1

Qcat. 11.04, 11:00

1. Recall

$$\mathfrak{C} : Set_\Delta \rightleftarrows Cat_\Delta : N$$

Basic objects

$\mathfrak{C}(\Delta^n)$ with objects $\{0, \dots, n\}$

$Map(a, b) = (\Delta^1)^{(a,b)}$, $(a,b) = \{a+1, \dots, b-1\}$

composition $(\Delta^1)^{(a,b)} \times (\Delta^1)^{(b,c)} \to (\Delta^1)^{(a,c)}$

defined by inserting 0 at place b.

2. Recall that Cat_Δ has Bergner model structure:

a) $\pi_0 : Cat_\Delta \to Cat$ replaces $Map(a,b)$ with $\pi_0(Map(a,b))$

b) w.equivalences are f: WE on Maps, $\pi_0(f)$ eq.

c) fibrations are f: fib on Maps + lifting of equivalences with a fixed source/~~target~~.

Thm (Joyal) Set_Δ has a model structure with

1) f is WE iff $\mathfrak{C}(f)$ is [cat. equiv.]

2) f is cof iff injective.

Rem S is fibrant iff it is Qc.

Thm $(\mathfrak{C}, N)$ is a Quillen equivalence

Today some indications about the proofs will be given; extra structures as inner Fun will be discussed. Connection to Kan model structure indicated. Analog of functor π_0 for Qcat will be ~~discussed~~ presented.

The second talk will be devoted to limits.

3. Equivalences in $X \in Set_\Delta$.

3.1 The functor $\pi_0 : Set_\Delta \to Cat$ is left adjoint to $N : Cat \to Set_\Delta$. Thus, can be defined by

$$\pi_0(\Delta^n) = (0 \to 1 \to \dots \to n),$$

+ commutes with colimits.

3.2 Otherwise, $\pi_0(X)$ is the category whose objects are X_0, whose arrows are generated by X_1, with the relations given for each $\sigma \in X_2$ by

$$d_1\sigma = d_0\sigma \circ d_2\sigma$$

3.3 Another description: $\pi_0(X) = \pi_0(\mathfrak{C}(X))$ where the rhs π_0 has been already described.

An even better description can be given for QC (proof more or less elementary)

3.4. Let X be a QC. Define $\pi(X)$ as follows (at the end this will be $\pi_0(X)$)

$Ob(\pi(X)) = X_0$

$Hom_{\pi(X)}(a,b) \doteq X(a,b)/\sim$ where $X(a,b) =$

$= \{ x \in X_1 \mid d_1 x = a,\ d_0 x = b \}$ and the equivalence is given as follows: $x \sim y$ iff $\exists$ triangle in X_2 s.t. $d_0 s = id_b$, $d_1 s = y$, $d_2 s = x$

$$s: \begin{tikzcd} & \cdot \arrow[dr, "id_b"] & \\ \cdot \arrow[ur, "x"] \arrow[rr, "y"'] & & \cdot \end{tikzcd}$$

Lemma 1. Above is an equivalence relation for $X \in Qcat$.

2. Relation

$$z \circ x = y \iff \exists s : d_0 s = z,\ d_1 s = y,\ d_2 s = x$$

defines a composition so that $\pi(X)$ becomes a cat.

3. $\pi(X) = \pi_0(X)$ naturally.

Note: One could have defined another equivalence relation:

$$\begin{tikzcd} & \cdot \arrow[dr, "x"] & \\ \cdot \arrow[ur, "id"] \arrow[rr, "y"'] & & \cdot \end{tikzcd}$$

Fortunately, they are ~~equivalen~~ the same.

Proposition For a QC X the following is equivalent:

1) X satisfies RLP wrt $\Lambda^n_i \to \Delta^n$, $i = 0, \dots, n-1$
2) — " — $i = 1, \dots, n$
3) X is Kan
4) (X is QC and) $\pi_0 X$ is a groupoid

Scheme of pf: 4) $\Leftrightarrow$ 1) is nontrivial, but the rest is obvious: 4) $\Leftrightarrow$ 2) by duality and 3) = 1) + 2).
In effect, 1) $\Rightarrow$ 4) is quite obvious since 1) implies any arrow is left invertible in $\pi_0(X)$. The converse is more involved.

Corollary Any QC has a $\overbrace{\text{maximal}}^{\text{unique}}$ Kan subset — this is the collection of all n-simplices whose all edges are equivalences.

4. Function spaces.

We know that analogs of 'weak functors' are played in our theory by (strictly) simplicial maps. Thus, the following notion is important

4.1. **Def** $B, C \in Set_\Delta$. $Fun(B, C)$ represents the functor

$$A \mapsto Hom(A \times B, C)$$

There is a much simpler formula in case X is a QC

<u>Def</u> Let X be a QC, $a, b \in X$. Define

$$\mathrm{Hom}^R_X(a,b)_n = \{ s : \Delta^{n+1} \to X \mid s|_{\Delta^{\{0,\dots,n\}}} = a,\ s(n+1) = b \}$$

Another version:

$$\underline{\mathrm{Hom}}_X(a,b)_n = \{ s : \Delta^n \times \Delta^1 \to X \mid s|_{\Delta^n \times \{0\}} = a,\ s|_{\Delta^n \times \{1\}} = b \}$$

One has as well $\mathrm{Hom}^L_X(a,b)$ defined dually to Hom^R_X.

Embeddings

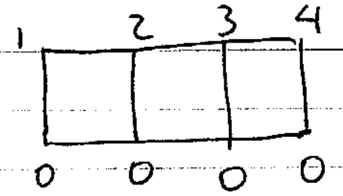
$$\star \quad \mathrm{Hom}^R_X(a,b) \longrightarrow \underline{\mathrm{Hom}}_X(a,b) \longleftarrow \mathrm{Hom}^L_X(a,b)$$

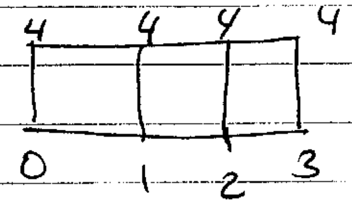
are defined by canonical maps

$$\Delta^{n+1} \xleftarrow{R} \Delta^n \times \Delta^1 \xrightarrow{L} \Delta^{n+1}$$

which are ~~retractions of two different embeddings~~
(should be maps of cosimplicial objects to make sense)

Easily defined on the level of posets:

R: 

L: 

<u>Thm</u> (announcement) all spaces in $\star$ are Kan; all maps are homotopy equivalences and everything is equivalent to $\mathrm{Map}_{\mathcal{L}(X)}(a,b)$

<u>Note</u> The above theorem, together with the explicit description of inner anodyne morphisms is the basis for the proof of both Joyal model structure and Quillen equivalence $(\mathfrak{L}, N)$.

<u>Some hints to the proof</u>

Define a cosimplicial object in Set_Δ by the formula

$$\Delta^n_* = \Delta^{n+1} \quad \text{(the nerve of the category}$$

obtained by adding a terminal object $*$)

Define, furthermore, J^n as the colimit

$$\cancel{Q^n = \mathrm{Map}_{\mathfrak{L}}} \qquad \begin{array}{ccccc} & & \{x\} & & \\ & \nearrow & & \searrow & \\ \Delta^n & & & & J^n \\ & \searrow & & \nearrow & \\ & & \Delta^n_* & & \end{array}$$

Finally, put
$$Q^n = \mathrm{Map}_{\mathfrak{L}(J^n)}(x, *)$$ the terminal object of Δ^n_*

Let us construct a map of cosimplicial objects

$$Q^\bullet \longrightarrow \Delta^\bullet$$

Since $\mathfrak{L}$ preserves colimits, Q^n is the Map simplicial set in the colimit category; thus $Q^\bullet$ is the quotient of the cube

$$\left(\Delta^1\right)^{(0,n+1)} = \mathrm{Map}_{\mathfrak{L}(\Delta^{n+1})}(0, n+1)$$

by the relation identifying any pair of compositions

$(i < n+1)$ $$\mathrm{Map}(0,i) \times \mathrm{Map}(i, n+1) \to \mathrm{Map}(0, n+1)$$

$$(\alpha, \beta) \sim (\alpha', \beta)$$

with the same $\beta \in \mathrm{Map}(i, n+1)$

The map $Q^n \to \Delta^n$ is defined via

$$\star : (\Delta^1)^{(0,n+1)} \to \Delta^n$$

- these simplicial sets are nerves of posets and the map $\star$ is defined on the level of posets by assigning to each subset $K \subseteq (0,n+1)$ its maximal element.

As usual, any cosimplicial space $Q^\bullet$ defines uniquely a left exact functor

$$X \longmapsto |X|_{Q^\bullet} \quad : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$$

so that $\Delta^\bullet$ defines the identity

<u>Prop</u> The canonical map

$$|X|_{Q^\bullet} \to X = |X|_{\Delta^\bullet}$$

is a homotopy equivalence

<u>Proof</u> Look how X is glued from simplices and follow what happens to $|X|_{Q^\bullet}$.

Let now $\mathcal{C} \in \mathrm{Cat}_\Delta$, $x, y \in \mathrm{Ob}\,\mathcal{C}$, $X = N(\mathcal{C})$

An n-simplex in $\mathrm{Hom}^R_X(x,y)$ is a map $J^n \to N(\mathcal{C})$ or $\mathcal{L}(J^n) \to \mathcal{C}$.

<u>Corollary</u> $\mathrm{Hom}^R_{N(\mathcal{C})}(x,y) = \{ n \mapsto \mathrm{Hom}(Q^n, \mathrm{Map}_{\mathcal{C}}(x,y)) \}$

Since we already believe that $Q^{\bullet}$ and $\Delta^{\bullet}$ define equivalent "geometric realizations" we can easily deduce that the ~~fu~~ adjoint functor

$$\mathrm{Sing}_{Q^{\bullet}}(S) = \{ n \mapsto \mathrm{Hom}(Q^{n}, S) \}$$

is also equivalent to identity (at least for Kan S).

(11) Qcat 11.04, 14:00

(Co) Limits

0. Introduction

0.1 given a functor $F: K \to \mathcal{C}$, its colimit is an object $X \in \mathcal{C}$ together with a collection of arrows $u_a : F(a) \to X$ for each $a \in K$ such that $u_a = u_b \circ F(\varphi)$ for each $\varphi : a \to b$. The above data should satisfy the well-known universal property: given a collection (Y, v_α) as above, there exists a unique map $X \to Y$ compatible with u_α and v_α.

0.2. A special case $K = \emptyset$ yields the notion of ~~term~~ initial object. A general case can be expressed through this special notion passing to the functor category $\mathrm{Fun}(K, \mathcal{C}) \ni F$ and to its undercategory $\mathrm{Fun}(K, \mathcal{C})_{F/}$.

0.3. Of course, the notion of limit can be obtaind by dualization.

All the above can be done for quasicategories. Our aim is to find such notion which will be unique up to (unique?) equivalence instead up-to-a-unique-isomorphism of the classical notion.

1. Recall: opposite simplicial set X^{op} has the same sets of n-simplices:

$$X^{op}_n = X_n ,$$

but different faces and degeneracies, so that for $\varphi \in X_1$ $d_0 \varphi^{op} = (d_1 \varphi)^{op}$,
$d_1 \varphi^{op} = (d_0 \varphi)^{op}$.

More formally, the functor $op: Set_\Delta \to Set_\Delta$ is defined by its restriction to the standard simplices where it inserts the total ordering of the vertices.

2. Over- and under categories.

2.1 Join

$$S * T(J) = \coprod_{J = I' \sqcup I''} S(I') \times T(I'')$$

where by agreement $S(\emptyset) = *$, $\forall i' \in I', i'' \in I''$ $i' < i''$

Example $S = N(C)$, $T = N(D) \Rightarrow S * T = N(C * D)$
where $C * D$ join of categories:
$Ob(C * D) = Ob\, C \sqcup Ob\, D$
$Hom(c, c')$ and $Hom(d, d')$ as they were,
$Hom(c, d) = *$, $Hom(d, c) = \emptyset$.

Example: $\mathcal{C} * \{*\}$ is adding a final object

Notation $S * \Delta^0 = S^{\triangleright}$, $\Delta^0 * S = S^{\triangleleft}$

Example $\Delta^m * \Delta^n = \Delta^{m+n+1}$

Lemma The functors $T \mapsto T * S$, $T \mapsto S * T$ considered as functors
$$Set_\Delta \to (Set_\Delta)_{S/},$$
commute with colimits.

Note One has natural maps $S \to S * T \leftarrow T$

Prop $S, T \in Qcat \Rightarrow S * T \in Qcat$ (Exercise)

(13)

2.2. Given $p: K \longrightarrow S$ denote

$$\begin{tikzcd} \mathrm{Hom}_p(Y*K, S) \arrow[r] \arrow[d] & \mathrm{Hom}(Y*K, S) \arrow[d, "K \to Y*K"] \\ \{*\} \arrow[r, "p"] & \mathrm{Hom}(K, S) \end{tikzcd}$$

this is the simplicial set of extensions of $p: K \to S$ to $Y*K \to S$.

<u>Prop</u> The functor $Y \mapsto \mathrm{Hom}_p(Y*K, S)$ is representable

$$\mathrm{Hom}(Y, S_{/p}) = \mathrm{Hom}_p(Y*K, S)$$

<u>Pf</u> Obvious, $(S_{/p})_n = \cancel{\mathrm{Hom}_p(\Delta^n * \Delta^n, S)}\ \mathrm{Hom}_p(\Delta^n * K, S)$

2.3. Dually, $S_{p/}$ is defined as representing the functor $Y \longmapsto \mathrm{Hom}^p(K*Y, S)$

<u>Thm</u> if $S \in Qcat$ then $S_{/p} \in Qcat$

for any $p: K \to S$

<u>Hint of a proof</u> Similarly to what was formulated about inner anodyne morphisms, can be done for left anodyne morphisms.

This implies

2.1.2.2 [<u>Prop</u> $S \in Qcat \Rightarrow S_{p/} \to S$ is a left fibration.

Then the composition $S_{p/} \to S \to *$ is an inner fibration $\Rightarrow S_{p/}$ is a Qcat.

3. Initial object

Let $x \in S$. We will write $S_{/x}$ instead of $S_{/p}$, $p : \Delta^0 \to S$ defined by x.

Def $x \in S$ is strongly final if

$$S_{/x} \longrightarrow S$$

is a trivial fibration.

This means the existence of a dotted map

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & S_{/x} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

or

$$\begin{tikzcd} \Delta^n \coprod_{\partial\Delta^n} (\partial\Delta^n * \Delta^0) \arrow[r] \arrow[d] & S \\ \Delta^n * \Delta^0 \arrow[ur, dashed] & \end{tikzcd}$$

An easy calculation shows this is just

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r, "f"] \arrow[d] & S \\ \Delta^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

where the map f carries $(n+1)$ to x.

Prop Let $S \in \mathrm{Qcat}$, $x \in S$. x is strongly final iff $\mathrm{Hom}^R_X(z,x)$ is contractible $\forall z$.

Proof

$\mathrm{Hom}^R_S(z,x)$ is the fiber of $p : S_{/x} \to S$.

Since p is right fibration, it is a trivial Kan iff all fibers are contractible (chapter 2)

Prop $S \in \mathcal{Q}cat$, $F \subseteq S$ the full subcategory defined by final vertices. Then F is contractible Kan complex (or empty).

Proof – obvious, using the notion of strict finality:

$$(n>0) \qquad \begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d] & F \arrow[r] & S \\ \Delta^n \arrow[urr, dashed] & & \end{tikzcd}$$

because f carries n to a final object. the map $\Delta^n \to S$ automatically has image in F.

4. Limits

Let $\quad p: K \to S \quad$ be given

Def A colimit of p is an initial object of $S_{p/}$; a limit is an ~~exist~~ terminal obj of $S_{/p}$.

Thus, a colimit is an object of $S_{p/}$ that is a map $K^{\triangleright} \to S$ extending $p: K \to S$. The meaning is clear.

The limit of $p: K \to S$, if it exists, is unique in the higher sense: the full subcategory of initial objects of $S_{p/}$ is a Kan contractible set.

5. colimits vs homotopy colimits

Thm I, $\mathcal{C}$ fibrant simplicial categories

$F: I \to \mathcal{C}$ a simpl. functor.

Assume $C \in Ob\,\mathcal{C}$ together with $\{\eta_i : F(i) \to C\}_{i \in I}$

be a collection of compatible maps. Then (1) ~ (2):

(1) $\{\eta_i\}$ exhibits C as a homotopy colimit of F

(2) $\overline{f} : N(J)^{\triangleright} \to N(\mathcal{C})$ defined by its restriction $f = N(F)$ on $N(J)$ and extended to $\overline{f}$ via $\{\eta_i\}$, is a homotopy colimit diagr.

Our aim is to understand what does the theorem claim.

5.1. Hcolim for simplicial categories.

Let $f : I \xrightarrow{f} I'$ be a simplicial functor, let $\mathcal{C}$ be a simplicial category. One has

$$\mathcal{C}^{I} \overset{\xrightarrow{f_!}}{\underset{\xrightarrow[f_*]{}}{\xleftarrow{f^*}}} \mathcal{C}^{I'}$$

where $\mathcal{C}^I$ is the (simplicial) category of simplicial functors $I \to \mathcal{C}$ and $f^*(F) = F \circ f$.
We are mostly interested with the case $I' = *$ so that $f^*(C)$ is the constant functor $C : I \to \mathcal{C}$.

A compatible collection $\{\eta_i : F(i) \to C\}$ is nothing but a morphism

$$\eta : F \longrightarrow f^*(C)$$

in $\mathcal{C}^I$. This means:

- For each object $i \in I$ an arrow $\eta(i) : F(i) \to C$ is given
- For each pair of objects i, j in I the composition

$$\mathrm{Map}_I(i,j) \longrightarrow \mathrm{Map}_{\mathcal{C}}(F(i), F(j)) \xrightarrow{\eta_j} \mathrm{Map}_{\mathcal{C}}(F(i), C)$$

sends the whole $\mathrm{Map}_I(i,j)$ to η_i.

A morphism $\eta : F \longrightarrow f^*(C)$ exhibits C as a homotopy colimit if $\forall A \in \mathcal{C}$ the maps

$$\mathrm{Map}_{\mathcal{C}}(C, A) \longrightarrow \mathrm{Map}_{\mathcal{C}}(F(i), A)$$

exhibit $\mathrm{Map}_{\mathcal{C}}(C, A)$ as a homotopy limit of the diagram

$$\{\mathrm{Map}_{\mathcal{C}}(F(i), A)\}_{i \in I} \quad \text{in } \mathrm{Set}_\Delta .$$

The latter is a simplicial functor $I^{op} \to \mathrm{Set}_\Delta$

How to construct $\overline{f} : N(I)^{\triangleright} \longrightarrow N(\mathcal{C})$ out of $\overline{f} = NF$ and $\{\eta_i\}$?

In other words, how to complete the diagram

$$\begin{tikzcd}
\mathcal{L}(\Delta^n) \arrow[r] \arrow[d] & I \arrow[d, "F"] \\
\mathcal{L}(\Delta^{n+1}) \arrow[r, dashed] & \mathcal{C}
\end{tikzcd}$$

This is very easy: for example, for $n=1$ the problem is to complete an arrow $A \to B$ in $\mathcal{C}$ to a homotopy commutative triangle

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[d] \\ & C \end{tikzcd}$$

. We can complete it with the triange, commutative on the nose.

The theorem is nontrivial, and it basically says that some homotopy commutative diagrams can be strictified. The following important general claim is in the heart of the proof.

4.2.4.4 __Thm__ $S \in Set_\Delta$, $\mathcal{C} \in Cat_\Delta$, $u: \mathfrak{C}(S) \to \mathcal{C}$ an equivalence, A a combinatorial simplicial model category, Then the map

$$N((A^{\mathcal{C}})^\circ) \to Fun(S, N(A^\circ))$$

is a categorical equivalence where A° is the simplicial set of fibrant cofibrant objects of A, $(A^{\mathcal{C}})^\circ$ (projective or injective model structure)

In the above theorem A can be replaced by special subcategories ($\mathcal{C}$-chunks)

More explicitly, $\mathrm{Fun}(B,C)_n = \mathrm{Hom}(B\times\Delta^n, C)$

Recall: S Kan $\Rightarrow$ $\mathrm{Fun}(K,S)$ is Kan. We have

<u>Theorem</u> S is QC $\Rightarrow$ $\mathrm{Fun}(K,S)$ is QC as well.
This fact is very important since allows one to construct "∞-categories of functors".
Proof is nontrivial; it is based on the notion of inner cofibration.

Let us make an attempt to directly prove the theorem. Deduce immediately ~~too~~ from:
S satisfies RLP wrt $\Lambda^n_i \times K \to \Delta^n \times K$. $(i\neq 0,n)$

[<u>Definition</u> f is inner anodyne if it has LLP wrt all inner fibrations.

By definition, $\Lambda^n_i \to \Delta^n$ $(i\neq 0,n)$ is inner anodyne.
Thus, it suffices to know that inner anodyne morphisms are <u>stable</u> under the direct product to a s.set.

<u>Notation</u> Let M be a set of arrows. The class generated by M, denoted $[M]$, is $\mathrm{LLP}(\mathrm{RLP}(M))$.

[T]2.3.2.1 <u>Theorem</u> The following 3 collections generate the same class in Set_Δ (inner anodyne)

1. A_1: $\Lambda^n_i \subset \Delta^n$, $i\neq 0,n$

2. A_2: given by

$$\begin{tikzcd}
\partial\Delta^m \times \Lambda^2_1 \arrow[r] \arrow[d] & \partial\Delta^m \times \Delta^2 \arrow[d] \\
\Delta^m \times \Lambda^2_1 \arrow[r] & \Delta^m \times \Delta^2
\end{tikzcd}$$

(the map from the coproduct to $\Delta^m \times \Delta^2$)

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3. The collection A_3

$$\begin{tikzcd} S\times\Lambda^2_1 \arrow[r] \arrow[d] & S\times\Delta^2 \arrow[d] \\ S'\times\Lambda^2_1 \arrow[r] & S'\times\Delta^2 \end{tikzcd}$$

for all $S\subset S'$

Idea of a proof Any morphism is A_3 can be presented as a composition of morphisms obtained by cobase cohange from A_2. Thus,

$$[A_2]=[A_3]$$

Then, $\Lambda^n_i\subset\Delta^n$ is a retract of the map from

$$\begin{tikzcd} \Lambda^n_i\times\Lambda^2_1 \arrow[r] \arrow[d] & \Lambda^n_i\times\Delta^2 \arrow[d] \\ \Delta^n\times\Lambda^2_1 \arrow[r] & \Delta^n\times\Delta^2 \end{tikzcd}$$

finally, eben worse calculation shows any arrow from A_2 belongs to $[A_1]$. $\square$

Cor S is a QC iff

$$\mathrm{Fun}(\Delta^2,S)\to\mathrm{Fun}(\Lambda^2_1,S)$$

is an acyclic Kan fibration.

Cor $A\to A'$ inner anodyne, $j: B\to B'$ cof $\Rightarrow$ the result is inner anodyne. In particular, $\mathrm{Fun}(K,S)$ is a QC if S is a QC.

Combining the above,

$K \in Set_\Delta$, $S \in QC$ $\rightsquigarrow$ $Fun(K,S)^{Kan}$, the maximal Kan subset.

More explicitly, a one-simplex
$$f : K \times \Delta^1 \longrightarrow S$$
is an equivalence if it carries any one-simplex from the LHS to an equivalence in S

Advertizement: $K, S \mapsto Fun(K,S)^{Kan}$ is the correct definition for function space in the model category of QC.

Note The fact that $Fun(K,S)$ is not necessarily Kan stresses that $Fun(-,-)$ does not provide Joyal model structure with a simplicial enrichment.

5. Joyal model structure.

$X \in Set_\Delta$, $a,b \in X_0$ $\rightsquigarrow$ $Map(a,b)$ - ?

Formally correct answer is: $Map_{\mathfrak{L}(X)}(a,b)$

where $Ob\,\mathfrak{L}(X) = X_0$, so a,b are objects in $\mathfrak{L}(X)$.
This is good but too abstract.