DIFFERENTIAL GEOMETRY. LECTURES 7-8, 16-19.06.08

Vector bundles can be dealt with as vector spaces — one can perform with them the same basic operations: direct sum, tensor product, et cetera.

We will first recall the standard operations with vector spaces.

3.3. **Operations with vector spaces.**

3.3.1. Direct sum. If V, W are two vctor spaces, their direct sum is defined as the set of pairs (v, w) with $v \in V,\ w \in W$. The vector space operations, sum and multiplication by a scalar, are defined componentwise:

$$(v, w) + (v', w') = (v + v', w + w'), \quad c(v, w) = (cv, cw).$$

It is convenient to have also a "coordinate-dependent" description: if $X = \{x_1, \ldots, x_n\}$ is a basis of V and $Y = \{y_1, \ldots, y_m\}$ is a basis of W then $X \cup Y$ is a basis of $V \oplus W$.

3.3.2. The dual space

Let V be a vector space. The dual space $V^\vee$ is defined as the set of linear maps $V \longrightarrow F$ to the base field F (we are mainly interested in the case $F = \mathbb{R}$ or $\mathbb{C}$). If V is finite dimensional than $\dim V^\vee = \dim V$. Moreover, if $x_1, \ldots, x_n$ is a basis in V, the collection of functionals x_i^* defined by the formulas $x_i^*(x_j) = \delta^{i,j}$ (Kronecker's delta) defines a basis in V^* called *the dual basis.* Note that the notation is slightly misleading since in order to define x_i^* one need all of x_j mato be chosen.

3.3.3. The space of linear maps

Given two vector spaces, V and W, one defines $\mathrm{Hom}(V, W)$ as the space of linear maps from V to W. In case $W = F$ we get back the dual space, $\mathrm{Hom}(V, F) = V^*$. The linear space operations are defined as usual:

$$(f + g)(v) = f(v) + g(v), \quad (cf)(v) = cf(v).$$

If $X = \{x_1, \ldots, x_m\}$ is a basis of V and $Y = \{y_1, \ldots, y_n\}$ is a basis of W then one can choose the following set $f_{i,j},\ i = 1, \ldots, n,\ j = 1, \ldots, m$ as a basis for $\mathrm{Hom}(V, W)$:

$$f_{i,j}(x_k) = \delta^{j,k} y_i.$$

As we know well, linear maps between vector spaces can be written as matrices once one chooses bases here and there. In this way the map $f_{i,j}$ identifies with the matrix having 1 at line i, column j, and zero otherwise.

If $\dim V = m,\ \dim W = n$, one has $\dim \mathrm{Hom}(V, W) = mn$.

3.3.4. Tensor product

There is another operation with vector space, giving a vector spaces of the same "size" as $\mathrm{Hom}(V, W)$ for finite dimensional V, W, but with a equal role of V and W (recall that $\mathrm{Hom}(V, F) = V^*$ even though $\mathrm{Hom}(F, V) = V$ which is not really the same.

The definition goes as follows. First of all, for any triple of vector spaces, V, W, U, one defines a bilinear map $f : V \times W \longrightarrow U$ as a map satisfying the linearity property along each one of the arguments: for all $x \in V$ the map $f(x, \ )$ is linear, as well as $f(\ , y)$ is linear for all $y \in W$.

Denote the collection of bilinear maps from $V \times W$ to U as $\mathrm{Bil}(V, W; U)$. An important observation follows: if x_i and y_j are bases in V and W respectivewly, a bilinear map $f : V \times W \to U$ is uniquely defined by its value on pairs of basis elements $f(x_i, y_j)$.

If $f : V \times W \to U$ is bilinear and if $g : U \to U'$ is any linear map, the composition $g \circ f : V \times W \to U'$ is bilinear. Thus, any linear map $g : U \to U'$ defines a map

$$\mathrm{Bil}(V, W; U) \longrightarrow \mathrm{Bil}(V, W; U').$$

Now we say that a bilinear map $F : V \times W \to X$ is universal if any bilinear map $f : V \times W \to U$ can be presented as $g \circ F$ for a *unique* linear map $g : X \to U$.

We will see in a moment that such universal bilinear map exists. But before we see this, let us check that (if it exists) it is essentially unique.

In fact, if $F : V \times W \to X$ and $G : V \times W \to Y$ are both universal bilinear maps, there exists a unique $f : X \to Y$ and a unique $g : Y \to X$ such that $f \circ F = G,\ g \circ G = F$. This implies that $g \circ f \circ F = F$ and, once more by uniqueness, that $g \circ f = \mathrm{id}_X$. Similarly, we get $f \circ g = \mathrm{id}_Y$ and this is precisely the sence of our phrase "essentially unique".

Let us present such universal bilinear map. Choose, as usual, the bases $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ in V and W respectively. Define $X = \mathrm{Span}\{e_{i,j}, i = 1, \ldots, m,\ j = 1, \ldots, n\}$ and denote

$$F : V \times W \to X$$

by the formula

$$F(x_i, y_j) = e_{i,j}.$$

(The above formula uniquely extends to a bilinear map.)

Since the universal bilinear map is essentially unique, it deserves a special name. We denote $X = V \otimes W$ and the image $F(v, w)$ by $v \otimes w$. If $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ are bases in V and in W respectively, the elements $x_i \otimes y_j$ for a basis of $V \otimes W$. Bilinearity means that if $u = \sum a_i x_i$ and $w = \sum b_j y_j$ then

$$v \otimes w = \sum_{i,j} a_i b_j x_i \otimes y_j.$$

3.3.5. Symmetric powers

If V is a vector space, one defines $T^d(V) = V \otimes \ldots \otimes V$ (d factors). If $\{x_i\}$ is a basis of V, the expressions

$$x_{i_1} \otimes \ldots \otimes x_{i_d}$$

form a basis of $T^d(V)$. If $\dim V = n$, one has $\dim T^d(V) = n^d$.

The space $T^d(V)$ has two interesting quotients, the symmetric and the exterior powers.

We define the d-th symmetric power of V, $S^d(V)$, as the quotient of $T^d(V)$ by the subspace generated by the differences

$$v_1 \otimes \ldots \otimes v_d - v_{s(1)} \otimes \ldots \otimes v_{s(d)}$$

for any v_i and all $s \in S_d$.

A linear basis for $S^d(V)$ can be described as follows. Let $x_1, \ldots, x_n$ form a basis of V. A basis of $S^d(V)$ is formed by monomials

$$x_1^{d_a} \ldots x_n^{d_n}$$

satisfying the condition $d = d_1 + \ldots + d_n$.

An especially important for us is the symmetric square $S^2(V)$. It is easy to see that linear maps from $S^2(V)$ to U are the same as symmetric bilinear maps $V \times V \longrightarrow U$.

If $x_1, \ldots, x_n$ form a basis of V, the products $x_i x_j = x_j x_i$ form a basis of $S^2(V)$. If $\dim V = n$, one has $\dim S^2(V) = \frac{n(n+1)}{2}$.

3.3.6. Exterior powers

Exterior powers $\wedge^d(V)$ are defined as the quotient of $T^d(V)$ by the vector subspace generated by the tensors

$$\ldots \otimes v \otimes \ldots \otimes v \otimes \ldots$$

If $x_1, \ldots, x_n$ form a basis of V, the products $x_{i_1} \wedge \ldots \wedge x_{i_d}$ with $i_1 < \ldots < i_d$ form a basis of $\wedge^d(V)$. If $\dim V = n$, one has $\dim \wedge^d(V) = \begin{pmatrix} n \\ d \end{pmatrix}$. In particular, $\dim \wedge^n(V) = 1$.

3.3.7. Change of base.

All the constructions above are *functorial* in the sense that if one has a linear map $V \to V'$ (or a pair $V \to V'$ and $W \to W'$), this canonically defines a linear map of the resulting vector space.

In what follows we will assume, when needed, that V has a base $\{x_i\}$, W has a base $\{y_j\}$, and similarly for V', W' the bases are denoted $\{x'_i\}$, $\{y'_j\}$.

For instance, given linear maps $\alpha : V \to V'$ and $\beta : W \to W'$, one defines a linear map $\alpha \oplus \beta : V \oplus W \to V' \oplus W'$ by the formula

$$\alpha \oplus \beta(v, w) = (\alpha(v), \beta(w)).$$

If α and β are given, for a specific choice of the bases, by matrices A and B, the automorphism $\alpha \oplus \beta$ is defined by the block-diagonal matrix $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$.

Similarly, a linear map $\alpha : V \to V'$ defines an adjoint map of $\alpha^\vee : V'^\vee \to V^\vee$; it is given in the dual bases by the transposed matrix A^t. Note that the assignment $\alpha \mapsto \alpha^\vee$ reverts compositions: one has

$$\alpha_1^\vee \circ \alpha_2^\vee = (\alpha_2 \circ \alpha_1)^\vee.$$

This is why it is more useful sometimes to use the inverse of the transpose: the assignment $\alpha \mapsto (\alpha^\vee)^{-1}$ preserves compositions.

Of course, this is only possible if α is an isomorphism.

If $\alpha : V \to V'$ is an isomorphism and $\beta : W \to W'$ is arbitrary, a linear map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V', W')$ is defined, by assigning to $f : V \longrightarrow W$ the composition $\beta \circ f \circ \alpha^{-1}$. If $B = (b_{i,j})$, $A^{-1} = (c_{i,j})$ then the resulting automorphism of $\mathrm{Hom}(V, W)$ sends $f_{i,j}$ to

$$\sum b_{r,i} c_{j,s} f'_{r,s}. \tag{1}$$

Let us look what happens with the tensor product.

If $\alpha : V \to V'$ and $\beta : W \to W'$ are given, the bilinear map

$$V \times W \longrightarrow V' \otimes W'$$

sending (v, w) to $\alpha(v) \otimes \beta(w)$, defines by universality a linear map $V \otimes W \longrightarrow V' \otimes W'$ sending $v \otimes w$ to $\alpha(v) \otimes \beta(w)$. We will denote this linear map by $\alpha \otimes \beta$. If α is given by a matrix A and β by B, the map $\alpha \otimes \beta$ carries $x_i \otimes y_j$ to

$$\alpha(x_i) \otimes \beta(y_j) = \sum_r a_{r,i} x'_r \otimes \sum_s b_{s,j} y'_s = \sum_{r,s} a_{r,i} b_{s,j} x'_r \otimes y'_s. \tag{2}$$

Note the difference between the formulas (1) and (2)!

Iterating the above formulas, we can get a map $T^n(V) \to T^n(V')$ induced by a map $\alpha : V \to V'$. We denote the resulting map as $T^n(\alpha)$.

It is interesting to describe the map $\wedge^n V \to \wedge^n V$, $n = \dim V$, induced by an endomorphism $\alpha : V \to V$.

Note that the vector space $\wedge^n V$ is one-dimensional and if $x_1, \ldots, x_n$ is a basis of V, $\wedge^n V$ has a basis consisting of the only element $x_1 \wedge x_2 \wedge \ldots \wedge x_n$. The image of this element under the map $\wedge^n \alpha$ is $\alpha(x_1) \wedge \alpha(x_2) \wedge \ldots \wedge \alpha(x_n)$. In order to understand what we got, we have to transform this expression using the standard properties, multilinearity and skew-symmetricity. Let α be given by a matrix A

in the basis of x_j. This means that $\alpha(x_j) = \sum_i a_{i,j} x_i$. Then, opening brackets we will have

$$\alpha(x_1) \wedge \ldots \wedge \alpha(x_n) = \sum_{s:[1,n]\to[1,n]} a_{s(1),1} a_{s(2),2} \ldots a_{s(n),n} \cdot x_{s(1)} \wedge \ldots \wedge x_{s(n)} =$$

$$\sum_{s\in S_n} \operatorname{sign}(s) a_{s(1),1} a_{s(2),2} \ldots a_{s(n),n} \cdot x_1 \wedge x_2 \wedge \ldots \wedge x_n = \det(A) x_1 \wedge x_2 \wedge \ldots \wedge x_n.$$

3.4. **Operations with vector bundles.** The operations with vector bundles are defined according to the same recipee.

Let, for instance, $\mathbb{V} \to X$ and $\mathbb{W} \to X$ be two vector bundles. Choose an open cover $X = \cup U_i$ such that the restrictions $\mathbb{V}|_{U_i}$ and $\mathbb{W}|_{U_i}$ are trivial. The latter means there exist isomorphisms

$$\eta_i : U_i \times V_i \longrightarrow \mathbb{V}|_{U_i}, \quad \zeta_i : U_i \times W_i \longrightarrow \mathbb{W}|_{U_i},$$

where V_i and W_i are vector spaces (all isomorphich to $\mathbb{R}^m$ or to $\mathbb{R}^n$ respectively). The vector bundles $\mathbb{V}$ and $\mathbb{W}$ are defined by the maps

$$\theta_{i,j} : U_{i,j} \longrightarrow \operatorname{Hom}(V_i, V_j), \quad \xi_{i,j} : U_{i,j} \longrightarrow \operatorname{Hom}(W_i, W_j)$$

satisfying the cocycle conditions.

Now we are ready to define, starting with these data, new vector bundles. For instance, the direct sum $\mathbb{V} \oplus \mathbb{W}$ is obtained gluing trivial bundles $U_i \times (V_i \oplus W_i)$ along the cocycles

$$\theta_{i,j} \oplus \xi_{i,j} : U_{i,j} \longrightarrow \operatorname{Hom}(V_i \oplus W_i, V_j \oplus W_j)$$

sending $x \in U_{i,j}$ to the linear map $\theta_{i,j}(x) \oplus \xi_{i,j}(x)$.

The three cocycle conditions for $\theta \oplus \xi$ are verified immediately.

Similar formulas define the operations $\mathbb{V} \otimes \mathbb{W}$, $S^n(\mathbb{V})$, $\wedge^n(\mathbb{V})$.

A minor difference appears in the definition of the dual vector bundle $\mathbb{V}^\vee$ and the vector bundle $\operatorname{Hom}(\mathbb{V}, \mathbb{W})$ (which is the same as $\mathbb{V}^\vee \otimes \mathbb{W}$ in the case (we are only interested in) of finite-dimensional vector bundles. If $\mathbb{V}$ is iven, as above, by and $\theta_{i,j} U_{i,j} \longrightarrow \operatorname{Hom}(V_i, V_j)$, then the cocycle defining $\mathbb{V}^\vee$ is defined by the cocycle $(\theta^\vee_{i,j})^{-1}$.

3.4.1. Base change

There is a special operation with vector bundles which replaces the base manifold X.

Let $\pi : \mathbb{V} \to X$ be a vector bundle and let $f : Y \to X$ be a smooth map. We define a vector bundle $f^*(\mathbb{V})$ on Y as follows.

As a topological space, $f^*(\mathbb{V})$ is the subspace of the direct product $Y \times \mathbb{V}$ determined by the condition

$$f^*(\mathbb{V}) = \{(y, v) \in Y \times \mathbb{V} | f(y) = \pi(v)\}. \tag{3}$$

The procetion to the first factor defines a continuous map $\pi' : f^*(\mathbb{V}) \longrightarrow Y$.

We will prove that $f^*(\mathbb{V})$ is a smooth manifold, that π' is a smooth map, simultaneously with presenting the map $\pi' : f^*(\mathbb{V}) \longrightarrow Y$ with the structure of a vector bundle. In fact, let U be an open subset on X such that $\mathbb{V}|U$ is trivial that is isomorphic to a direct product $U \times \mathbb{R}^n$. Then put $V = f^{-1}(U)$. This is an open subset of Y. The preimage ${\pi'}^{-1}(V)$ can be easily identified with $V \times \mathbb{R}^n$ and this proves simultaneously everything.

3.4.2. Standard tensor bundles.

The cotangent bundle T^*X is, by definition, the vector bundle dual to the tangent bundle $T(X)$. The sections of T^*X are called *differential 1-forms.*

For given p, q the tensor product $T^p(TX) \otimes T^q(T^*X)$ is denoted $T^p_q(X)$. A section of T^p_q is called a (p, q) tensor field on X.

Thus, $(1, 0)$ tensor fields are vector fields, and $(0, 1)$ tensor fields are 1-forms.

Skew-symmetric $(0, n)$ vector fields are called n-forms. They are the same as the sections of the bundle $\wedge^n T^*X$.

3.5. **Morphisms of vector bundles versus local operators.** Let $\mathbb{V}$, $\mathbb{W}$ be two vector bundles. Any morphism $f : \mathbb{V} \longrightarrow \mathbb{W}$ of vector bundles induces a linear map of the spaces of sections

$$\Gamma(f) : \Gamma(\mathbb{V}) \to \Gamma(\mathbb{W}). \tag{4}$$

Recall that one can multiply sections by smooth functions on X: if $s \in \Gamma(\mathbb{V})$ and $\alpha \in C^\infty(X)$ then αs is also a section of $\mathbb{V}$.

Remark. This makes $\Gamma(\mathbb{V})$ into a *module* over the commutative ring $C^\infty(X)$.

The map $\Gamma(f) : \Gamma(\mathbb{V}) \to \Gamma(\mathbb{W})$ induced by a morphism of vector bundles f preserves the multiplication by functions:

$$\Gamma(f)(\alpha s) = \alpha \Gamma(f)(s).$$

There are, however, very interesting maps $\Gamma(\mathbb{V}) \to \Gamma(\mathbb{W})$ which do not come from morphisms of vector bundles.

Here is a typical example.

Let $\mathbf{1}$ denote the trivial one-dimensional bundle $\mathbf{1} = X \times \mathbb{R}$. One has $\Gamma(\mathbf{1}) = C^\infty(X)$. We will define now a very interesting operator

$$d : C^\infty(X) \longrightarrow \Gamma(T^*X)$$

assigning, in local coordinates, to a function $f \in C\infty(X)$ the 1-form

$$df = \sum \frac{\partial f}{\partial x_i} dx_i, \tag{5}$$

where dx_i is the dual basis to $\frac{\partial}{\partial x_i}$.

There are two ways of proving the definition makes sense. The first, direct proof, is to check the formula above is compatible on intersection of the charts: we know the cocycle defining the vector bundle T^*X (the corresponding matrix is inverse to the adjoint of the Jacobi matrix) and we have to check the formula (5) changes similarly under a change of coordinates.

Another way (for the lazybones) is to prove that $\Gamma(T^*)$ can be expressed as $\mathrm{Hom}_{C^\infty}(\Gamma(TX), C^\infty)$, the set of C^∞-linear map from $(\Gamma(TX)$ to C^∞ and to take into account that each vector field acts on the functions.

3.6. **De Rham complex.** Let X be a manifold. We denote $\Omega^k = \Gamma(\wedge^k(T^*M))$ - this is the space of differential k-forms on X. In particular, $\Omega^0 = C^\infty(X)$.

We intend to define a collection of maps $d : \Omega^k \to \Omega^{k+1}$ satisfying some nice properties, in particular, $d \circ d = 0$, such that for $k = 0$ this amounts to the differential of a function defined above.

This collection of vector spaces and maps is called the *de Rham complex.*

One defines *de Rham cohomology* by the formula

$$H^k_{DR}(X) = \mathrm{Ker}(d : \Omega^k \to \Omega^{k+1})/\,\mathrm{Im}(d : \Omega^{k-1} \to \Omega^k).$$

We mimic the definition of d in the local coordinates. If $x_1, \ldots, x_n$ are local coordinates given by a chart $\phi : D \longrightarrow U$, a general k-form has a basis consisting of expressions

$$dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

with coefficients from C^∞. Thus, it is enough to define what is

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_k}). \tag{6}$$

One should not be very astonished to have as an answer the following expression.

$$d(f)dx_{i_1} \wedge \ldots \wedge dx_{i_k} = \sum \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \tag{7}$$

Exercises.

1. Prove, using a local calculation, that $d \circ d = 0$.
2. Prove that a map $\mathbb{V} \to \mathbb{W}$ is the same as a global section of the bundle $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$.
3. Calculate the de Rham cohomology H^0 and H^1 of the circle.