DIFFERENTIAL GEOMETRY. LECTURE 6, 05.06.08

3. Vector bundles

3.1. **Definition.** The tangent bundle TX of a smooth manifold X is an example (may be, the most important example) of a vector bundle. Informally speaking, a vector bundle on a smooth manifold X is a family of vector spaces parametrized by the points of X.

We start with a provisional notion.

3.1.1**. Definition.** *A family of vector spaces over* X is a smooth manifold $\mathbb{V}$ endowed with a smooth map $\pi : \mathbb{V} \longrightarrow X$ and with a structure of a vector space (in our course this will be always a real vector space) on each fiber $\mathbb{V}_x := \pi^{-1}(x)$.

3.1.2. Let $\pi_1 : \mathbb{V}_1 \to X$ and $\pi_2 : \mathbb{V}_2 \to X$ be two families of vector spaces over X. A map $f : \mathbb{V}_1 \longrightarrow \mathbb{V}_2$ of families of vector spaces is a map of smooth manifolds satisfying the condition

$$\pi_1 = \pi_2 \circ f,$$

such that for each $x \in X$ the map of the fibers $f_x : \mathbb{V}_{1x} \to \mathbb{V}_{2x}$ is linear.

3.1.3. Restriction

Let $\pi : \mathbb{V} \to X$ is a family of vector spaces and let U be an open subset of X. Then the restriction $\pi^{-1}(U) \longrightarrow U$ is a family of vector spaces over U. This is a special case of the pullback construction we will study later.

3.1.4. Trivial family Let V be a vector space. A trivial family of vector spaces over X with fiber V is the family isomorphic to the direct product $X \times V$, with the map π given by the projection to the first factor.

Now we are ready to define vector bundles.

3.1.5**. Definition.** A vector bundle over X is a family of vector spaces $\mathbb{V}$ such that for any $x \in X$ there exists an open neighborhood U of x in X such that the restriction $\mathbb{V}|_U$ is a trivial family.

3.1.6**. Example.** For any X and for any vector space V the trivial family $X \times V$ is a vector bundle. It is called the trivial vector bundle.

3.1.7**. Example.** The tangent bundle TX of X is a vector bundle over X. In fact, for any chart $\phi : D \to U \subset X$ the restriction of the tangent bundle to U is isomorphic to $D \times \mathbb{R}^n$ with the isomorphism given by the map $(\phi, D\phi)$.

For any vector bundle $\mathbb{V}$ over X a smooth map $X \to \mathbb{V}$ sending each point $x \in X$ to $0 \in \mathbb{V}_x$ is defined. This map is called *the zero section* of $\mathbb{V}$.

3.1.8**. Example.** Here is the simplest example of a nontrivial vector bundle. Let $X = S^1$ and let $\mathbb{V}$ be the Möbius band. If $\mathbb{V}$ were isomorphic to $X \times \mathbb{R}^1$, the complement $\mathbb{V} \setminus X$ to the zero section would have two connected components. A small experiment with the scissors shows his is not the case.

3.2. **Trivialization.** Let $\mathbb{V} \to X$ be a vector bundle. Choose an open cover $X = \cup_i U_i$ such that $\mathbb{V}|_{U_i}$ is trivial. Choose the isomorphisms

$$\eta_i : U_i \times \mathbb{R}^n \longrightarrow \mathbb{V}|_{U_i}. \tag{1}$$

The isomorphisms η_i and η_j, do not, in general, coincide on $U_i \cap U_j$ — otherwise we would be able to glue all η_i together to get an isomorphism $X \times \mathbb{R}^n \longrightarrow \mathbb{V}$.

Denote $U_{i,j} = U_i \cap U_j$. Consider the difference between η_i and η_j, that is an isomorphism

$$\eta_j^{-1} \circ \eta_i : U_{i,j} \times \mathbb{R}^n \longrightarrow U_{i,j} \times \mathbb{R}^n. \tag{2}$$

This map has to be identity on $U_{i,j}$, so uniquely determined by a map $U_{i,j} \longrightarrow \mathrm{GL}(n, \mathbb{R})$. Denote the above defined map

$$\theta_{i,j} : U_{i,j} \longrightarrow GL(n, \mathbb{R}). \tag{3}$$

The following properties of the maps $\theta_{i,j}$ are easily verified:

$$\theta_{i,i} \text{ is constant with value id} \in \mathrm{GL}(n, \mathbb{R}). \tag{4}$$

$$\theta_{i,j} = \theta_{j,i}^{-1}. \tag{5}$$

$$\theta_{j,k} \circ \theta_{i,j} = \theta_{i,k} \tag{6}$$

— the equality of the restrictions to $U_{i,j,k} = U_i \cap U_j \cap U_k$.

Vice versa, given an open cover $X = \cup\, U_i$, a collection of functions $\theta_{i,j} : U_{i,j} \longrightarrow \mathrm{GL}(n, \mathbb{R})$ satisfying the above conditions, one can "glue" $\mathbb{V}$ from these data as explained below.

3.2.1. Gluing a manifold from open subsets .

The idea is the following. Given a topological space X with an open cover $X = \cup U_i$, one can present X as a result of gluing of U_i along $U_{i,j} = U_i \cap U_j$.

So, we start with a collection of topological spaces U_i, open subsets $U_{i,j} \in U_i$ form each pair i, j. In order to glue $U_{i,j}$ with $U_{j,i}$ we need a homeomorphism $\phi_{i,j} : U_{i,j} \to U_{j,i}$. We have requirements on $\phi_{i,j}$ similar to the requirements on $\theta_{i,j}$ above.

(7) $$\phi_{i,i} = \mathrm{id}_{U_i}.$$
(8) $$\phi_{i,j} = \phi_{j,i}^{-1}.$$
(9) $$\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$$

— the equality of the restrictions to $U_{i,j} \cap U_{i,k}$.

The topological space X is defined as the quotient of the disjoint union $\coprod U_i$ by the equivalence relation defined by the requirements that $x \in U_{i,j}$ is equivalent to $\phi_{i,j}(x) \in U_{j,i}$. Note that the properties of $\phi_{i,j}$ really mean that this is an equivalence relation!

One has therefore the canonical projection

$$\coprod U_i \longrightarrow X.$$

The topology on X is defined as usual: a subset $U \in X$ is open iff its preimage in $\coprod U_i$ is open.

The maps $U_i \to X$ are defined as the compositions

$$U_i \longrightarrow \coprod U_i \longrightarrow X.$$

The images of U_i obviously cover the whole X. Let us check that for each i the map $U_i \to X$ is a homeomorphism to an open subset.

This requires checking that the map $U_i \to X$ is open and one-to-one.B This easily follow from the properties (7).

Assume now that all U_i are manifolds and $\phi_{i,j}$ are diffeomorphisms. Let us take the union of all charts of all U_i. Since U_i cover X, the charts cover the whole X. The charts belonging to the same U_i are obviously compatible. The charts belonging to different U_i are compatible since $\phi_{i,j}$ are diffeomorphisms. Thus, gluing smooth manifolds gives automatically a smooth manifold.

3.2.2. Gluing vector bundles

We are now given a smooth manifold X, an open cover $X = \cup U_i$, and a collection of smooth maps $\theta_{i,j} : U_{i,j} \longrightarrow \mathrm{GL}(n, \mathbb{R})$ satisfying the conditions (4).

We see that X can be presented as the result of gluing U_i along diffeomorphisms $\phi_{i,j} : U_{i,j} \to U_{j,i}$.

We define $\mathbb{V}$ as the result of gluing of the trivial bundles $\mathbb{V}_i = U_i \times \mathbb{R}^n$. We define subsets $\mathbb{V}_{i,j} = U_{i,j} \times \mathbb{R}^n$ and the gluing maps $\Phi_{i,j} : \mathbb{V}_{i,j} \longrightarrow \mathbb{V}_{j,i}$ by the formula

(10) $$\Phi_{i,j}(x, v) = (\phi_{i,j}(x), \theta_{i,j}(x)(v)).$$

One has to check a few things:

1. $\Phi_{i,j}$ satisfy (7). This gives a manifold $\mathbb{V}$.
2. The projection $\mathbb{V} \longrightarrow X$ is correctly defined.

3. This is a vector bundle.
4. Its restriction to U_i gives $U_i \times \mathbb{R}^n$.

This is more or less obvious.

3.2.3. Equivalence

It is worthwhile to understand when do two collections $\theta_{i,j}$ and $\theta'_{i,j}$ define isomorphic vector bundles.

Recall that $\theta_{i,j}$ are defined by the choice of a trivialization which is a collection of isomorphisms (1). Let

$$\eta'_i : U_i \times \mathbb{R}^n \longrightarrow \mathbb{V}|_{U_i}. \tag{11}$$

be another choice of trivialization of $\mathbb{V}$. We can compare them by defining

$$\eta'^{-1}_i \circ \eta_i : U_i \times \mathbb{R}^n \to U_i \times \mathbb{R}^n$$

which is uniquely defined by a smooth function

$$\alpha_i : U_i \longrightarrow \mathrm{GL}(n, \mathbb{R}).$$

This allows to express $\theta'_{i,j}$ via $\theta_{i,j}$ and α_i by the formula

$$\theta'_{i,j} = \alpha_j \theta_{i,j} \alpha_i^{-1}. \tag{12}$$

We have therefore proven the following

3.2.4. **Theorem.** *The trivializations $\{\theta_{i,j}\}$ and $\{\theta'_{i,j}\}$ define isomorphic vector bundles if and only if there exist a collection of smooth maps*

$$\alpha_i : U_i \to \mathrm{GL}(n, \mathbb{R})$$

such that the condition (12) is satisfied.

Note without proof the following important

3.2.5. **Theorem.** *Any vector bundle on the standard disc is trivial. Therefore, any vector bundle has a trivialization over any atlas.*

Homework.

1. Write down trivialiation maps $\theta_{i,j}$ for the tangent bundle of a smooth variety X.
2. Using trivialization for the tangent bundle of a circle obtained in Problem 1, prove using Theorem 3.2.4 that it is trivial.
3. Let M_n be the result of gluing of two short sides of a rectangle after twisting it n half-twists (so that M_0 is a cylinder and M_1 is the Möbius band). We consider M_n as a vector bundle over the circle. Prove that M_n is isomorphic to M_0 if n is even and to M_1 if n is odd.