SM CATEGORIES AND THEIR REPRESENTATIONS

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1. Introduction

A monoidal category $\mathcal{C}$ is a category with a bifunctor $\otimes : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ endowed with an associativity constraint a which is a functorial isomorphism between two compositions

$$x, y, z \mapsto (x \otimes y) \otimes z \text{ and } x \otimes (y \otimes z)$$

satisfying the pentagon identity.

One requires the existence of unit object which is an object $\mathbf{1}$ together with functorial isomorphisms $\mathbf{1} \otimes x \to x$, $x \otimes \mathbf{1} \to x$, compatible with one another.

Any monoidal category is equivalent to a strict monoidal category, but one has a bad feeling assuming all monoidal categories are strict.

A symmetric monoidal (or SM or tensor) category is a monoidal category together with a commutativity constraint which is an isomorphism of the bifunctors $\otimes \longrightarrow \otimes \circ\tau$ where τ is the permutation $(x, y) \mapsto (y, x)$. Of course, there is a long list of compatibilities.

Once more, any SM category is equivalent to the one with strict monoidal structure. We prefer, however, to present another way of looking at SM categories.

2. SM versus PT

A SM category assigns to each finite set I a functor $\otimes^I : \mathcal{C}^I \to \mathcal{C}$ (in particular, the empty set defines an object in $\mathcal{C}$).

The compatibility between different $\otimes^I$ is described as follows.

For any map $a : I \to J$ of finite sets a natural isomorphism a

$$\otimes^I \longrightarrow \otimes^J \circ \prod_{j\in J} \otimes^{a^{-1}(j)}$$

which is associative in an obvious sense.

It is not completely obvious how to define functors between SM categories using this approach. We will leave this question for a while to introduce an important generalization of SM categories called PT (pseudotensor) categories. The notion of a functor between PT categories will be straightforward. This will allow us to define a SM functor as a PT functor satisfying some natural restrictions.

2.1. **Pseudotensor category.** There is a very important weaker notion, that of pseudotensor category.

A pseudotensor (PT) category M has a class of objects which will be denoted as $[M]$; for any finite collection of objects $c : I \to [M]$ and for any $d \in [M]$ a set $M(c, d)$ of "multi-arrows".

The composition is defined for any map $f : I \to J$ of finite sets and collections of objects

$$c : I \to [M], \quad d : J \to [M], e \in [M].$$

This is a map

$$M(d, e) \times \prod_{j \in J} M(c_j, d(j)) \longrightarrow M(c, e) \tag{1}$$

satisfying the obvious (strict) associativity condition. Here in the formula (**??**) c_j denotes the restriction of $c : I \to [M]$ to $f^{-1}(j)$.

We also assume the existence of identity maps $\mathrm{id}(a) \in M(\{a\}, a)$ for each object a (and a singleton I).

Note that among surjective maps there are permutations $s : I \to I$ commuting with c. These permutations form a subgroup $\mathrm{Aut}(c)$ and so an action of this subgroup on $M(c, d)$ is defined.

More generally, any isomorphism $\theta : I \to J$ defines an isomorphism, for each $c : J \to [M]$, from $M(c, d)$ to $M(c\theta, d)$. We denote this isomorphism by θ^*.

The notion of PT category is a generalization of the notion of operad.

In fact, a PT category with one object is precisely what is called operad (in the category of sets).

This is why PT categories have another name — that of *colored operads*. Objects of the PT category are colors and the operations take into account the color of of the arguments as well as the color of the result.

2.1.1. Unitality

In our definition finite sets I are allowed to be empty and all maps $f : I \to J$ are allowed. This is a so-called *unital* version of the notion, in the sense that it corresponds to SM categories with a unit object (see below).

From the point of view of operads this just means that our colored operad algebras are allowed to have constants (nullary operations).

A non-unital version also makes sense — one has to require only that the set I in the definition is nonempty, and to consider only surjective maps $f : I \to J$ (to ensure nonempty fibers).

2.1.2. Non-symmetric versions

If one replaces finite sets with totally ordered finite sets and the maps of finite sets with the monotone maps, we get a notion of PM (pseudomonoidal) category which generalizes the notion of monoidal category in the same way PT categories generalize the notion of SM category. As in the symmetric case, there are unital and nonunital versions of the notion.

Pseudomonoidal categories are multicolored versions of *asymmetric operads.*

2.1.3. SM categories

Any SM category defines a PS category as follows: We define $M(\{A_i\}_{i\in I}, B)$ to be $\mathrm{Hom}(\otimes^I A_i, B)$ in the obvious notation.

Let M be a PT category. Let us look what makes M a SM category. First of all, $[M]$ together with the operations defined by the singletons, give a category. Furthermore, the composition operations with the identity id_d gprovide a functor structure on the assignment $d \mapsto M(\{c(i)\}, d)$.

Assume that the functors $d \mapsto M(\{c(i)\}, d)$ are representable for each collection $c : I \to [M]$. We can denote a/the representing object by $\otimes^I c(i)$. Then for each map $f : I \to J$ a canonical map

$$\otimes^I c(i) \longrightarrow \otimes^J \left(j \mapsto \otimes^{f^{-1}(j)} c(i) \right) \tag{2}$$

is defined. Assume that all maps (**??**) are isomorphisms. Then we get a SM structure on M.

This structure is not completely unique because of the choice we have had in representing the functors $\otimes^I$. Thus, PT structure on a category, even if it comes from a SM structure, is more canonical than the latter.

This explains why the notion of a functor between SM categories can be better understood using the language of PT categories.

Note that the non-unital version of PT category corresponds to the non-unital version of tensor categories.

2.2. **Pseudo-tensor functors.** Given two PT categories M and N, a PT functor is defined in an obvious way: this is a map $f : [M] \to [N]$ together with a compatible collection of maps

$$M(c, d) \to N(f \circ c, f(d))$$

for each $c : I \to [M]$.

Extend below

If we apply the notion of PT functor to SM categories, we will get what is conventionally called *a lax SM functor.*

2.3. **SM functors.** Let M and N be two tensor categories and let $f : M \to N$ be a pseudo-tensor functor. This means that a compatible collection

$$\mathrm{Hom}(\otimes^I a_i, b) \to \mathrm{Hom}(\otimes^I f(a_i), f(b))$$

is given. By naturality, this is the same as a compatible collection of morphisms

$$\otimes^I f(a_i) \longrightarrow f(\otimes^I a_i). \tag{3}$$

A PT functor $f : A \longrightarrow B$ is called a tensor functor if the collection (**??**) is the collection of isomorphisms.

2.4. **Examples.**

2.4.1. PROPs

An example of tensor category, a close relative of various Cob_n, is the 1960' PROP.

PROP is a tensor category whose objects are the natural numbers. If you choose your favorite type of algebras (sure, these are Frobenius algebras), you have an appropriate PROP defined by the property that the collection of maps from m to n is the collection of all "universal" operations from $A^{\otimes m}$ to $A^{\otimes n}$ where A is an algebra of your chosen type.

For instance, operad algebras have an appropriate PROP (PROP generated by an operad). A cyclic operad (for those who know what is this) define even three different PROPs (algebra version as above, metric algebra version and Casimir algebra version). Thus, we have a sequence of generalizations

Operads → PROPs → SM categories → PT categories = Color operads.

The composition is in no sense identity.

2.5. **Algebras over PROPs.** These are SM functors from a PROP $\mathcal{P}$ to the tensor category of vector spaces. In particular, bialgebras, Lie bialgebras, Frobenius algebras etc are tensor functors to `Vect` from an appropriate PROP.

More generally, for a SM category M M-algebras are SM functors from M to `Vect`. If $f : M \to N$ is a SM functor, one defines $f^* : \mathtt{Alg}(N) \to \mathtt{Alg}(M)$ by restriction of scalars.

Let us show that one cannot expect this functor to have left adjoint. In effect, let, for instance, N be the PROP for Hopf algebras and let M be the "empty" PROP with symmetric groups acting on each of the objects and no morphisms between different objects (an even better description: the tensor category of finite sets and isomorphisms, with the tensor structure given by disjoint union).

The algebras over M are just the vector spaces, so the functor $f_!$ applied to V would be the free Hopf algebra generated by V. We know that it does not exist.

2.6. **Representations of PT categories.** A representation of a PT category M is a PT functor $M \to$ `Vect`. Thus, if M is SM category, its representation is a weaker form of M-algebra.

The behavior of representations is the one we would like to have for algebras.

The following theorem is an abstract nonsense.

2.6.1. **Theorem.** *Let $f : M \to N$ be a PT functor. There is a pair of adjoint functors*

$$f_! : \mathtt{Rep}(M) \rightleftarrows \mathtt{Rep}(N) : f^*$$

where f^* is the restriction of scalars functor, assigning to $X : N \to \mathtt{Vect}$ the composition $f^*(X) = X \circ f : M \to N \to \mathtt{Vect}$.

In the special case M has one object, that is M is just an operad, a representation of M is just an operad algebra. In general, there is a notion of free representation — the one generated by a collection of vector spaces numbered by colors.

Let us describe free representations in a more detail. Let $V : [M] \to \mathtt{Vect}$ be a collection of vector spaces numbered by the colors.

The free algebra $F(V)$ is the collection of vector spaces $F(V)_d$, $d \in [M]$, described as follows. Collections $c : I \to [M]$ form a groupoid denoted $Fin/[M]$. To each $c \in Fin/[M]$ we assign the vector space

$$M(c,d) \otimes \bigotimes_{i\in I} V_{c(i)}.$$

This gives rise to a functor $\mathfrak{F}(V)_d : Fin/[M] \to \mathtt{Vect}$; its colimit is the component $F(V)_d$ of the free representation of M generated by V.

The functor $f_!$ can be explicitly described as follows. It carries the free M-representation generated by a collection V to the free N-representation generated by the same collection. For a general M-representation A one can present it as a quotient $F_M(V)/I$ of a free M-representation by an ideal, and then $f_!(A)$ is the quotient of $F_N(A)$ by the ideal generated by I.

3. DG version

The notions of PT category and SM category presented above have their enriched counterparts. One can use as the basic category any SM category (and, it seems, any PT category).

We restrict ourselves with the category of complexes over a fixed commutative ring k, with the tensor product of complexes as the SM operation.

There are much more general versions, when all categories are enriched over a certain SM model category, see [**?**].

From now on we are talking about colored operads instead of PT categories.

3.1. **Theorem.** *Let M be a dg colored operad over a commutative ring $k \supset \mathbb{Q}$. The category $\mathtt{Rep}(M)$ has a model category structure with quasiisomorphisms as weak equivalences and surjections as fibrations.*

A map $f : M \to N$ of colored dg operads (we assume $[M] = [N]!$) is called weak equivalence if for all $c : I \to [M], \quad d \in [M]$ the map $M(c,d) \to N(c,d)$ is a quasiisomorphism.

3.2. **Theorem.** *Let $f : M \to N$ be a quasiisomorphism of dg colored operads over $k \supset \mathbb{Q}$. Then the pair of adjoint functors $(f_!, f^*)$ is a Quillen equivalence.*

The proof of a somewhat more general result is given below.

3.3. **Free algebras, limits and the like.** Let M be a dg PT category and let as above $[M]$ be the collection of objects of M. We will consider $[M]$ as a discrete PT category with the identity presenting the only existing morphisms.

The embedding $[M] \to M$ gives rise to the pair of adjoint functors

$$F : \mathtt{Rep}([M]) \rightleftarrows \mathtt{Rep}(M) : \sharp,$$

where f is the free representation functor described above and $\sharp$ is the forgetful functor. Thus, limits in $\mathtt{Rep}(M)$ exist and commute with $\sharp$; colimits in $\mathtt{Rep}(M)$ exist and can be described as appropriate quotient of the free representation generated by the colimit of the diagram in $\mathtt{Rep}([M])$ obtained by applying $\sharp$ to the original diagram in $\mathtt{Rep}(M)$.

3.4. **Σ-splitting.** Let M be a PT category. We define a new PT category M^Σ which "remembers the order of operands". The objects of M^Σ and of M are the same: $[M^\Sigma] = [M]$.

Let $c : I \to [M]$ be a collection of objects of M numbered by a finite set I and let $d \in [M]$. We define

$$M^\Sigma(c,d) = \bigoplus_{\theta : I \simeq \langle n \rangle} M(c,d), \tag{4}$$

where $\langle n \rangle = \{1, \dots, n\}$ is the standard (totally ordered) n-element set. Note that the choice of θ is equivalent to the choice of a total order on I.

Let us define the composition. Let $f : I \to J$ be a map of sets and let $c : I \to [M]$ and $d : J \to [M]$ be collections. The map

$$M^\Sigma(d,e) \times \prod_j M^\Sigma(c_j, d(j)) \longrightarrow M^\Sigma(c,e) \tag{5}$$

is defind as follows. Choice of a total order on J together with a choice of total orders on each fiber $f^{-1}(j)$ defines a lexicographical total order on I: if two elements of I belong to different fibers, we compare the fibers, and if they belong to the same fiber, we compare them inside the fiber. With the described above choice of the orderings, the corresponding component of the map (**??**) is given by the composition (**??**) for M.

For example, if M is the operad for commutative algebras, M^Σ is the operad for associative algebras.

3.4.1. **Remark.** The operation $M \mapsto M^\Sigma$ described above can be better understood in the context of PM categories (asymmetric color operads). One has an obvious forgetful functor

$$\sharp : \mathtt{PTcat}([M]) \longrightarrow \mathtt{PMcat}([M])$$

assigning to any color operad its asymmetric counterpart. The functor $\sharp$ admits a left adjoint functor which we denote $M \mapsto M^\Sigma$; if M is an asymmetric color operad, the color operad M^Σ is defined by the formula

$$M^\Sigma(c,d) = \bigoplus_{\theta: I \to \langle n \rangle} M((c,\theta),d), \tag{6}$$

where the pair (c,θ) describes a colored collection c numbered by the totally ordered set (I,θ).

The endofunctor $M \mapsto M^\Sigma$ described in the paper is in fact the composition of this pair of adjoint functors.

3.4.2. One has a canonical map

$$\pi : M^\Sigma \longrightarrow M$$

summing up the components corresponding to different orderings (this is the standard adjunction map in terms of Remark above).

One defines Σ-splitting as a collection of splittings $t = t^{c,d} : M(c,d) \longrightarrow M^\Sigma(c,d)$ of the canonical map π described above, satisfying the properties (SPL), (INV), (COM) which will be specified later on. We will usually omit the superscript (c,d) from the notation.

A σ-splitting t is defined by a collection of its components $t_\theta : M(c,d) \to M(c,d)$ numbered by different orderings of I.

The first two requirements for Σ-splitting are

(SPL) The map t splits π, that is $\sum_\theta t_\theta = \mathrm{id}$.

(INV) For any isomorphism $f : c' \to c$ (that is, a bijection $f : I' \to I$ satisfying $c' = c \circ f$) the induced isomorphism $f^* : M(c,d) \to M(c',d)$ commutes with t. The latter means that

$$f^* \circ t_\theta = t_{\theta f} \circ f^*.$$

The last requirement of Σ-splitness is a weak form of compatibility of the splitting with the compositions.

Let $c : I \to [M],\ d : J \to [M],\ a : K \to [M], a' : K' \to [M]$ be finite collections in M. Let $f : I \to J$ be a map of finite sets and let $\phi : a \to a'$ be an isomorphism of collections (that is, a bijection $\phi : K \to K'$ such that $a = a' \circ \phi$).

Gluing the above data, one gets collections $c \sqcup a : I \sqcup K \to [M]$ and $d \sqcup a' : J \sqcup K' \to [M]$, as well as a map of finite sets $f \sqcup \phi : I \sqcup K \to J \sqcup K'$.

The requirement (COM) describes a compatibility of the splitting with the composition in M

$$M(d \sqcup a', e) \otimes \bigotimes_{j \in J} M(c_j, d(j)) \longrightarrow M(c \sqcup a, e) \tag{7}$$

induced by the morphism $f \sqcup \phi$.

We are now able to formulate the third requirement of Σ-splittings.

(COM) The following diagram is commutative.

$$\begin{array}{ccc}
M(d \sqcup a', e) \otimes \bigotimes_{j \in J} M(c_j, d(j)) & \longrightarrow & M(c \sqcup a, e) \\
\downarrow t & & \downarrow t \\
\bigoplus_{\eta: J \sqcup K' \simeq \langle |J|+|K| \rangle} M(d \sqcup a', e) \otimes \bigotimes_{j \in J} M(c_j, d(j)) & & \bigoplus_{\theta: I \sqcup K \simeq \langle |I|+|K| \rangle} M(c \sqcup a, e) \\
\downarrow q & & \downarrow q \\
\bigoplus_{k \in K'} M(d \sqcup a', e) \otimes \bigotimes_{j \in J} M(c_j, d(j)) & \longrightarrow & \bigoplus_{k \in K} M(c \sqcup a, e)
\end{array} \tag{8}$$

The upper vertical arrows in the diagram are defined by splitting of $M(d \sqcup a', d)$ $(c \sqcup a, e)$ respectively. In order to define the lower vertical arrows we will introduce the following notation. For each ordering η of the set $J \sqcup K'$ we denote by $\min_{K'}(\eta)$ the smallest element of the subset K' of $J \sqcup K'$. In the same manner we define $\min_K(\theta)$. Now the maps q send each η-component (resp., θ-component) to the corresponding $\min_{K'}(\eta)$-component (resp., $\min_K(\theta)$-component).

3.4.3. **Remark.** There is another (stronger) version of Σ-splitness where q is replaced with a projection to the sum over orderings of K' (resp., of K). It seems more satisfactory aesthetically; in this formulation the condition (INV) is its special case for $I = J = \emptyset$.

This stronger version is the definition used in [**?**].

3.4.4. **Example.** In the case $k \supset \mathbb{Q}$ the map

$$t_\theta(m) = \frac{1}{n!} m$$

defines a Σ-splitting.

3.4.5. **Example.** Let N be an asymmetric color operad and let $M = N^\Sigma$. The canonical map of asymmetric operads $N \to M^\sharp$ defines a map of operads $t : M \to M^\Sigma$ splitting the canonical map $M^\Sigma \to M$. Any such map satisfies obviously the conditions (SPL), (INV), (COM).

3.4.6. Color operad for color operads

Color operads on a fixed set of objects S can be also described as algebras over a certain color operad which we will denote $\mathtt{Op}(S)$. The objects of $\mathtt{Op}(S)$ are triples $(I, c : I \to S, d \in S)$; operations are generated by the compositions assigned to collections $(c : I \to S, d : J \to S, f : I \to J, e \in S)$. The relations between the operations are defined by the strict associativity of the operations.

There are other variants of this operad: one can consider nonunital versions, as well as asymmetric operads.

It is worthwhile to understand when do such color operads are Σ-split. We suspect that the color operad for nonunital color operads (operads with no constant term) comes from an asymmetric operad, and is, therefore, Σ-split. We suspect the same is true form the operad for asymmetric operads.

It is also clear that in case $k \supset \mathbb{Q}$ the color operad for color operads is Σ-split.

This would give a uniform proof for the existing results about model category structure on the category of DG operads, see [?], [?].

3.5. **Proof of ??.** In this subsection we prove the following result which by **??** immediately implies Theorem **??**.

3.5.1. **Theorem.** *Let M be a Σ-split dg colored operad over a commutative ring k. The category $\mathtt{Rep}(M)$ has a model category structure with quasiisomorphisms as weak equivalences and surjections as fibrations.*

A standard idea of proving statements similar to **??** is to use a pair of adjoint functors with a model category and to "transfer" the model structure to the category in question.

We will use the following easy result (can be found in [?], sec. 2).

3.5.2. **Theorem.** *Given a pair of adjoint functors*

$$F : C(k) \rightleftarrows \mathcal{C} : R$$

where $C(k)$ is the category of complexes over a commutative ring k, satisfying the conditions:

- T1. *$\mathcal{C}$ has finite limits and arbitrary colimits; the functor R commutes with filtered colimits.*
- T2. *For any $A \in \mathcal{C}$ the map $R(A) \to R(A \coprod F(H))$ where H is the standard contractible complex concentrated in the degrees $n-1$ and n, is a quasiisomorphism.*

Then $\mathcal{C}$ admits a model structure with the following classes of arrows.

- *f is a weak equivalence in $\mathcal{C}$ iff $R(f)$ is a quasiisomorphism.*
- *f is a fibration in $\mathcal{C}$ iff $R(f)$ is surjective.*
- *f is a cofibration iff it satisfies the LLP with respect to all trivial fibrations.*

Moreover, the classes of cofibrations are generated by $F(k[-n]) \to F(H)$ and the classes of trivial cofibrations are generated by $F(0) \to F(H)$.

In the above theorem the category $C(k)$ can be replaced with $C(k)^S$ where S is an arbitrary set.

Let M be pseudotensor category over $C(k)$ (a DG pseudotensor category). We apply the above theorem to the categories $\mathcal{C} = \mathtt{Rep}(M)$ and $C(k)^{\mathrm{Ob}(M)}$. The functor R is the forgetful functor; F is the free algebra functor.

The forgetful functor commutes with filtered colimits. Thus, we have to check only that the map $A \to A \coprod F(H_a)$ is a quasiisomorphism for H_a standard contractible complex concentrated at a color $a \in [M]$.

3.5.3. Extending homotopy to a free algebra Let $V = \{V_d | d \in [M]\}$ be a collection of complexes, $\alpha : V \to V$ an endomorphism and h a homotopy, that is a degree -1 map satisfying the condition

$$dh = \mathrm{id}_V - \alpha.$$

The endomorphism α induces an endomorphism $F(\alpha) : F(V) \to F(V)$; we will present an explicit homotopy between $\mathrm{id}_{F(V)}$ and $F(\alpha)$ which we will denote $F(h)$. The homotopy $F(h)$ will be based on a Σ-splitting of M. Thus, this construction always works in the case $k \supset \mathbb{Q}$. The condition of Σ-splitness cannot be completely ignored since, for example, free commutative algebras of a contractible complex are not contractible in positive characteristic.

Recall that one has a morphism of PT categories $\pi : M^\Sigma \to M$ identical on the objects, as well as a Σ-splitting $t : M(c,d) \to M^\Sigma(c,d)$.

We are now ready to define a homotopy H on $F(V)$. Recall that $F(V)$ is the direct limit of collections $F_c(V)$ where $c : I \to [M]$ and

$$F_c(V)_d = M(c,d) \otimes \bigotimes_i V_{c(i)}.$$

We will define a degree -1 endomorphism H of each separate $F_c(V)_d$ compatible with the isomorphisms $c \to c'$ of collections. It is given by the composition

$$M(c,d) \otimes \bigotimes_i V_{c(i)} \xrightarrow{t} \bigoplus_{\theta : I \simeq \langle n \rangle} M(c,d) \otimes \bigotimes_i V_{c(i)} \xrightarrow{S} \bigoplus_{\theta : I \simeq \langle n \rangle} M(c,d) \otimes \bigotimes_i V_{c(i)} \xrightarrow{\pi} M(c,d) \otimes \bigotimes_i V_{c(i)}, \tag{9}$$

with the map S being defined at the θ-component as

$$S_\theta = \sum_i \mathrm{id}_M \otimes \alpha^{i-1} \otimes h \otimes \mathrm{id}^{n-i}.$$

3.5.4.

In order to check that the morphism $A \to A\coprod F(H_a)$ is a quasiisomorphism for a standard contractible H_a, one proceeds as follows.

Let $A'=A\oplus H_a$. Then $A\coprod F(H_a)$ can be described as the quotient of $F(A')$ by the ideal generated by the kernel of the natural map $F(A) \longrightarrow A$.

Let $\alpha : A'\to A'$ be zero on H_a and id_A on A. Let $h:A'\to A'$ be a degree -1 map vanishing on A such that $dh=\mathrm{id}-\alpha$. Then h defines a homotopy $F(h)$ on $F(A')$ extending h.

Let $\mathcal{I}$ the the kernel of the natural projection $F(A)\to A$ and let $\mathcal{J}$ be the ideal in $F(A')$ generated by $\mathcal{I}$. We check below that $H(\mathcal{J})\subset\mathcal{J}$ and this induces a homotopy on the quotient $F(A')/J=A\coprod F(H_a)$.

3.5.5. Action of $F(h)$ on $F(A')$

Some relevant notation. For $c:I\to[M]$ and $n\geq 0$ we define $c^{*n}:I\sqcup\langle n\rangle\to[M]$ by the formula

$$c^{*n}(i)=c(i) \text{ for } i\in I;\ c^{*n}(k)=a \text{ for } k\in\langle n\rangle;$$

The free representation $F(A')$ with $A'=A\oplus H_a$ is the colimit of the complexes

$$M(c^{*n},e)\otimes\bigotimes_i A_{c(i)}\otimes H_a^{\otimes n}$$

The homotopy $F(h)$ is defined by the components S_θ numbered by the total ordering θ of the set $I\sqcup\langle n\rangle$. Since h vanishes on A and α is identity on A and vanishes on H_a, the map S_θ has form

$$S_\theta=\mathrm{id}_M\otimes\mathrm{id}_A\otimes\mathrm{id}^{\otimes k-1}\otimes h\otimes\mathrm{id}^{n-k}$$

where the homotopy h is applied to the $k:=\min_{\langle n\rangle}(\theta)$ component of H_a.

3.5.6. End of the proof

We keep the notation of **??**.

The ideal $\mathcal{J}$ in $F(A')$ generated by $\mathcal{I}$, is spanned by the expressions

$$u\otimes\delta\otimes\bigotimes_{i\in I-\{0\}} b_i\otimes\bigotimes_{k\in\langle n\rangle} x_k \tag{10}$$

where $c:I\to[M],\ 0\in I,\ c(0)=c_0,\ \delta\in\mathcal{I}_{c_0},\ u\in M(c^{*n},e),\ b_i\in A_{c(i)}$ and $x_k\in H_a$.

We will now explicitly calculate the image of (**??**) under the homotopy $F(h)=\sum_\theta S_\theta\circ t_\theta$ to make sure it belongs to $\mathcal{J}$.

Let

$$t(u)=\sum_{\theta:I^{*n}\to\langle|I|+n\rangle} t_\theta(u).$$

We claim that $F(h)$ carries (**??**) to the sum

$$\sum_\theta u_\theta \otimes \delta \otimes \bigotimes_{i\in I-\{0\}} b_i \otimes \bigotimes_{k\in\langle n\rangle} x_{\theta,k}, \tag{11}$$

where

$$x_{\theta,k} = \begin{cases} x_k, & k \neq \min_{\langle n\rangle}(\theta) \\ h(x_k), & k = \min_{\langle n\rangle}(\theta) \end{cases} \tag{12}$$

It is sufficient to check the formula (**??**) in case δ is a monomial in $F(A)$:

$$\delta = m \otimes \bigotimes_{j\in J} a_j \tag{13}$$

with $m \in M(d, c_0),\ d : J \to [M],\ a_j \in A_{d(j)}$.

Replace δ in (**??**) with the expression (**??**). We get a monomial

$$z := u \circ m \otimes \bigotimes_{j\in J} a_j \otimes \bigotimes_{i\in I-\{0\}} b_i \otimes \bigotimes_{k\in\langle n\rangle} x_k, \tag{14}$$

where $u \circ m$ denotes the composition of u and m belonging to $M(c \circ d, e)$ where $c \circ d : I - \{0\} \sqcup J \to [M]$ is the restriction of $c \sqcup d$, whose image under $F(h)$ is given by the formula

$$F(h)(z) = \sum_{\eta: I-\{0\}\sqcup J \simeq \langle |I|+|J|-1\rangle} S_\eta \circ t_\eta. \tag{15}$$

By the axiom (COM) of Σ-splitness applied to the surjection $I - \{0\} \sqcup J \longrightarrow I$ sending the elements of J to 0 and the elements of $I - \{0\}$ to themselves, we deduce that $F(h)(z)$ is equal to (**??**).

3.6. **Comparison.**

3.6.1.

Let $f : M \to N$ be a morphism of PT categories. By the general nonsense, a pair of adjoint functors

$$f_! : \mathtt{Rep}(M) \rightleftarrows \mathtt{Rep}(N) : f^*$$

is defined.

The functor f^* is just the forgetful functor. The functor $f_!$ carries free representations to free representations, etc.

3.6.2. **Theorem.** *The pair of functors $(f_!, f^*)$ is Quillen pair. That is, $f_!$ preserves cofibrations and trivial cofibrations (or, what is equivalent, f^* preserves fibrations and trivial fibrations).*

In effect, this is clearly obvious.

As a result, one has an adjoint pair of derived functors

$$\mathbf{L}f_! : \mathrm{Ho}(\mathtt{Rep}(M)) \rightleftarrows \mathrm{Ho}(\mathtt{Rep}(N)) : f^* = \mathbf{R}f^*$$

between the corresponding homotopy categories.

3.6.3. PT equivalence

Let $f : M \to N$ be a PT morphism. If one forgets the multivaried operations, M and N become dg categories which we denote as M_1 and N_1. The PT morphism f induces a functor $f_1 : M_1 \to N_1$.

A morphism $f : M \to N$ is called equivalence if for each $c : I \to [M]$ and $d \in [M]$ the morphism $M(c,d) \to N(f \circ c, f(d))$ is a quasiisomorphism and the functor $H_0(f_1) : H_0(M_1) \to H_0(N_1)$ is an equivalence of categories.

The latter condition means essential surjectivity of f: for any $n \in [N]$ there exists an object $m \in [M]$ and a homotopy equivalence between $f(m)$ and n in N_1.

3.6.4. **Theorem.** *Assume $f : M \to N$ is a PT equivalence of Σ-split PT categories* preserving the Σ-splitting. *Then the Quillen pair $(f_!, f^*)$ is a Quillen equivalence.*

The proof of the theorem is the direct generalization of Theorem 4.7.4 of [**?**].

Let $a : B \to C$ be a map of N-representations. If a is a quasiisomorphism, $f^*(a)$ is as well quasiisomorphism. The converse is also valid because of the essential surjectivity of f. In this case, in order to prove that the pair $(f_!, f^*)$ is a Quillen equivalence, it is sufficient to prove that if A is a cofibrant M-representation, the natural map

$$A \longrightarrow f^*(f_!(A))$$

is a quasiisomorphism.

First of all, we can assume that A is standard cofibrant, that is, obtained from the initial representation by a sequence of attachments of generators killing cycles. Next, we reduce everything to the case A is finitely generated — this is possible since $f_!$ and f^* commute with filtered colimits.

From now on A is a finitely generated standard cofibrant M-prepresentation. The well-ordered set of generators of A is given by a map

$$v : I \to [M]$$

from the set of indices to the set of objects of M.

For each $d \in [M]$ the complex A_d has an increasing filtration numbered by the lexicografically ordered set of multi-indices $m : I \to \mathbb{N}$, or, what is the same, by isomorphism classes of surjective maps $\hat{m} : J \to I$ (the correspondence assigns to a surjection $\mu : J \to I$ a map $m : I \to \mathbb{N}$ defined by the formula $m(i) = |\mu^{-1}(i)|$).

The corresponding associated graded piece will be isomorphic to a quotient of $M(v \circ \hat{m}, d)$ by the automorphism group of the collection $v \circ \hat{m} : J \to [M]$.

Thus, in order to prove the theorem, we have to establish that the map f induces a quasiisomorphism of the quotients of $M(v \circ \hat{m}, d)$ and of $N(v \circ \hat{m}, d)$ by the automorphism group of the collection $v \circ \hat{m} : J \to [M]$.

Tthe latter map is a retract of the corresponding map induced by $f^\Sigma : M^\Sigma \to N^\Sigma$ which is a quasiisomorhism under the conditions of the theorem.

3.7. **Homotopy algebras.** Temporarily k is a field.

Let M be a SM dg category. A representation A of M is called a homotopy algebra if the structure maps

$$\otimes^I A(x_i) \to A(\otimes^I x_i)$$

are all quasiisomorphisms. A representation quasiisomorphic to a homotopy algebra, is itself a homotopy algebra. We denote $\mathrm{Ho}^{\mathrm{ha}}(M)$ the full subcategory of the homotopy category of M-representations whose objects are homotopy M-algebras.

Let $f : M \to N$ be a weak equivalence of Σ-split SM categories. According to Theorem **??** the pair of adjoint functors $(f_!, f^*)$ between the categories of representations of M and N, is a Quillen equivalence. The functor $f^* : \mathtt{Rep}(N) \to \mathtt{Rep}(M)$ preserves homotopy algebras.

This proves the following

3.7.1. **Theorem.** *Let $f : M \to N$ be a SM functor, which is a PT equivalence. Assume that f preserves Σ-splittings. Then the pair $(f_!, f^*)$ induces an equivalence of the homotopy categories of homotopy algebras*

$$\mathbf{L}f_! : \mathrm{Ho}^{\mathrm{ha}}(M) \rightleftarrows \mathrm{Ho}^{\mathrm{ha}}(N) : \mathbf{R}f^*.$$

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