(1) April, 25. Homotopy limits vs limits in QC

1. Derived functors

1.1. Let (M, W) be a category with weak equivalences. Given a functor

$$F : M \to C$$

we define its derived functors by ~~a~~ universal properties:

$$\begin{tikzcd} & Ho(M) \arrow[dr] & \\ M \arrow[ur, "Q"] \arrow[rr, "F"] & & \mathcal{C} \end{tikzcd}$$

$RF : Ho(M) \to C$ together with a natural transformation

$$\theta : F \to RF \circ Q$$

is a right derived functor if for any pair $(G, \tau : F \to G \circ Q)$ there exists a unique natural transformation $\alpha : RF \to G$ making the diagram

$$\begin{tikzcd} F \arrow[r, "\theta"] \arrow[dr, "\tau"] & RF \circ Q \arrow[d, "\alpha"] \\ & G \circ Q \end{tikzcd}$$

By a standard reasoning (RF, θ), if exists, is unique up to unique isomorphism.

1.2. ~~How~~ A typical situation: $F : M \to N$ a functor between two categories with WE. Then the above definition should be applied to the composition

$$\begin{tikzcd} & Ho(M) \arrow[drr, dashed, "RF"] & & \\ M \arrow[ur, "Q"] \arrow[r, "F"] & N \arrow[r, "Q"] & Ho(N) \end{tikzcd}$$

(2) 1.3. As we saw above, derived functors are independent of model structure which can/should be chosen so that it becomes easy to calculate the derived functor.

<u>Prop</u> Let M be model, $F: M \to C$. Assume F carries tf of fibrant objects of M to isom. of C. Then RF exists (and is calculated using fibrant resolutions $X \to F(X)$ (which are tc).

<u>1.4. Remark.</u> The condition does not necessarily hold: if you choose the injective model structure on the category of complexes, you can't expect to be able to calculate derived tensor products with ~~co~~fibrant replacements (everything is cofibrant)

<u>Cor</u> Let $F: M \rightleftarrows N: G$ be a Quillen pair of functor between the model categories M and N. Then the derived pair

$$LF: Ho(M) \rightleftarrows Ho(N): RG$$

exists.

2. Functor categories

2.1 Given a functor $f : \mathcal{C} \to \mathcal{D}$, and a category M, the following three adjoint functors ~~are~~ can be defined

$$\begin{tikzcd} M^{\mathcal{C}} \arrow[r, shift left=3, "f_!"] \arrow[r, shift right=3, "f_*"'] & M^{\mathcal{D}} \arrow[l, "f^*"'] \end{tikzcd}$$

The most interesting case is that of $D = *$. Then $f^*(x)$ is the constant functor with value $x \in M$; for ~~$M \in M$~~ $F \in M^{\mathcal{C}}$ one has

$$f_!(F) = \operatorname{colim}(F)$$
$$f_*(F) = \lim(F).$$

If M has ~~notion~~ notion of WE, $M^{\mathcal{C}}$ has as well the notion of WE. Thus, we may ask whether a derived functor of colim or of lim exists, and how to calculate it.

<u>Problems</u>: 1. $M^{\mathcal{C}}$ has no model structure in general.

~~2. Even if it has, the pairs $(f_!, f^*)$ and (f^*, f_*) are not necessarily Quillen pairs.~~

2. Even when it has, fibrant replacement functor is not very explicit.

④ <u>Injective and projective model structures</u>

Let M be a model category.

<u>Def</u> $\alpha : F \to G$ in $M^{\mathcal{C}}$ is called

1) injective cofibration if $\forall x \in \mathcal{C}$ $\alpha(x) : F(x) \to G(x)$ is cof.
2) projective fibration if $\forall x \in \mathcal{C}$ $\alpha(x) : F(x) \to G(x)$ is fib.
3) weak equivalence if $\forall x \in \mathcal{C}$ $\alpha(x) : F(x) \to G(x)$ is WE.

<u>Prop</u> Let M be a <u>combinatorial</u> model cat. Then $M^{\mathcal{C}}$ has two model structures:

1) injective model structure, defined by WE and inj cofibrations
2) projective model structure, defined by WE and projective fib.

Note that injective fibrations are arrows satisfying RLP wrt trivial injective cofibrations; projective cofibrations are arrows satisfying LLP wrt trivial projective fibrations. These two classes of morphisms are not easy to describe.

<u>Prop</u> M combinatorial, $f : C \to D$ a functor.

(1) The pair $(f_!, f^*)$ is a Quillen pair between projective model structures

(2) The pair (f^*, f_*) is a Quillen pair between injective model structures.

Thus, in order to calculate Rf_* or $R\lim$ one has to find a fibrant replacement $F \to \widetilde{F}$ of a functor, and then calculate $\lim \widetilde{F}$.

This proves the existence of $R\lim$ at least when M is combinatoral; but does not provide a good way of calculating it since ~~pr~~ fibrant objects in the injective model structure are not easy to construct.

5

A useful generalization

We are now in the simplicial case.

Given $S : \mathcal{C} \to Set_\Delta$, $F : \mathcal{C} \to M$, define

$$\mathrm{Fun}\ \cancel{} (S, F)$$

as the inferse limit of all $\mathrm{Fun}(S(x), F(y))$ along all arrows $\alpha : x \to y$.

Recall that if M is a simplicial model category, $S \in Set_\Delta$, $x \in M$ $\rightsquigarrow$ $\mathrm{Fun}(S, x) \in M$ ('simplicial path spaces') representing the functor

$$y \longmapsto \mathrm{Hom}_{Set_\Delta}(S, \mathrm{Map}(y, x)) .$$

<u>Examples</u> $S = *$ $\Rightarrow$ $\mathrm{Fun}(S, F) = \lim F$

As we saw above, one can calulate the derived functors of

$$\mathrm{Fun} : \left(Set_\Delta{}^{\mathcal{C}}\right)^{op} \times M^{C} \longrightarrow M$$

using the fibrant replacement in the injective model structure in the second argument.

But one can equally calculate it using the projective model structure in both argument, cofibrant replacement for the first and a ~~the~~ fibrant replacement for the second argument (compare to the functor Hom in homological algebra, or to calculation of Rf_* via Cech complexes in sheaf theory).

<u>Lemma</u> The functor $S : \mathcal{C} \to Set_\Delta$,

$$S(x) = N(\mathcal{C}/x),$$

is a cofibrant resolution of the constant functor

$$x \mapsto *,$$

in the projective model structure.

<u>Def</u> For $F : \mathcal{C} \to M$ the homotopy limit is defined by the formula

$$\operatorname{holim} F = \operatorname{Fun}(N(\mathcal{C}/-), F)$$

<u>Claim</u> $R\lim(F) = \operatorname{holim} \tilde{F}$ where $\tilde{F}$ is a fibrant replacement of F in the projective model structure (i.e., $F(x) \to \tilde{F}(x)$ is a fibrant replacement for all $x \in \mathcal{C}$).

3. Simplicial functor categories.

Let both $\mathcal{C}$, M be simplicial categories. Denote by $M^{\mathcal{C}}$ the category of <u>simplicial</u> functors $F : \mathcal{C} \to M$.

One has the similar projective and injective model structures and the similar property of the functors corresponding to $f : \mathcal{C} \to \mathcal{C}'$

<u>Prop</u> - The pair $(f_!, f^*)$ is a Quillen pair wrt projective model structures.

- The pair (f^*, f_*) is a Quillen pair wrt injective model structures.

(7)

Characterization of homotopy limits.

M simplicial + combinatorial.

Let $F \in M^{\mathcal{C}}$.

Def A diagram

$$\begin{array}{ccc} & & \lim F \\ & & \downarrow \\ G & \longrightarrow & \lim F' \end{array} \qquad \bigstar$$

given by a weak equivalence $F \to F'$ and a morphism $G \longrightarrow \lim F'$, represents a homotopy limit of F if for a fibrant replacement of F' in the injective model structure, $F' \to F''$, the composition $G \to \lim F' \longrightarrow \lim F''$, is a weak equivalence.

We want to formulate a criterion ~~of~~ for a diagram $\bigstar$ to represent a homotopy limit.

Let m be a cofibrant object in M. For any $F \in M^C$ denote F_m : ~~$Set_\Delta \to$~~ $C \to Set_\Delta$ by the formula

$$F_m(c) = Map_M(m, F(c)).$$

Similarly, $G_m = Map(m, G)$.

Prop A diagram $G \to \lim F' \leftarrow \lim F$ represents a homotopy limit of F iff its evaluation at each cofibrant object $m \in M$

$$G_m \to \lim F'_m \leftarrow \lim F_m$$

represents a homotopy limit ~~in~~ of F_m.
(Here G ~~and~~, F, F' are assumed (projectively) fibrant)

Proof 1. Replace F' with injectively fibrant F''.
The induced map $F'_m \to F''_m$ is an equivalence; moreover, F''_m is injectively fibrant in $Set_\Delta^{\mathcal{C}}$: this follows from the lemma below

Lemma Let $A \to B$ be a trivial injective cofibration in $Set_\Delta^{\mathcal{C}}$; then $A \otimes m \to B \otimes m$ is a trivial injective cofibration in $\mathcal{M}^{\mathcal{C}}$ $\boxtimes$

Thus, we have to check that $G \to \lim F$ with injectively fibrant F and fibrant G is a weak equivalence iff for any cofibrant $m \in M$

$$Map(m, G) \longrightarrow Map(m, \lim F) = \lim Map(m, F)$$

is a weak equivalence.
This is obvious.

Holim for simplicial categories

Let ~~F~~ M be a fibrant simplicial category $\mathcal{C}$ a simplicial category (or $f: \mathcal{C} \to \mathcal{C}'$ a simpl. functor). Let $G \in M$, $F \in M^{\mathcal{C}}$ a map ~~$G \to \lim$~~

$$\eta : f^*G \longrightarrow F' \xleftarrow{\sim} F$$

is a homotopy right Kan extension (or homotopy limit if $\mathcal{C}' = *$) if $\forall m \in M$ the induced map

$$\eta_m : G_m \to f_* F'_m$$

exhibits $G_m \in Set_\Delta^{\mathcal{C}'}$ as homotopy right Kan extension of $F'_m \in Set_\Delta^{\mathcal{C}}$.

In other words, $G \in M$ is a homotopy limit if

$$Map(m, G) \to \lim \{Map(m, F(c))\}$$

exhibits $Map(m, G)$ as a homotopy limit of $Map(m, F(c))$.

This notion reduces to the model category setting if we replace $\mathcal{M}$ with $\mathcal{M}^\circ = \{$fibrant cofibrant objects of $\mathcal{M}\}$.

Comparison with the QC notion

Thm $\mathcal{C}$, $\mathcal{J}$ fibrant simplicial categories, $F: J \to \mathcal{C}$ a simplicial functor. Given an object $c \in \mathcal{C}$ and a family

$$\eta_i : c \longrightarrow F(i)$$

of compatible arrows, the following are equivalent:

(1) η_i exhibit c as a homotopy limit of F

(2) Let f ~~NF~~: $NJ \longrightarrow N\mathcal{C}$ be the simplicial nerve,

$$\bar{f} : (NJ)^{\triangleleft} \longrightarrow N\mathcal{C} \qquad \text{the extension defined}$$

by η_i. Then $\bar{f}$ is a limit diagram

Comments 1. How do η_i define $\bar{f}$? Of course,

$\bar{f}(*) = c$. Furthermore, η_i allow one to extend $\bar{f}$ to 1-simplices. Furthermore, the 2-simplex

$$\begin{tikzcd} & i \arrow[d, "\alpha"] \\ * \arrow[ur] \arrow[r] & j \end{tikzcd}$$

goes to the (strictly commutative) diagram
$$\begin{tikzcd} & F(i) \arrow[d, "F(\alpha)"] \\ c \arrow[ur, "\eta_i"] \arrow[r, "\eta_j"'] & F(j) \end{tikzcd}$$

Furthermore, the 3-simplex

$$\begin{tikzcd} & i \arrow[r] \arrow[d] & j \arrow[dl] \\ * \arrow[ur] \arrow[urr] \arrow[r] & k & \end{tikzcd}$$

gives rise to the 3-simplex in $\mathcal{C}$

$$\begin{tikzcd} & F(i) \arrow[dr] \arrow[dd] & \\ c \arrow[ur] \arrow[rr] \arrow[dr] & & F(j) \arrow[dl] \\ & F(k) & \end{tikzcd}$$

where all "higher "homotopies" are trivial.

2. Therefore, the theorem claims that, if a homotopy limit exists for a simplicial diagram F, the limit for its nerve f can be chosen in a very special form.

Key result needed in the proof — comparison of construction of diagram categories in the simplicial and in the quasi-categorical setting

M a combinatorial simplicial model category.
$\mathcal{C}$ a small simplicial category.

<u>Definition</u> A chunk of M is a full subcat. $\mathcal{U} \subseteq M$ with the properties

(a) $A \in \mathcal{U}$, $\{\varphi_i : A \to B_i\}$ finite collection of arrows in $\mathcal{U}$. $\exists$ factorisation

$$A \xrightarrow{p} \bar{A} \xrightarrow{q} \prod B_i \qquad \begin{array}{l} p \in \text{triv.cof} \\ q \in \text{fib}, \ \bar{A} \in \mathcal{U}, \end{array}$$

functorial on $\{\varphi_i\}$ via a simplicial functor

(b) Dually,

$$\coprod B_i \xrightarrow{p} \bar{A} \xrightarrow{q} A \qquad \begin{array}{l} p \ \text{cof}, \ q \in TF, \\ \bar{A} \in \mathcal{U} \end{array}$$

<u>Def</u> $\mathcal{U}$ is a $\mathcal{C}$-chunk of M if

(1) $\mathcal{U}$ is a chunk of M

(2) $\mathcal{U}^{\mathcal{C}}$ is a chunk of $M^{\mathcal{C}}$ (in the projective model str)

Prop $S \in Set_\Delta$, $\mathcal{C} \in \mathcal{C}at_\Delta$, $u : \mathfrak{C}(S) \tilde{\to} \mathcal{C}$

M combinatorial simp. model, $\mathcal{U} \subset M$ a $\mathcal{C}$-chunk.

The map

$$\star : \quad N((\mathcal{U}^{\mathcal{C}})^\circ) \longrightarrow \mathrm{Fun}(S, N(\mathcal{U}^\circ))$$

is a categorical equivalence.

Explanation By definition, $\mathcal{U}^\circ = \mathcal{U} \cap M^\circ$ (fibrant cofibrant), the same for $(\mathcal{U}^{\mathcal{C}})^\circ$.

The 0-simplices of the LHS are fibrant cofibrant functors $\mathcal{C} \to M$ with values in $\mathcal{U}$. They of course define $\mathfrak{C}(S) \to \mathcal{C} \to M^\circ$ that is maps $S \to N(\mathcal{U}^\circ)$.

Next, $M^{\mathcal{C}}$ is a simplicial category: for a simplicial ~~categ~~ set K and a simplicial functor $F : \mathcal{C} \to M$ the tensor product $K \otimes F : \mathcal{C} \to M$ is defined by the formula $(K \otimes F)(c) = K \otimes F(c)$.

The map $\star$ assigns to an n-simplex of the LHS, that is a map $\sigma : \mathfrak{C}(\Delta^n) \to (\mathcal{U}^{\mathcal{C}})^\circ$, the map $S \times \Delta^n \to N(\mathcal{U}^\circ)$ adjoint to the composition

$$\mathfrak{C}(S \times \Delta^n) \to \mathfrak{C}(S) \times \mathfrak{C}(\Delta^n) \xrightarrow{u \times \sigma} \mathcal{C} \times (\mathcal{U}^{\mathcal{C}})^\circ \xrightarrow{ev} \mathcal{U}^\circ$$