1. Even though the idea of replacing an object X of a category $\mathcal{C}$ with the corresponding representable functor

$$h_X(Y) = \mathrm{Hom}_C(Y, X)$$

seem to be too general, and even though replacement of functors on schemes with functors on affine schemes seems to be a merely mild simplification, presentation of schemes as functors on the category of affine schemes (= of commutative rings) is in the heart of algebraic geometry.

For, what is an algebraic variety defined by the equation

$$x^2 + y^2 = 1 \quad ?$$

Of course, this is $\mathrm{Spec}\ \mathbb{Z}[x,y]/(x^2+y^2-1)$. But, even more natural answer would be

"this is the functor assigning to a commutative ring A the collection of pairs (x,y) satisfying the equation"

Then the description via Spec is just a presentation of the functor.

Similarly, algebraic groups are better defined via the functors they represent: everybody understands that $GL(n)$ is the functor

$$A \mapsto GL(n, A)$$

and this description is much more understandable than desription of the corresponding commutative Hopf algebra.

2. According to faithfully flat descent, presheaves represented by schemes are sheaves in fpqc topology. However, the notion of sheaf is too general to give a meaningful generalization of scheme.

Instead, one defines properties of morphisms of sheaves via base change and reduction to the case of properties of morphisms of schemes, and requires the existence of covering with schemes. Here are precise definitions.

A morphism of sheaves $f: \mathcal{F} \to \mathcal{G}$ is schematic if for any base change $\mathrm{Spec}(A) \to \mathcal{G}$, the fiber product is representable by a scheme.

A morphism is flat / smooth / étale if such base change is flat / smooth / étale over $\mathrm{Spec}(A)$.

A sheaf $\mathcal{F}$ is called an algebraic space if it ~~can~~ admits an étale surjective cover with a scheme, plus some extra techical conditions on the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ (schematicity + quasicompactness).

One can vary the topology on Aff (smooth, étale ...) as well as the requirement on the covering by a scheme (smooth, flat, ...)

3. Not all meaningful functors on affine schemes are representable. Look at the functor

$$A \longmapsto \mathrm{Pic}(A)$$

from affine schemes to abelian groups assigning to A the group of classes of invertible modules over A.

One can glue invertible modules, but not in the way which would yield a sheaf. In a more detail, to get a sheaf we have to be able, given invertible modules L_i on elements of a covering $X = \cup U_i$ such that

$$L_i\big|_{U_i \cap U_j} \approx L_j\big|_{U_i \cap U_j},$$

to be able to present an invertible sheaf L on the whole X such that $L_i \approx L|_{U_i}$.

This is not the way we glue sheaves.

So, there cannot exist a "Picard scheme" Pic representing the functor

$$X \longmapsto Pic(X) = Hom(X, Pic)$$

<u>Remark</u> There is, however, another Picard scheme,

$$Pic_X$$

for a fixed scheme X, representing the functor

$$T \longmapsto Pic(X \times T) / Pic(T)$$

But that is another story.

<u>4. Stacks.</u> The previous example of the functor

$$X \longmapsto Pic(X)$$

is worth studying. We saw that to be a sheaf, a functor $F: Aff^{op} \to Set$ has to have a certain gluing property.

But, even though one can glue invertible sheaves as well, this is not the gluing property making $X \mapsto Pic(X)$ a sheaf.

The gluing of invertible sheaves requires not the isomorphism classes only, but also isomorphisms between the sheaves. That is, we have to assign to an affine scheme X not the set / group $Pic(X)$ of isomorphism classes of invertible sheaves, but the whole groupoid $\mathcal{Pic}(X)$ of invertible sheaves on X.

Thus, various concrete interesting problems leed us to consider analogs of sheaves with values in groupoids (insted of sets), with the gluing property formulated similarly to the way it is formulated for glueing sheaves.

Stacks appear naturally when we assign to a scheme a category or a groupoid rather than a set (of isomorphism classes).

A small remark.
An analog of presheaf for stacks is given by a functor

$$\mathrm{Aff}^{op} \to \mathcal{G}rp$$

to the category of groupoids. This is quite unnatural object as for a pair of maps $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ one has usually a natural isomorphism, but not an equality

$$(\beta\alpha)^* \xrightarrow{\sim} \alpha^* \circ \beta^* .$$

This has been formalized by Grothendieck in the notion of 'catégorie fibrée en groupoïdes', which corresponds to Lurie's right fibration of simplicial sets. We will intentionally ignore this difference since any catégorie fibrée can be strictified.

Here is a general definition of stack of groupoids

A functor $\mathcal{F} : (\mathcal{C}, \tau)^{op} \to Grp$ is a stack if

(1) for any $x \in \mathcal{C}$ and any pair of objects $a, b \in \mathcal{F}(x)$ the functor assigning to each

$$f : y \to x$$

the set $\mathrm{Hom}_{\mathcal{F}(y)}(f^*a, f^*b)$,

is a sheaf on $(\mathcal{C}/x, \tau)$.

(2) Let $R \subset h_x$ be a covering of $x \in \mathcal{C}$.

Define the groupoid $\mathcal{F}_R(x)$ of objects given R-locally as follows:

- its objects consist of collections

$$x_\alpha \in \mathcal{F}(y) \qquad \forall \; \alpha : y \to x \text{ in } R$$

together with isomorphisms

$$\gamma^*(x_\beta) \xrightarrow{\sim} x_\alpha ,$$

for each commutative triangle
$$\begin{tikzcd} y \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & x \\ z \arrow[ur, "\beta"'] & \end{tikzcd}$$
$\beta \in R$,

satisfying the obvious compatibility condition.

- its morphisms are compatible collections of maps $x_\alpha \longrightarrow y_\alpha$.

Now we are ready to formulate the object gluing condition:

For any covering sieve R of x the obvious functor

$$\mathcal{F}(x) \longrightarrow \mathcal{F}_R(x)$$

is an equivalence.

A stack is a groupoid analog of a sheaf, and, as for sheaves on Aff, also here not all stacks have "geometric meaning".

Similarly to algebraic spaces, geometric meaning is provided by an existence of a good covering of a stack $\mathcal{F}$ by a scheme, good means étale, smooth, or something similar. It is worthwhile to formulate this covering property, together with a less obviously meaningful properties of the diagonal $\mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F}$, in one package.

Here it is.

Let $F_\bullet$ be a simplicial object in $\mathcal{C}$ (or, more generally, a simplicial presheaf such that all F_n are coproducts of representables).

$F_\cdot$ is called a groupoid if the maps

$$F_n \longrightarrow F_1 \underset{F_0}{\times} \cdots \underset{F_0}{\times} F_1$$

are isomorphisms (and the fiber products exist...)

$F_\cdot$ is called étale groupoid if all face maps (actually, only $F_1 \rightrightarrows F_0$ make difference) are étale (of course, smooth or flat or anything else can be defined in this way).

Definition A stack $\mathcal{F}: Aff^{op} \to Grp$ is algebraic if it is equivalent to the stack associated to the simplicial presheaf defined by an étale / smooth groupoid (extra ~~separated~~ properness or quasicompactness of $F_1 \longrightarrow F_0 \times F_0$ is required).

6. Examples ① Moduli stack of curves. Its points should parametrize smooth projective curves of a given genus g. Here is its description as a functor:

A family of smooth genus g curves with base X is a smooth map $\pi: Y \to X$, proper (=projective), whose fibers are genus g curves. The functors $\mathcal{M}_g$ assigns to a scheme X the groupoid of families of smooth genus g curves.

Stack properties are an exercise. To prove $\mathcal{M}_g$ is an algebraic (DM) stack, one needs

a scheme $\tilde{M}$ with an affine covering $\tilde{M} \to \mathcal{M}_g$.
A reasonable way to do so is to find an extra structure for curves (e.g, level structure) such that the corresponding moduli problem is already representable by a scheme. One can be even more radical in the complex-analytic situation, choosing *framing* (see definition below) as the extra structure, and getting the Teichmüller space as a covering.

Recall

Definition Let X be a compact Riemann surface of genus g, S a compact topological oriented genus g 2-dimensional manifold. A framing on X is an isotopy class of orientation-preserving homeomorphism $S \to X$.

Framings of X form a torsor over the modular group $\Gamma_g = \pi_0(\mathrm{Diff}_+(S))$.

② Moduli of principal G-bundles

X scheme. Define the functor

$$\mathcal{M}_{G,X}(S) = \text{groupoid of principal } G\text{-bundles } \{E \xrightarrow{p} X \times S\}$$

Thm X curve $\Rightarrow$ $\mathcal{M}_{G,X}$ is an algebraic stack. It is smooth of dimension $\dim(G)(g-1)$ if G is reductive.

The best way of proving smoothness is to look at infinitesimal neighborhood of points which is described by deformation theory.

7. Deformation theory

We will live for simplicity over a field k of char $=0$. If X is a scheme and $x \in X$ a k-point, the formal neighborhood at x is defined by the restriction of h_X to the category of local artinian k-algebras.

There are Schlessinger's conditions for pro-representabily of a functor

$$F : \mathrm{art}(k) \longrightarrow \mathrm{Set}$$

Here they are:

1) $F(k) = *$
2) $F(k[\varepsilon]/\varepsilon^2)$ is a finite dimensional vector space
3) F preserves fiber products:
$$F(A \times_B C) \xrightarrow{\sim} F(A) \times_{F(B)} F(C)$$

If X is a scheme and $x \in X(k)$, the corresponding functor is defined as

$$F(A) = \text{fiber of } \begin{array}{c} X(A) \\ \downarrow \\ X(k) \end{array} \text{ at } x \in X(k)$$

There is another version of Schlessinger's theorem, describing a weaker condition of existence of a hull for a functor $F : \mathrm{art}(k) \to \mathrm{Set}$. This corresponds to a big extent to existence of a smooth covering by a scheme of the corresponding stack.

Schlessinger's conditions to existence of a hull are
1), 2) and
3) in case the map $C \to B$ is surjective with one-dimensional kernel ('small map').
4) F carries surjective maps to surjective maps

In late 80-ies it was pointed out by Deligne, Drinfeld and Feigin that for many deformation problems appearing 'in real life', there exist a very nice description of the deformation functor via a certain dg Lie algebra (called tangent Lie algebra)

For such a dg Lie algebra $\mathfrak{g}$ $(\mathfrak{g}^i = 0, i<0)$

$$\mathfrak{g}: \quad \mathfrak{g}^0 \to \mathfrak{g}^1 \to \mathfrak{g}^2 \to \cdots$$

One has to define $F_{\mathfrak{g}}(A)$ as the quotient of the set $MC(\mathfrak{m} \otimes \mathfrak{g}) = \{x \in \mathfrak{m} \otimes \mathfrak{g}^1 \mid dx + \frac{1}{2}[x,x] = 0\}$ modulo the action of nilpotent group $\exp(\mathfrak{m} \otimes \mathfrak{g}^0)$

For each deformation problem the construction of $\mathfrak{g}$ requires some imagination; once it is done, we have tools to study the moduli space in an infinitesimal neighborhood.

Examples:

1) A is an associative k-algebra.

$C^\cdot(A,A)$ Cohomological Hochschild complex,

$$C^n = \mathrm{Hom}(A^{\otimes n}, A). \quad \text{Then } \mathfrak{g}^n = C^{n+1}, \; n \geq 0$$

so that

$$\mathfrak{g} = \left(C^\cdot(A,A)[1] \right)_{\geq 0}$$

2) X smooth scheme $\Rightarrow$ $\mathfrak{g} = R\Gamma(X, T_X)$

where T is the tangent vector bundle on X.

<u>Remark</u> Lie algebra structure on the above complexes (sometimes defined up to ...) is an important issue which we completely ignore.

Looking at the definition of functor

$$F_{\mathfrak{g}}(A) = MC(\mathfrak{m} \otimes \mathfrak{g})/\sim$$

one can easily guess that passage to quotient is what ~~an~~ make sometimes the functor $F_{\mathfrak{g}}$ non-representable. Thus, a more basic object is the functor

$$F^1_{\mathfrak{g}} : \mathrm{art}(k) \longrightarrow \mathrm{grp}$$

to groupoids, which should describe the formal completion of the moduli stack

$$\mathcal{F} : \mathrm{Aff}_k^{op} \longrightarrow \mathrm{grp}$$

at a point.

An important property of the functors $F_{\mathfrak{g}}$, $F^1_{\mathfrak{g}}$ is that they depend (up to equivalence) only on the quasiisomorphism class of $\mathfrak{g}$ (provided it is concentrated in degrees ≥ 0).

8. Higher stacks and derived stacks

We are prepared enough to understand how the definition goes. The main problem, however, is to understand why do we need them.

I will try to explain this, looking at the corresponding deformation problems.

We saw that passage to stacks was needed when we were studying the functors to groupoids

$$\mathcal{F} : Aff^{op} \longrightarrow \underline{Grp} :$$

the existence of non-trivial automorphisms of objects destroys the gluing property once we pass from a groupoid to the set of its isomorphism classes.

Thus, it is clear that we will have to move further to higher groupoids and higher stacks if our deformation problem is that of a 'higher object', for instance, of an additive or an abelian category.

In effect, categories form a 2-category, so one will assign to an affine scheme a 2-groupoid of deformations of a given category (whatever this may mean) with a given base.

All this is not very persuasive, since one has to be a higher category fan to be sure ~~deformations~~ moduli of categories or of higher categories appear in everyday life.

I will try to present an argument in the context of deformation theory.

Recall that deformations of an associative algebra A are described by the dg Lie algebra

$$\mathfrak{g} = C^\bullet(A,A)[1]_{\geq 0}$$

Recall as well that quasi-isomorphic dg Lie algebra concentrated in degrees ≥ 0, generate equivalent deformation functors (with values in groupoids).

Let $A = k[x_1, \dots, x_n]$. The cohomology of $C^\bullet(A,A)$ is well-known to be isomorphic to $\Lambda^\bullet_A T$ where T is the Lie algebra of vector fields on A. According to Kontsevich formality theorem, the dg Lie algebra $C^\bullet(A,A)[1]$ is equivalent to $\Lambda^\bullet_A T[1]$

But this is not true for the nonnegative parts!

$$C^\bullet(A,A)[1]_{\geq 0} \not\simeq \Lambda^\bullet_A T[1]_{\geq 0}.$$

The difference is quite essential:

as a result, any Poisson bracket can be quantized, but sometimes the quantized algebra cannot be glued out of the local data.

What can be glued is the corresponding category of modules.

Thus, Kontsevich's formality is applicable to deformations of the category of A-modules better than to deformations of A itself.

The difference is precisely in the (-1) terms of $C^{\cdot}(A,A)[1]$ which is A.

We have here a dg Lie algebra $\mathfrak{g}$ with $\mathfrak{g}^i = 0$ for $i < -1$, and it allows one to describe a functor

$$F^2_{\mathfrak{g}} : \mathrm{art}(k) \longrightarrow 2\text{-grp}$$

so that the objects of $F^2_{\mathfrak{g}}(R)$ are given, as before, by MC elements of $(m \otimes \mathfrak{g})^1$, morphisms by $(m \otimes \mathfrak{g})^0$, and 2-morphisms by $(m \otimes \mathfrak{g})^{-1}$.

This may explain why functors with values in Kan simplicial sets are a meaningful generalization of stacks. How to formally define them, we already know:

<u>Definition</u> A higher stack is a functor

$$\mathcal{F} : (\mathcal{C}, \tau)^{op} \longrightarrow Set_{\Delta}$$

to simplicial sets, satisfying the following conditions:

1) $\mathcal{F}(X)$ is Kan for all $X \in \mathcal{C}$

2) for any hypercovering $V_\bullet \to U$ the map

$$\mathcal{F}(U) \to \operatorname{holim} \{ n \mapsto \mathcal{F}(V_n) \}$$

is a weak equivalence.

In other words, a (higher) stack $\mathcal{F}$ is just a simplicial presheaf which is fibrant in the model category structure described 2 weeks ago.

Derived stacks do not generalize usual stacks; they extend the higher stacks: they are functors

$$\mathcal{F} : (dAff)^{op} \longrightarrow Set_{\Delta}$$

from a larger category to simplicial sets. Any derived stack has a restriction to Aff which is a "conventional" higher stack. Thus, we cannot give an example where derived stacks would not exist were they not derived.

Thus, our argument of the usefulness of derived stacks will be of a different nature.

Let us once more go back to deformation theory defined by a dg Lie algebra $\mathfrak{g}$

For each local artinian k-algebra (R, m) with $R = k \oplus m$ a Kan simplicial set ("∞-groupoid")

$$F^{\infty}_{\mathfrak{g}}(R) \quad \text{is assigned.}$$

Roughly speaking, its objects live in $m \otimes \mathfrak{g}^1$, its 1-arrows live in $m \otimes \mathfrak{g}^0$, etc.

We see that $\mathfrak{g}^i$ for $i > 2$ have ~~do not~~ no influence on $F^{\infty}_{\mathfrak{g}}(R)$.

Recall the empyrical fact that deformation problems usually have a dg Lie algebra attached which describes deformations. But the same dg Lie algebra defines ~~deformat~~ a Kan simplicial set also for any

(R, m) which is artin, local, and dg. One should only replace

$m \otimes \mathfrak{g}^1$ with $(m \otimes \mathfrak{g})^1$, $m \otimes \mathfrak{g}^0$ with $(m \otimes \mathfrak{g})^0$ et cetera. Fortunately, if (R, m) is non-~~neg~~ positively graded, the corresponding $F^{\infty}_{\mathfrak{g}}(R)$ depends on the quasiisomorphism type of R.

Thus, if there is a meaningful deformation problem described by a dg Lie algebra, one should be able to define a deformation functor on a greater class of algebras:

$$F_{\mathfrak{g}}^{\infty} : dgart(k)^{\leq 0} \longrightarrow Set_{\Delta}.$$

This functor should be homotopy-invariant, that is to satisfy the following:

Let $f : R \to R'$ be a quasiisomorphism in ~~of~~ $dgart(k)^{\leq 0}$. Then the induced map

$$F_{\mathfrak{g}}^{\infty}(R) \to F_{\mathfrak{g}}^{\infty}(R')$$

is a weak equivalence.

Definition ~~A f~~ In what follows

$dAff$ is the category dual to the category of simplicial commutative algebras.

A functor

$$\mathcal{F} : dAff^{op} \longrightarrow Set_{\Delta}$$

is called a derived stack if

(1) $\mathcal{F}(X)$ is Kan $\forall\, X \in dAff$

(2) $\mathcal{F}$ carries weak equivalence to a w.e.

(3) for each hypercovering $\{V_{\bullet} \to U\}$ (see below) the map

$$\mathcal{F}(U) \to \operatorname{holim}\{n \mapsto \mathcal{F}(V_n)\}$$

is an equivalence.

We have to define étale hypercovering

Necessary steps:
Model structure on simplicial algebras (we and fibrations are as in simplicial sets).

$\underline{\mathrm{Hom}}(A,B)$ simplicial Hom, $\mathbb{R}\underline{\mathrm{Hom}}(A,B)$ the derived functor.

For $f: A \to B$ cotangent complex $\mathbb{L}_f$ is defined as

$$\mathbb{L}_f = \Omega_{B'/A} \otimes^{L}_{B'} B$$

where $A \to B' \to B$ is the decomposition of f into a cofibration followed by a trivial fibration.

Then f is formally étale if $\mathbb{L}_f = 0$;
f is homotopically finitely presented if

$$\mathbb{R}\underline{\mathrm{Hom}}_A(-, B) \quad \text{commutes}$$

with filtered colimits.

Finally, f is étale if it satisfies both properties.