**DIFFERENTIAL GEOMETRY. LECTURES 9-10, 23-26.06.08**

Let us provide some more details to the definintion of the de Rham differential.

Let $\mathbb{V},\ \mathbb{W}$ be two vector bundles and assume we want to define an operator

$$a : \Gamma(\mathbb{V}) \to \Gamma(\mathbb{W}). \tag{1}$$

Assume $s \in \Gamma(\mathbb{V})$. In order to define a section $a(s)$ of $\mathbb{W}$, it is enough to define a collection of sections $a(s)_i$ of $\mathbb{W}$ over open subsets $U_i$ of $X$ covering $X$, so that the restrictions $a(s)_i|_{U_{i,j}}$ and $a(s)_j|_{U_{i,j}}$ coincide. This is usually what one does in order to construct an operator (1).

Let us construct in this way the de Rham differential

$$d : C^\infty(X) \longrightarrow \Omega^1(X).$$

In our case $\mathbb{V} = \mathbf{1},\ \mathbb{W} = T^*X$ and we choose the cover of $X$ by the charts of an atlas.

If $\phi : D \longrightarrow U \subset X$ is a map and $s \in C^\infty(X)$ the function $s|_U$ is a function of the standard variables $x_1, \ldots, x_n$ in $\mathbb{R}^n$ and we put $ds = \sum_i \frac{\partial(s\phi)}{\partial x_i} dx_i$.

Let us check that the section of $T^*X$ over $U$ so obtained coincide at the intersections of the charts.

If $\phi_1$ and $\phi_2$ are two charts as above, we will have

$$\omega_1 = \sum_j \frac{\partial(s \circ \phi_1)}{\partial x_j}, \tag{2}$$

$$\omega_2 = \sum_j \frac{\partial(s \circ \phi_2)}{\partial x'_j}, \tag{3}$$

— two sections of $T^*X$ over $U_1$ and $U_2$ respectively, written in the coordinates of $\phi_1$ and $\phi_2$. Compatibility means that the transition function $\theta = \phi_2^{-1} \circ \phi_1$ expressing the dependence of $x'_j$ on $x_j$, carries $\omega_1$ to $\omega_2$. Since $s \circ \phi_1 = s \circ \phi_2 \circ \theta$, one has

$$\omega_1 = \sum_j \frac{\partial(s \circ \phi_2 \circ \theta)}{\partial x_j} dx_j == \sum_j \sum_i \frac{\partial(s \circ \phi_2)}{\partial x'_i} \frac{\partial x'_i}{\partial x_j} dx_j$$

and this is exactly what is supposed to be.

3.7. **More about tensors.** Similarly to the above, one can check that the operator $d : \Omega^k \to \Omega^{k+1}$ defined in the previous lecture, makes sense.

It is, however, more clear what happens, if one uses a multiplicative structure of the de Rham complex.

Note two "algebraic" operations with tensors. Let us agree to use the following notation. $T^p_q(X)$ or just $T^p_q$ is the bundle of $(p,q)$-tensors. The sections of this bundle are called the $(p,q)$-tensors; we denote $\mathcal{T}^p_q = \Gamma(T^p_q)$. We have already had special cases of this notion: $\mathcal{T}^1_0$ is just $\mathcal{T}$ and $\mathcal{T}^0_1$ is $\Omega^1$.

**1. Product.** Since $T^p_q = T^p(TX) \otimes T^q(T^*X)$, there is an obvious map $T^p_q \otimes T^{p'}_{q'} \longrightarrow T^{p+p'}_{q+q'}$. This gives a product operation

$$\mathcal{T}^p_q \times \mathcal{T}^{p'}_{q'} \longrightarrow \mathcal{T}^{p+p'}_{q+q'}.$$

**2. Contraction.** The map $T^*X \otimes TX \longrightarrow \mathbf{1}$ is defined by pairing $T^*_x(X)$ with $T_xX$. This allows one to define a collection of maps $T^p_q \longrightarrow T^{p-1}_{q-1}$, for each choice of a vector and a covector argument.

Passing to the sections, we have contraction operations

$$c : \mathcal{T}^p_q \longrightarrow \mathcal{T}^{p-1}_{q-1}.$$

The exterior power $\wedge^k V$ can be identified with the antisymmetric part of $T^kV$: the natural projection $T^kV \longrightarrow \wedge^k V$ is split by the map

$$x_1 \wedge \ldots \wedge x_k \mapsto \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(k)}. \tag{4}$$

The natural map $\wedge^k V \otimes \wedge^l V \longrightarrow \wedge^{k+l} V$ is defined so that the projection from $T^kV$ to $\wedge^k V$ preserves the multiplication.

*Exercise. Check that the map (4) does not preserve the multiplication.*

Now we can formulate the property characterizing the de Rham differential.

3.7.1. **Theorem.** *There is a unique collection of linear maps $d : \Omega^k \longrightarrow \Omega^{k+1}$ satisfying the following properties.*

- *Its restriction to $\Omega^0 = C^\infty$ is as described above.*
- *It satisfies a (skew) Leibniz formula*

  $$d(\omega \cdot \omega') = d(\omega) \cdot \omega' + (-1)^k \omega \cdot d(\omega'),$$

  *where $\omega \in \Omega^k$.*
- *$d \circ d = 0$.*

*Proof.* Let us prove first of all uniqueness. Assume such collection of operators exists. Then in local coordinates one has to have

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \tag{5}$$

— because of Leibniz formula and the condition $dd = 0$. This proves uniqueness. Let us prove existence. It consists of two parts. First of all, we present formulas in local coordinates satisfying the required properties. Then we have to prove that the formulas agree at the intersections of the charts. Fortunately, we do not have to do this since the formulas have to agree because of the uniqueness property we have already proven.

Let us now define the differential in the local coordinates by the formula (5) and let us check it satisfies the required properties. First of all, Leibniz formula. One has

$$fdx_{i_1}\wedge\ldots\wedge dx_{i_k}\wedge gdx_{j_1}\wedge\ldots\wedge dx_{j_l}=fgdx_{i_1}\wedge\ldots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\ldots\wedge dx_{j_l}$$

so that it is enough to check the Leibniz formula for $k=l=0$, that is to check

$$d(fg)=df\cdot g+fdg.$$

This is a standard calculus claim. *Exercise. Find out why did we need the mysterious sign* $(-1)k$ *to appear.*

Now, let us check $dd=0$. Recall that $df=\sum\frac{\partial f}{\partial x_i}dx_i$ so

$$dd(fdx_{i_1}\wedge\ldots\wedge dx_{i_k})=d(\sum\frac{\partial f}{\partial x_i}dx_i\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k})= \tag{6}$$

$$\sum_{i,j}\frac{\partial^2 f}{\partial x_i\partial x_j}dx_j\wedge dx_i\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k}=0, \tag{7}$$

□

since $dx_i\wedge dx_i=0$ and

$$\frac{\partial^2 f}{\partial x_i\partial x_j}=\frac{\partial^2 f}{\partial x_j\partial x_i}.$$

3.8. **Morphisms of vector bundles, continued.** This is a second attempt to start telling this topic.

Let $F:\mathbb{V}\to\mathbb{W}$ be a morphism of vector bundles over $X$. Recall that this means that $F$ is a smooth map commuting with the projection to $X$ and such that for each $x\in X$ the map of fibers $\mathbb{V}_x\to\mathbb{W}_x$ is linear (recall that the fibers are vector spaces).

The map $F$ defines by composition the map of the corresponding sections which we will denote

$$\Gamma(F):\Gamma(\mathbb{V})\longrightarrow\Gamma(\mathbb{W}).$$

Note that one can multiply sections by smooth functions, and that $\Gamma(F)$ obviously preserves this structure

$$\Gamma(F)(fs)=f\Gamma(F)s. \tag{8}$$

As we already know, there are interesting maps $\Gamma(\mathbb{V})\to\Gamma(\mathbb{W})$ which do not satisfy this property, and, therefore, they do not come from morphisms of vector bundles. Such is, for instance, the de Rham differential.

Today we will see more examples of this sort. Still, interesting examples of operators satisfy the following property which we call *locality.*

3.8.1. **Definition.** An operator $f:\Gamma(\mathbb{V})\to\Gamma(\mathbb{W})$ is local if for each $s\in\Gamma(\mathbb{V})$ and for each open subset $U\subset X$ the equality $s|_U=0$ implies $f(s)|_U=0$.

De Rham differential is obviously local since zero function has zero derivatives.

In what follows we need the following basic fact which we leave without proof.

3.8.2. **Lemma.** *Let $U', U$ be two open subsets of $X$ such that $\bar{U}' \subset U$. Then there exists a smooth function $\alpha$ on $X$ having support in $U$ such that $\alpha|_{U'} = 1$.*

Recall that support of a function $f$ is defined as the closure of the set $\{x | f(x) \neq 0\}$.

3.8.3. **Corollary.** *Let $U' \subset U$ be open subsets of $X$, so that $\bar{U}' \subset U$ and let $\mathbb{V}$ be a vector bundle on $X$. For any section $s \in \Gamma(\mathbb{V}|_U)$ there exists a section $t \in \Gamma(\mathbb{V})$ such that $s|_{U'} = t|_{U'}$.*

*Proof.* Choose a function $\alpha$ as in Lemma 3.8.2 and define $t(y) = \alpha(y)s(y)$ for $y \in U$ and $t(y) = 0$ for $y \ni U$. □

3.8.4. **Lemma.** *Assume that an operator $f : \Gamma(\mathbb{V}) \longrightarrow \Gamma(\mathbb{W})$ is $C^\infty$-linear. Then $f$ is local.*

*Proof.* Let $s|_U = 0$. We want to check that $f(s)|_U = 0$. It is sufficient to prove that for each $x \in U$ there exists a neighborhood $U' \ni x$ such that $f(s)|_{U'} = 0$. We can assume $\bar{U}' \subset U$. Then there exists a smooth function $\alpha$ on $X$ such that $\alpha|_{U'} = 1$ and the support of $\alpha$ is in $U$. Under these conditions $\alpha s = 0$ therefore $f(\alpha s) = \alpha f(s) = 0$. Thus, $f(s)|_{U'} = 0$ as required. □

Local operators $f : \Gamma(\mathbb{V}) \to \Gamma(\mathbb{W})$ satisfy a very special property: one can uniquely extend them to sections over open subsets, see the following theorem.

3.8.5. **Theorem.** *Let $f : \Gamma(\mathbb{V}) \longrightarrow \Gamma(\mathbb{W})$ be a local operator. There is a unique collection of operators*

$$f_U : \Gamma(\mathbb{V}|_U) \longrightarrow \Gamma(\mathbb{W}|_U)$$

*compatible with the restrictions.*

Of course, this is precisely the property which makes local operators so important.

*Proof.* Let $U$, $U'$ be two open subsets of $X$ such that $\bar{U}' \subset U$. We define an operator $f_{U,U'} : \Gamma(\mathbb{V}|_U) \longrightarrow \Gamma(\mathbb{W}|_{U'})$ as follows. Let $s \in \Gamma(\mathbb{V}|_U)$. By 3.8.3 there exists a section $t \in \Gamma(\mathbb{V})$ such that $t|_{U'} = s|_{U'}$. We define

$$f_{U,U'}(s) := f(t)|_{U'}.$$

The definition makes sense since, if $t'|_{U'} = t|_{U'}$, then by locality of $f$ one has $f(t')|_{U'} = f(t)|_{U'}$. By the construction, $f_{U,U'}$ is the only operator compatible with $f$, that is making the diagram below

$$\begin{array}{ccc} \Gamma(\mathbb{V}) & \xrightarrow{f} & \Gamma(\mathbb{W}) \\ \downarrow & & \downarrow \\ \Gamma(\mathbb{V}|_U) & \xrightarrow{f_{U,U'}} & \Gamma(\mathbb{W}|_{U'}) \end{array} \tag{9}$$

commutative.

We can now present $U$ as a union of subsets $U'$ as above. The maps $f_{U,U'_1}$ and $f_{U,U'_2}$ coincide on the intersection $U'_1 \cap U'_2$ because of uniqueness property. Gluing the sections $f_{U,U'}(s)$ for various $U'$ we get a section over $U$ which we denote $f_U(s)$. The uniqueness of the construction is obvious. ☐

We will now prove the following.

3.8.6. **Theorem.** *An operator $f : \Gamma(\mathbb{V}) \to \Gamma(\mathbb{W})$ comes from a morphism of vector bundles if and only if it is $C^\infty$-linear, that is it satisfies (8).*

*Proof.* We will first of all prove the claim in case $\mathbb{V}$ is a trivial bundle, and then will use a trivialization of $\mathbb{V}$. In the second part we will use Theorem 3.8.5.

Assume $\mathbb{V}$ is trivial, that is isomorphic to $X \times \mathbb{R}^k$ for some $k$. Let $s_i$, $i = 1, \ldots, k$, be constant sections of $\mathbb{V}$ corresponding to a fixed basis of $\mathbb{R}^k$. Any section of $\mathbb{V}$ can be uniquely presented as a linear componation

$$s = \sum a_i s_i$$

where $a_i \in C^\infty(X)$. Thus, a $C^\infty$-linear map $f : \Gamma(\mathbb{V}) \to \Gamma(\mathbb{W})$ is uniquely defined by $k$ sections $t_i = f(s_i) \in \Gamma(\mathbb{W})$. These sections uniquely define a morphism $X \times \mathbb{R}^k \to \mathbb{W}$ by the formula

$$F(x, \sum c_i s_i(x)) = (x, \sum c_i t_i(x)).$$

Choose a trivialization $X = \cup U_i$ of $\mathbb{V}$. On each one of $U_i$ we have by Theorem 3.8.5 an operator $f_{U_i} : \mathbb{V}|_{U_i} \to \mathbb{W}|_{U_i}$. By the first part of the proof this gives rise to a unique map $F_i : \mathbb{V}|_{U_i} \to \mathbb{W}|_{U_i}$ for each $i$. The maps $F_i$ are compatible on the intersections by uniqueness. That gives the result. ☐

3.9. **Integral curves.** Vector fields are just sections of the tangent bundle. However, they have a special meaning (since the tangent bundle is a very special bundle).

Let $s$ be a vector field on a manifold $X$. One can look for a curve $\gamma : (a, b) \longrightarrow X$ satisfying the equation

$$\gamma'(t) = s(\gamma(t)). \tag{10}$$

The theory of ODE claims the following.

3.9.1. **Theorem.** *For any $x \in X$ there exists an interval $(a, b)$ containing zero and a unique curve $\gamma : (a, b) \longrightarrow X$ satisfying the equation (10) and $\gamma(0) = x$.*

The size of the segment $(a, b)$ may depend on $x$. However, for each $x$ there exists a neighborhood $U \ni x$ and a *common* segment $(a, b)$ such that the the differental equation (10) has solution in $(a, b)$ for any initial condition $\gamma(0) = y,\ y \in U$.

Fix such $U$ and $(a, b)$. Uniqueness of the solution of the differential equation allows one to define a smooth map $\Theta : (a, b) \times U \longrightarrow X$. We will write $\Theta_t(x)$ instead of $\Theta(t, x)$.

3.9.2. **Theorem.** *Assume $t, s, t + s$ belong to $(a, b)$. Then the maps $\Theta_{t+s}$ and $\Theta_t \circ \Theta_s$ coincide at the common doman of definition. In particular, $\Theta_t$ is a local diffeomorphism for small $t$.*

*Proof.* $\Theta_s(x)$ is defined as $\gamma(s)$ where $\gamma$ satisfes the differential equation 10 with the initial condition $\gamma(0) = x$. Therefore $\Theta_t \circ \Theta_s(x)$ is $\delta(t)$ where $\delta$ satisfies the same equation with initial condition $\delta(0) = \gamma(s)$. By the uniqueness of the solution of ODE the functions $\delta(t)$ and $\gamma(t + s)$ coincide.

In particular, for small $t$ one has $\Theta_t \circ \Theta_{-t} = \Theta_1 = \mathrm{id}$. □

3.10. **Lie derivative.** In this subsection we define, for each vector field $s \in \mathcal{T}$, a local operator

$$L_s : \mathcal{T}^p_q \longrightarrow \mathcal{T}^p_q.$$

This operator is called Lie derivative.

**3.10.1. The case** $p = q = 0$ As we know, any vector field $s \in \mathcal{T}$ defines a derivation on the space of smooth functions. This is $L_s$ in the case $p = q = 0$.

**3.10.2. The case** $p = 1,\ q = 0$ We have $\mathcal{T}^1_0 = \mathcal{T}$ and the operator $L_s$ is defined by the formula

$$L_s(t) = [s, t].$$

3.10.3. **Theorem.** *There exists a unique collection of operators*

$$L_s : \mathcal{T}^p_q \longrightarrow \mathcal{T}^p_q$$

*satisfying the following conditions*

- *$L_s$ is as defined above in the cases $p = q = 0$ (on functions) and $p = 1,\ q = 0$ (on vector fields).*
- *$L_s$ satisfies Leibniz rule:*

$$L_s(u \cdot v) = L_s(u) \cdot v + u \cdot L_s(v).$$

- *$L_s$ commutes with (all) contractions: if $c : \mathcal{T}^p_q \longrightarrow \mathcal{T}^{p-1}_{q-1}$ is a contraction, one has*

$$L_s(c(u)) = c(L_s(u)).$$

*Proof.* As usual, we will first prove uniqueness, and then take care of existence. Let us first of all show that the contraction axiom allows one to determine the action of $L_s$ on 1-forms.

Let $\omega \in \mathcal{T}_1^0 = \Omega^1$ and let $t \in \mathcal{T}$. One has to have $L_s(t \cdot \omega) = [s,t] \cdot \omega + t \cdot L_s(\omega)$, so by contraction condition

$$s(\langle t, \omega \rangle) = \langle [s,t], \omega \rangle + \langle t, L_s(\omega) \rangle$$

We claim that $L_s(\omega)$ is uniquely defined by this formula. In fact, $L_s(\omega)$ is a section of $T^*X = \mathrm{Hom}(TX, \mathbf{1})$ that is (by Exercise 2 of Lecture 7) a map $TX \longrightarrow \mathbf{1}$ that is a $C^\infty$-linear map from $\mathcal{T}$ to $C^\infty$. The formula precisely defines this map.

Now, having defined $L_s$ on $C^\infty$, $\mathcal{T}$ and $\Omega^1$, we have it (locally) for all tensors. This proves $L_s$ (if exists) is defined uniquely.

The best way to prove existence of $L_s$ is to present another definition and then check that it satisfies the required properties. We will sketch this definition without entering into details.

The vector field $s \in \mathcal{T}$ defines a collection of diffeomorphisms $\Theta_t$ of $X$ for small $t$. Any diffeomorphism $\Theta : X \to Y$ defines a map $\Theta^* : \mathcal{T}_q^p(Y) \longrightarrow \mathcal{T}_q^p(X)$.

In particular, for $u \in \mathcal{T}_q^p$ the assignment $t \mapsto \Theta_t^*(u)$ defines a path in $\mathcal{T}_q^p$. The tangent vector at $t = 0$ is an element of $\mathcal{T}_q^p$ which is denoted $L_s(u)$.

Let us calculate $L_s(f)$ where $f$ is a smooth function. By definitnion, $\Theta_t(f)$ is a function assigning to $x \in X$ the value $f(\Theta(t,x))$. By definition this coincides with $\langle f, s \rangle(x)$.

In this way one can check that $L_s$ satisfies all the requirements (we will not do this).

Alternatively, one can follow the way we used in dealing with de Rham differential. This means that one has to make explicit calculations in order to check that in the local coordinates the operators $L_s$ defined as in the first part of the proof, satisfy all requirements. Then uniqueness would imply that the formulas coincide on the intersection of the charts.

We leave this to an interested reader. We will just indicate that if $s = \sum s_i \frac{\partial}{\partial x_i}$ then $L_s(\frac{\partial}{\partial x_i}) = -\sum_j \frac{\partial s_j}{\partial x_i} \frac{\partial}{\partial x_j}$ and $L_s(dx_i) = \sum_j \frac{\partial s_i}{\partial x_j} dx_j$. This allows one (after very lengthy calculations) to prove the existence locally, and, because of the uniqueness, also globally.

□