE 10.

## 10. Jordan decomposition: theme with variations

**10.1.** Recall that $f \in \mathrm{End}(V)$ is semisimple if $f$ is diagonalizable (over the algebraic closure of the base field). Equivalently, this means that $V$ admits a basis of eigenvectors. Equivalently, this means that the minimal polynomial for $f$ has distinct roots. This formulation is convenient to prove that if $W$ is an invariant subspace of $V$ and $f$ is semisimple, then $f|_W$ is semisimple as well.

**Proposition 10.1.1.** *Let $x \in \mathrm{End}(V)$.*

1. *There exist unique elements $s, n \in \mathrm{End}(V)$ such that $x = s + n$, $s$ is semisimple, $n$ is nilpotent and $[s, n] = 0$.*
2. *There exist polynomials $p, q \in k[t]$ with no constant term, such that $s = p(x)$, $n = q(x)$. Thus, $s$ and $n$ commute with every endomorphism commuting with $x$.*
3. *If $A \subseteq B \subseteq V$ and $x(B) \subseteq A$ then $s(B) \subseteq A$ and $n(B) \subseteq A$.*

*Proof.* Existence in 1. follows from the standard linear algebra theorem (Jordan decomposition).

Let $a_i$, $i = 1, \ldots, k$ be the eigenvalues of $x$ with multiplicities $m_i$. Then $V = \sum V_i$ and the characteristic polynomial of $x|_{V_i}$ is $(t - a_i)^{m_i}$. If we define $s$ by the condition $s|_{V_i} = a_i$ and $n = x - s$, we get a required decomposition.

The claim 2 seems to be ugly. It will, however, help us to prove the rest of the claims (including the uniqueness part of claim 1).

We claim there is a polynomial $p \in k[t]$ such that
$p \equiv a_i \mod (t - a_i)^{m_i}$ and $p \equiv 0 \mod t$ (the proof see below; this claim is called Chinese remander theorem).

Then for each $i$ $p(x) = a_i + (x - a_i)^{m_i} = a_i$. This proves that $p(x) = s$. If we put $q = t - p$, we get $q(x) = n$.

Thus, we have proven claim 2 for the specific decomposition $x = s + n$. Let us now prove uniqueness of the decomposition. Let $x = s + n = s' + n'$. Since $s'$ and $n'$ commute with $x$ and $s = p(x)$, $n = q(x)$, $s'$ commutes with $s$ and $n'$ commutes with $n$. Then one has

$$s - s' = n' - n.$$

The sum of two commuting nilpotent elements is nilpotent. The sum of two commuting semisimple elements is semisimple. A nilpotent semisimple element is zero. This together gives the uniqueness claim.

Finally, since $p, q$ have no constant term, the claim 3 follows. $\square$

Here is Chinese Remainder theorem.

**Lemma 10.1.2.** *Let $f_1, \ldots, f_k$ be pairwise coprime polynomials in $k[t]$. Let $a_i, \ldots a_k \in k[t]$. Then there exists a polynomial $p \in k[t]$ satisfying the equality*

$$p \equiv a_i \mod (f_i).$$

*Proof.* For each $i$ we have a canonical homomorphism

$$\pi_i : k[t] \to k[t]/(f_i).$$

Together they yield a homomorphism

$$\pi : k[t] \longrightarrow \prod_i k[t]/(f_i).$$

The kernel of $\pi$ consists of polynomials divisible by all $f_i$. Since all $f_i$ are pairwise coprime, the kernel of $\pi$ is the ideal generated by the product $f = \prod f_i$.

The map $\bar{\pi} : k[t]/(f) \to \prod_i k[t]/(f_i)$ induced by $\pi$ is injective. Since the dimensions of the source and the target as vector spaces over $k$ are the same, $\bar{\pi}$ is surjective. This implies there is a polynomial whose image under $\pi_i$ is $a_i$. $\square$

**Lemma 10.1.3.** *Let $x \in \mathrm{End}(V)$, $x = s + n$. Then $\mathrm{ad}_x = \mathrm{ad}_s + \mathrm{ad}_n$ is the Jordan-Chevalley decomposition of $\mathrm{ad}_x$.*

*Proof.* $\mathrm{ad}_s$ is semisimple and $\mathrm{ad}_n$ is nilpotent (see Lemma in Engel theorem). They commute:

$$\mathrm{ad}_s \circ \mathrm{ad}_n(f) = snf - sfn - nfs + fns = ad_n \circ \mathrm{ad}_s(f),$$

since $s$ and $n$ commute. $\square$

**10.2. Replicas.** The following trick is difficult to grasp.

Let $\phi : k \longrightarrow k$ be a $\mathbb{Q}$-linear map.

Let $s : V \to V$ be a semisimple endomorphism. This means that $V$ is uniquely decomposed as $V = \oplus V_i$ where $V_i$ is the eigenspace for $s$ with an eigenvalue $\lambda_i$. Then we define $\phi(s)$ as the semisimple endomorphism of $V$ acting on $V_i$ as $\phi(\lambda_i) \cdot \mathrm{id}$.

We will call $\phi(s)$ *a replica* of $s$. Choose a polynomial $P \in k[t]$ such that $P(0) = 0$, $P(\lambda_i) = \phi(\lambda_i)$. This is always possible since $\phi(0) = 0$. Then obviously $\phi(s) = P(s)$.

**Lemma 10.2.1.** *Let $s \in \mathrm{End}(V)$ be semisimple. Then for each $\phi$ one has*

$$\phi(\mathrm{ad}_s) = \mathrm{ad}_{\phi(s)}.$$

*Proof.* Choose a basis $x_1, \ldots, x_n$ of eigenvectors in $V$. Then $\mathrm{End}(V)$ has a basis $e_{ij}$ defined by the formula $e_{ij}(x_k) = \delta_{jk}x_i$ (Kronecker's delta). If $s(x_i) = \lambda_i x_i$, then

$$\mathrm{ad}_s(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}.$$

Since $\phi(\lambda_i - \lambda_j) = \phi(\lambda_i) - \phi(\lambda_j)$, the result follows. □

**Corollary 10.2.2.** *Let $u = s + n$ be the canonical decomposition of an endomorphism of $V$. Let $A \subset B \subset \mathrm{End}(V)$ be subspaces satisfying the condition $ad_u(B) \subset A$. Then for each $\phi : k \to k$ over $\mathbb{Q}$ one has $\mathrm{ad}_{\phi(s)}(B) \subset A$.*

*Proof.* We already know that $\mathrm{ad}_s$ is the semisimple part of $\mathrm{ad}_u$, so $\mathrm{ad}_s(B) \subset A$. Since $\phi(\mathrm{ad}_s) = \mathrm{ad}_{\phi(s)}$ is a polynomial without constant term of $\mathrm{ad}_s$, we are done. □

**Lemma 10.2.3.** *Let $u = s+n$ be a decomposition of an endomorphism of $V$ as above. If $\mathrm{Tr}_V(u\phi(s)) = 0$ for all $\phi : k \to k$ over $\mathbb{Q}$, then $s = 0$ that is $u$ is nilpotent.*

*Proof.* Choose a basis of $V$ so that $s$ is diagonal and $n$ is upper-triangular. The trace will be

$$\mathrm{Tr}(u\phi(s)) = \sum_i m_i \lambda_i \phi(\lambda_i),$$

where $m_i$ are the multiplicities of the respective eigenvalues. Choose $\phi$ whose image belongs to $\mathbb{Q}$. Then

$$0 = \phi(Tr(u\phi(s))) = \sum m_i \phi(\lambda_i)^2$$

which is possible only when $\phi(\lambda_i) = 0$. If this is valid for all $\phi : k \to \mathbb{Q}$, all eigenvalues are equal to zero. □

We are ready now to prove Cartan criterion.

**Theorem 10.2.4.** *Let $L \subset \mathfrak{gl}(V)$ be a Lie subalgebra. The following conditions are equivalent:*

- *$L$ is solvable.*
- *$\mathrm{Tr}_V(xy) = 0$ for $x \in L,\ y \in [L, L]$.*

*Proof.* First of all, we can assume that $k$ is algebraically closed. If $L$ is solvable, $V$ admits a basis for which $L$ consists of upper-triangulate matrices. Then $[L, L]$ consists of strictly upper-triangulate matrices, and the trace vanishes.

Let us now prove the converse. It suffices to check that $[L, L]$ consists of nilpotent elements: then by Engel theorem $[L, L]$ is a nilpotent Lie algebra and then $L$ is solvable.

Let $u \in [L, L]$. Write $u = s + n$ as above. According to the lemma, it suffices to check that $\mathrm{Tr}_V(u\phi(s)) = 0$ for all $\phi$.

Let $u = \sum c_i[x_i, y_i]$. We have

$$\mathrm{Tr}(u\phi(s)) = \sum c_i \mathrm{Tr}([x_i, y_i]\phi(s)) = \sum c_i \mathrm{Tr}(y_i[\phi(s), x_i]).$$

It remains now to prove that the brackets $[\phi(s), x_i]$ belong to $[L, L]$. This follows from Proposition 10.1.1(3) applied to $A = [L, L]$ and $B = L$.

□

**10.3.** Let $L$ be a semisimple Lie algebra. Each element $x \in L$ defines an endomorphism $\mathrm{ad}_x \in \mathrm{End}(L)$ which has a unique semisimple and nilpotent part

$$\mathrm{ad}_x = s + n.$$

We will see later that the elements $s$ and $t$ can be also expressed (in a unique way) as

$$s = \mathrm{ad}_{x_s}; \quad n = \mathrm{ad}_{x_n}.$$

The presentation $x = x_s + x_n$ is called *the abstract Jordan decomposition.*

The existence of such decomposition in a semisimple Lie algebra is a first step in the classification of semisimple Lie algebras.

**Lemma 10.3.1.** *Let $V$ be a finite dimensional algebra and $D = \mathrm{Der}(V)$. If $x = s + n \in D$ then $s \in D$, $n \in D$.*

*Proof.* Let $V = \oplus_a V_a$ be the decomposition of $V$ into generalized eigenspaces with respect to the eigenvalues of $x$. We claim that $V_a \cdot V_b \subseteq V_{a+b}$. In fact, if $v \in V_a$, $\quad w \in V_b$, so that

$$(x - a)^i v = 0, \quad (x - b)^j w = 0,$$

then

$$(1) \quad (x - a - b)^{i+j}(vw) = \sum_k \binom{i+j}{k} (x - a)^{i+j-k}(v)(x - b)^k(w) = 0.$$

In formula (1) we used the identity

$$(2) \qquad (x - a - b)^n(vw) = \sum_k \binom{n}{k} (x - a)^{n-k}(v)(x - b)^k(w)$$

which can be easily proven by induction. In fact, For $n = 1$ we have an obvious claim

$$(x - a - b)(vw) = (x - a)(v) \cdot w + v \cdot (x - b)(w).$$

ASsuming it for $n$, we can apply it to $(x - a - b)(vw)$ and, using the standard binomial identities, get the required formula.

The endomorphism $s$ has value $a$ on $V_a$. Therefore, Leibniz identity is obvious for $s$.

Finally, since $x, s \in D$, $n = x - s$ is in $D$ as well. $\square$

**Proposition 10.3.2.** *Let $L$ be semisimple. Then the map*

$$ad : L \longrightarrow \mathrm{Der}(L)$$

*is a Lie algebra isomorphism.*

*Proof.* $L$ is semisimple, therefore, has no center. Thus, $\mathrm{ad} : L \longrightarrow D = \mathrm{Der}(L)$ is injective. Identify $L$ with $\mathrm{ad}(L)$. We claim $L$ is an ideal in $D$. In fact, if $x \in L$ and $d \in D$ then $[d, \mathrm{ad}_x] = \mathrm{ad}_{d(x)}]$.

Let us check that the Killing form of $D$ restricted to $L$, gives the Killing form of $L$. Choose a base in $L$ and complete it to a base in $D$. Then one sees that for $x, y \in L$ one has

$$\mathrm{Tr}_D(\mathrm{ad}_x \mathrm{ad}_y) = \mathrm{Tr}_L(\mathrm{ad}_x \mathrm{ad}_y)$$

since $\mathrm{ad}_y(D) \subseteq L$ and the trace depends on diagonal elements only.

Now, use that the Killing form of $L$ is non-degenerate. This means that $L^\perp \cap L = 0$ which implies $D = L \oplus L^\perp$. By invariantness of the Killing form, we deduce that $L^\perp$ is an ideal. Therefore, $[L, L^\perp] = 0$ that is $D = L \times L^\perp$.

Finally, if $d \in L^\perp$ and $x \in L$ then $[d, \mathrm{ad}_x] = \mathrm{ad}_{d(x)}$ which implies that $d(x) = 0$. Thus, $d = 0$ and we are done. $\square$

**10.4.** We are now ready to deduce the main result.

**Proposition 10.4.1.** *Let $L$ be a semisimple Lie algebra, $x \in L$. Then there exist unique elements $x_s,\ x_n \in L$ such that*

- *$x = x_s + x_n$, and the three elements commute with each other.*
- *$\mathrm{ad}_{x_s}$ is semisimple and $\mathrm{ad}_{x_n}$ is nilpotent.*

*Proof.* $\mathrm{ad}_x$ is a derivation, therefore its semisimple and nilpotent parts are derivations by Lemma 10.3.1. Then by Proposition 10.3.2 the semisimple and nilpotent parts of $\mathrm{ad}_x$ come also from $L$. $\square$

Problem assignment, 8

1. Compute the basis in $\mathfrak{sl}_2$ dual to the standard basis $e, f, h$ with respect to the Killing form.
2. Let $L = L_1 \times L_2$ is a product of semisimple Lie algebras. Let $x \in L$ be presented $x = x^1 + x^2$ with $x^i \in L_i$. Prove that $x_s = x^1_s + x^2_s$.
3. Calculate the KLilling form for the two-dimensional non-abelian Lie algebra.