INTRODUCTION TO LIE ALGEBRAS.
LECTURE 1.

1. Algebras. Derivations. Definition of Lie algebra

1.1. **Algebras.** Let k be a field. An algebra over k (or k-algebra) is a vector space A endowed with a bilinear operation

$$a, b \in A \mapsto a \cdot b \in A.$$

Recall that bilinearity means that for each $a \in A$ left and right multiplications by a are linear transformations of vector spaces (i.e. preserve sum and multiplication by a scalar).

1.1.1. Some extra properties

An algebra A is called associative if $a \cdot (b \cdot c) = (a \cdot b) \cdot c)$.

An algebra A is commutative if $a \cdot b = b \cdot a$.

Usually commutative algebras are supposed to be associative as well.

1.1.2. Example

If V is a vector space, $\mathrm{End}(V)$, the set of (linear) endomorphisms of V is an associative algebra with respect to composition. If $V = k^n$ $\mathrm{End}(V)$ is just the algebra of $n \times n$ matrices over k.

1.1.3. Example The ring of polynomials $k[x]$ over k is a commutative k-algebra. The same for $k[x_1, \ldots, x_n]$, the algebra of polynomials of n variables.

1.1.4. Example

If V is a vector space, define an operation by the formula

$$a \cdot b = 0.$$

This is an algebra operation.

1.2. **Subalgebras, ideals, quotient algebras.** A linear map $f : A \to B$ of k-algebras is called *homomorphism* if $f(a \cdot b) = f(a) \cdot f(b)$ for each $a, b \in A$.

The image of a homomorphism is a *subalgebra* (please, give a correct definition). Kernel of f defined as $\{a \in A | f(a) = 0\}$ is *an ideal* in A. Here are the appropriate definitions.

Definition 1.2.1. A vector subspace $B \subseteq A$ is called a subalgebra if

$$a, b \in B \Longrightarrow a \cdot b \in B.$$

Definition 1.2.2. A vector subspace $I \subseteq A$ is called an ideal if

$$a \in A \& x \in I \Longrightarrow a \cdot x \in I \& x \cdot a \in I.$$

Lemma 1.2.3. *Let $f : A \to B$ be a homomorphism of algebras. Then* $\mathrm{Ker}(f)$ *is an ideal in A.*

Proof. Exercise. □

An important property of ideals is that one can form a quotient algebra "modulo I". Here is the construction.

Let A be an algebra and I an ideal in A. We define the quotient algebra A/I as follows.

As a set this is the quotient of A modulo the equivalence relation

$$a \sim b \text{ iff } a - b \in I.$$

Thus, this is the set of equivalence classes having form $a + I$, where $a \in A$.

Structure of vector space on A/I is given by the formulas

$$(a + I) + (b + I) = (a + b) + I; \quad \lambda(a + I) = \lambda a + I.$$

Algebra structure on A/I is given by the formula

$$(a + I) \cdot (b + I) = a \cdot b + I.$$

One has a canonical homomorphism

$$\rho : A \longrightarrow A/I$$

defined by the formula $\rho(a) = a + I$.

As usual, the following theorem (Theorem on homomorphism) is straighforward.

Theorem 1.2.4. *Let $f : A \longrightarrow B$ be a homomorphism of algebras and let I be an ideal in A. Suppose that $I \subseteq \mathrm{Ker}(f)$. Then there exists a unique homomorphism $\overline{f} : A/I \longrightarrow B$ such that $f = \overline{f} \circ \rho$ where $\rho : A \longrightarrow A/I$ is the canonical homomorphism.*

Moreover, $\overline{f}$ is onto iff f is onto; $\overline{f}$ is one-to-one iff $I = \mathrm{Ker}(f)$.

Proof. Exercise. □

1.3. **Derivations.** A linear endomorphism $d : A \longrightarrow A$ is called *derivation* if the following *Leibniz rule* holds.

$$d(a \cdot b) = d(a) \cdot b + a \cdot d(b).$$

The set of all derivations of A is denoted $\mathrm{Der}(A)$. This is clearly a vector subspace of $\mathrm{End}(A)$.

1.3.1. Composition Let $d, d' \in \mathrm{Der}(A)$ let us check that the composition dd' is not a derivation.

$$dd'(a \cdot b) = d(d'(a) \cdot b + a \cdot d'(b)) = d(d'(a) \cdot b) + d(a \cdot d'(b)) = dd'(a) \cdot b + d'(a) \cdot d(b) + d(a) \cdot d'(b) + a \cdot dd'(b) \tag{1}$$

which is not exactly what we need.

1.3.2. Bracket Thus, we suggest another operation. Given $d, d' \in \mathrm{Der}(A)$, define $[d, d'] = dd' - d'd$.

Theorem 1.3.3. *If $d, d' \in \mathrm{Der}(A)$ then $[d, d'] \in \mathrm{Der}(A)$.*

Proof. Direct calculation. □

1.3.4. Properties of this bracket

1. $[x, x] = 0$.
2. (Jacobi identity) $[[xy]z] + [[zx]y] + [[yz]x] = 0$

Exercise: check this.

1.4. **Definition of Lie algebra. First examples.** A Lie algebra is an algebra with an operation satisfying the properties 1.3.

The operation in a Lie algebra is usually denoted $[,]$ and called (Lie) bracket.

1.4.1. Anticommutativity The first property of a Lie algebra saying $[xx] = 0$ is called anticommutativity. In fact, it implies that $[xy] = -[yx]$ for all x, y.

Proof: $0 = [x + y, x + y] = [xx] + [xy] + [yx] + [yy]$. This implies $[xy] = -[yx]$. The converse is true if char $k \neq 2$. In fact, $[xx] = -[xx]$ implies that $2[xx] = 0$ and, if the charactersitic of k is not 2, this implies $[xx] = 0$.

1.4.2. Example Let $k = \mathbb{R}$, $L = \mathbb{R}$. We are looking for possible Lie brackets on L. Bilinearity and anticommutativity require

$$[a, b] = [a \cdot 1, b \cdot 1] = ab[1, 1] = 0.$$

Thus, there is only one Lie bracket on $L = \mathbb{R}$.

Definition 1.4.3. A Lie algebra L having a zero bracket is called *a commutative Lie algebra.*

1.4.4. Observation Fix a field k of characteristic $\neq 2$ and let $L = \langle e_1, \ldots, e_n \rangle$ be n-dimensional vector space over k. In order to define a bilinear operation, it is enough to define it on e_i:

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

(this is true for any type of algebra). Elements c_{ij}^k are called *structure constants* of L.

Since we want the bracket to be anti-commutative, one has to have

$$[e_i, e_j] = [e_j, e_i].$$

Bilinearity and this condition imply anti-commutativity of the bracket (check this formally!).

Suppose now we have checked already anticommutativity. To check Jacobi identity let us denote

$$J(x, y, z) = [[xy]z] + [[zx]y] + [[yz]x].$$

One observes that J is trilinear (linear on each one of its three arguments) and antisymmetric (it changes sign if one interchanges any two arguments).

Thus, in order to check $J(x, y, z)$ is identically zero, it is enough to check

$$J(e_i, e_j, e_k) = 0 \text{ for } 1 \leq i < j < k \leq n.$$

1.4.5. Example Suppose $\dim L = 2$. Suppose L is not commutative. Choose a basis $L = \langle e_1, e_2 \rangle$. One has

$$[e_1, e_1] = [e_2, e_2] = 0 \text{ and } [e_1, e_2] = -[e_2, e_1].$$

Let $[e_1, e_2] = y$. Then $y \neq 0$ and any bracket in L is proportional to y (by bilinearity).

Thus, it is convenient to take y as one of generators on L. Choose another one, say x. We have $L = \langle x, y \rangle$ and $[x, y] = \lambda y$. Since L

is not commutative, $\lambda \neq 0$. Thus change variables once more setting $x := x/\lambda$.

We finally get

$$L = \langle x, y \rangle \text{ and } [x, y] = y. \tag{2}$$

We have therefore proven that there are only two two-dimensional Lie algebras over k up to isomorphism: a commutative Lie algebra and the one described in (2).

1.4.6. Example The set of $n \times n$ matrices over k is an associative algebra with respect to the matrix multiplication. It becomes a Lie algebra if we define a bracket by the formula

$$[x, y] = xy - yx.$$

This Lie algebra is denoted $\mathfrak{gl}_n(k)$ (sometimes we do not mention the field k). Its dimension is, of course, n^2.

The Lie algebra $\mathfrak{gl}_n$ admits a remarkable Lie subalgebra.

Define $\mathfrak{sl}_n = \{a \in \mathfrak{gl}_n | \mathrm{tr}(a) = 0\}$.

Here $\mathrm{tr}(a) = \sum a_{ii}$ is the trace of a, the sum of the diagonal elements of a.

We claim this is a Lie subalgebra.

1.4.7. Proof

Recall that for each pair of matrices a, b one has

$$\mathrm{tr}(ab) = \mathrm{tr}(ba).$$

(Proof is just a direct calculation: both sides are equal to $\sum_{ij} a_{ij} b_{ji}$.)

Then $\mathrm{tr}([a, b]) = \mathrm{tr}(ab) - \mathrm{tr}(ba) = 0$. This proves that $\mathfrak{sl}_n$ is closed under the bracket operation.