DIFFERENTIAL GEOMETRY. LECTURE 5-6, 29.05-02.06.08

2.2.5. Vector fields on an open subset. Gluing

Let X be a smooth manifold, U an open subset of X (not necessarily a chart). Then, as we know, U is itself a smooth manifold. The tangent bundle of U identifies with an open subset (submanifold) of the tangent bundle of X: $TU \subset TX$. Any vector field s on X restricts to a vector field on U. We denote the restriction $s|_U$.

Let now X be covered by a collection of open sets U_i. If a collection of vector fields s_i on U_i is given, compatible in the sense that for each pair i, j one has

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

there exists a unique vector field s on X such that $s|_{U_i} = s_i$.

Remark. The above mentioned property means that the assignment

$$U \mapsto \mathcal{T}(U)$$

is a *sheaf*.

2.2.6. Vector fields as derivations

Let $s \in \mathcal{T}(X)$ and let $f \in C^\infty(X)$. We define a new function $s(f) : X \longrightarrow \mathbb{R}$ by the formula

$$s(f)(x) = \langle f, s(x) \rangle.$$

Let us make an explicit calculation for the case X is an open subset of $\mathbb{R}^n$. In this case $TX = X \times \mathbb{R}^n$ and a vector has form

$$s = \sum_i f_i e_i$$

where $f_i \in C^\infty(X)$ and e_i are the constant vector fields forming the standard basis of $\mathbb{R}^n$. Then, if $g \in C^\infty(X)$, one has

$$s(g) = \sum_i f_i \frac{\partial g}{\partial x_i}.$$

In particular, $s(f)$ is always smooth.

We see that the vector field e_i considered as an operator on $C^\infty(X)$, carries g to $e_i(g) = \frac{\partial g}{\partial x_i}$. This formula explains another convenient notation for the constant vector field e_i, $e_i = \frac{\partial}{\partial x_i}$.

Our calculation shows that if X is an open subset of $\mathbb{R}^n$, $s(f)$ is smooth for any $s \in \mathcal{T}(X)$, $f \in C^\infty(X)$.

This easily implies the similar statement for arbitrary X: in order to check $s(f)$ is smooth, we have to check that for any chart $\phi : D \to U \subset X$ the composition $s(f)|_U \circ \phi$ is smooth. This basically reduces the claim to the case of an open subset of $\mathbb{R}^n$ (see Problem 3 below).

Note that if a vector field s vanishes as an operator on $C^\infty(X)$, that is it satisfies the condition $s(f) = 0$ for all functions f, then $s = 0$. In fact, choose a chart $\phi : D \to U$ of U. The vector field s has form

$$s = \sum f_i \frac{\partial}{\partial x_i}$$

in the coordinates of D. Then, if one of the functions f_i is nonzero at any point $t \in U$, the value $s(g)$ for a function g chosen as below will be non-zero.

Here how we can choose g. Choose $\epsilon > 0$ so that the disc of radius 2ϵ with the center at t belongs to D. There exists a smooth function g on D satisfying the following properties:

- $g(x_1, \ldots, x_n) = x_i$ inside the disc of radius ϵ with the center at t.
- $g = 0$ outside the disc of radius 2ϵ with the center at t.

The function g described above is a restriction of a smooth function on X (which one?). This is why $s(g) = 0$ which gives a contradiction.

Definition. An operator $s : C^\infty(X) \longrightarrow C^\infty(X)$ is called *a derivation* if it satisfies the Leibniz rule:

$$s(fg) = s(f)g + fs(g).$$

One can prove that the set of vector fields $\mathcal{T}(X)$ coincides with the set of derivations of $C^\infty(X)$.

2.2.7. Bracket of vector fields

We will now define a binary operation (bracket)

$$\mathcal{T}(X) \times \mathcal{T}(X) \longrightarrow \mathcal{T}(X), \quad (s,t) \mapsto [s,t]$$

satisfying some remarkable properties.

Let us study, first of all, the case X is an open subset of $\mathbb{R}^n$.

Lemma. *Let X be an open subset in $\mathbb{R}^n$ and $s, t \in \mathcal{T}(X)$. There exist a unique vector field $u \in \mathcal{T}(X)$ such that for each $f \in C^\infty(X)$ one has*

$$u(f) = s(t(f)) - t(s(f)).$$

Proof. This is an easy (but important) calculation. Let

$$s = \sum f_i \frac{\partial}{\partial x_i}, \quad t = \sum g_i \frac{\partial}{\partial x_i}.$$

Then for a function $h \in C^\infty$

$$s(t(h)) - t(s(h)) = \sum_j (\sum_i f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i}) \frac{\partial h}{\partial x_j}$$

which uniquely defines the vector field u by the formula

$$u = \sum_j (\sum_i f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i}) \frac{\partial}{\partial x_j}$$

□

With a minimal effort, the above lemma can be proven for any manifold X.

Theorem. *Let $s, t \in \mathcal{T}(X)$. There exist a unique vector field $v \in \mathcal{T}(X)$ such that for each $f \in C^\infty(X)$ one has*

$$v(f) = s(t(f)) - t(s(f)).$$

Proof. For each chart $\phi : D \to U \subset X$ the above calculation shows that the formula

$$v(f) = s(t(f)) - t(s(f))$$

defines a (unique) vector field on U. We denote it v_U. Once more, by uniqueness, for any pair of charts $\phi_i : D_i \to U_i, i = 1, 2$, the vector fields $vU_i|_{U_1 \cap U_2}$ coincide. This implies that there is a (unique) vector field v on X such that $v|_U = v_U$ for any chart. This is the vector field we were looking for. □

2.3. **Tangent map.** A smooth map $f : X \to Y$ defines for each point $x \in X$ the map $Tf_x : T_x(X) \longrightarrow T_{f(x)}(Y)$. The maps Tf_x for different $x \in X$ "glue together" into a smooth map $Tf : TX \longrightarrow TY$. Details being very easy, we only sketch them below.

2.3.1. The linear map Df_x. Any curve $\gamma : (-\epsilon, \epsilon) \to X$ with $\gamma(0) = x$ give rise to a curve $\delta = f \circ \gamma : (-\epsilon, \epsilon) \to Y$ with $\delta(0) = f(x)$. We need a calculation in coordinates to make sure that this correspondence carries equivalent curves to equivalent curves and that the obtained map $T_x(X) \to T_{f(x)}(Y)$ is linear.

Choose maps containing x and $y := f(x)$ respectively. Let $\phi_1 \longrightarrow U_1 \ni x$ and $\phi_2 \longrightarrow U_2 \ni y$. Then the standard calculation with Chain rule shows the induced map from $T_x(X)$ to $T_y(Y)$, after identification with $\mathbb{R}^n$ and $\mathbb{R}^m$ via $D\phi_1$ and $D\phi_2$, is described by the Jacobi matrix $D(\phi_2^{-1} \circ \phi_1)$.

2.3.2. The map Df of the tangent bundles

We have already constructed a map of the sets $TX \to TY$ which will be denoted Tf. Let us check it is smooth. We will use the (generalized) charts

$(\phi, D\phi) : D \times \mathbb{R}^n \to \pi^{-1}(U)$ constructed in the definition of TX. The same calculation as above shows the map Tf is given in the chosen coordinates by the map $(\phi_2^{-1} \circ \phi_1, D(\phi_2^{-1} \circ \phi_1))$.

Homework.

1. Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^m$ be a smooth function. Assume that the set

$$X = \{x \in \mathbb{R}^N | F(x) = 0\}$$

is a smooth submanifold (eventhough we do not assume that the rank of DF equals m). Is this still true that

$$T_x(X) = \mathrm{Ker}(DF_x)?$$

2. Prove the equivalence of two definitions of $\langle f, v \rangle$ mentioned in Lecture 3.
3. Prove that for $s \in \mathcal{T}(X),\ f \in C^\infty(X)$ the function $s(f)$ is smooth.
4. Prove Leibniz rule: for $s \in \mathcal{T},\ f, g \in C^\infty(X)$

$$s(fg) = s(f)g + fs(g).$$