DIFFERENTIAL GEOMETRY. LECTURE 4, 26.05.08

2.1. **Tangent bundle.** We want now to glue all tangent spaces $T_x(X),\ x \in X$, into a big manifold TX endowed with a smooth map $\pi : TX \longrightarrow X$ such that $T_xX = \pi^{-1}(x)$.

In the case X is an open subspace of $\mathbb{R}^n$, we will get just the direct product $X \times \mathbb{R}^n$. In general we will have to "glue" TX from the tangent bundles corresponding to the charts.

We will proceed as follows. First of all, we have $TX = \coprod_{x \in X} T_x(X)$ as a set. Using the atlas of X, we will present a covering of TX by open subsets of $\mathbb{R}^{2n}$ where $n = \dim X$. This will define a topology on TX, as well as an atlas.

2.1.1. Topology on TX Let $\phi : D \longrightarrow U \subset X$ be a chart of X. Recall that for each $x \in U$ an isomorphism $D\phi : \mathbb{R}^n \longrightarrow T_x(X)$ is defined. This defines an injection

$$(\phi, D\phi) : D \times \mathbb{R}^n \longrightarrow \coprod_{x \in X} T_x(X) = TX$$

carrying the pair (t, v) to $(\phi(t), D\phi(v)) \in T_{\phi(t)}(X)$.

Note that

- The product $D \times \mathbb{R}^n$ is an open subset in $\mathbb{R}^{2n}$.
- The images of $(\phi, D\phi)$ cover the whole TX.

We declare a subset V of TX open if and only if for any chart ϕ in the atlas of X the preimage $(\phi, D\phi)^{-1}(V)$ is open in $D \times \mathbb{R}^n$.

2.1.2. The charts of TX

Of course, we would like to say that the maps $(\phi, D\phi) : D \times \mathbb{R}^n \to TX$ form the charts of TX. This is true, but requires checking that $(\phi, D\phi)$ is a homeomorphisms of $D \times \mathbb{R}^n$ with its image in TX and that this image is open in TX.

This is clearly a bijection. To prove it is a homeomorphism, we have to check that for any open subset V in $D \times \mathbb{R}^n$ its image $(\phi, D\phi)(V)$ is open in TX. By definition, this amounts to checking the following. Choose any (other) chart $\psi : D_1 \longrightarrow U_1 \subset X$. We have to check that the preimage

$$(\psi, D\psi)^{-1}(\phi, D\phi)(V) \qquad (1)$$

is open in $D_1 \times \mathbb{R}^n$.

We will get this simultaneously with the compatibility of the charts. Let as above $\phi : D \to U \subset X$ and $\psi : D_1 \to U_1 \subset X$ are two (compatible) charts of X. By definition, this means that the map

$$\psi^{-1} \circ \phi : \phi^{-1}(U \cap U_1) \longrightarrow \psi^{-1}(U \cap U_1)$$

is a diffeomorphism.

The intersection of the images of the maps $(\phi, D\phi)$ and $(\psi, D\psi)$ in TX is clearly $\coprod_{x \in U \cap U_1} T_x(X)$.

The inverse images of this set in $D \times \mathbb{R}^n$ and $D_1 \times \mathbb{R}^n$ are obviously $\phi^{-1}(U \cap U_1) \times \mathbb{R}^n$ and $\psi^{-1}(U \cap U_1) \times \mathbb{R}^n$.

The transition map $(\psi, D\psi)^{-1}(\phi, D\phi)$ from one space to the other is given by the formula

$$(t, v) \mapsto (\psi^{-1}\phi(t), D(\psi^{-1} \circ \phi)(v))$$

— this follows from Lemma 2.2.4 (Lecture 3). This is clearly a bijection since the Jacobi matrix of a diffeomorphism is nondegenerate. A small calculation below (Lemma 2.1.3) shows it is also a diffeomorphism.

We still have a small debt, that of checking that (1) is open. But now this set is the image of the open set $V \cap (\phi^{-1}(U \cap U_1) \times \mathbb{R}^n)$ under a diffeomorphism $(\psi, D\psi)^{-1}(\phi, D\phi)$. Therefore, it is open.

2.1.3. **Lemma.** *Let U be an open subset in $\mathbb{R}^n$ and let $f : U \longrightarrow \mathbb{R}^m$ be a smooth map. Define $F : U \times \mathbb{R}^n \longrightarrow \mathbb{R}^m \times \mathbb{R}^m$ by the formula*

$$F(t, v) = (f(t), Df_t(v))$$

where Df_t denotes, as usual, the Jacobi matrix of f at t. Then the Jacobi matrix of F at (t, v) is of form

$$\left(\begin{array}{c|c} Df_t & 0 \\ \hline * & Df_t \end{array}\right) \tag{2}$$

In particular, if f is a local diffeomorphism, than F is a local diffeomorphism as well.

Proof. Direct computation. The bottom left quadrant consists of expressions like

$$\sum_k \frac{\partial^2 f_i}{\partial x_j \partial x_k} v_k.$$

□

2.2. **Properties of Tangent bundles. Vector fields.** We start with (long overdue)

2.2.1. **Definition.** Let X, Y be smooth manifold. Their direct product $X \times Y$ is defined as follows.

- As a topological space, it is the direct product of X and Y.

- Each pair of charts $\phi : D_1 \to U_1 \subset X$ and $\psi : D_2 \to U_2 \subset Y$ defines a (generalized) chart
$$\phi \times \psi : D_1 \times D_2 \to U_1 \times U_2 \subset X \times Y$$
for $X \times Y$.

2.2.2. Open subsets of $\mathbb{R}^n$

If X is an open subset of $\mathbb{R}^n$, its tangent bundle is diffeomorphic to $X \times \mathbb{R}^n$. We will see later that the tangent bundle does not always have such a simple form.

2.2.3. **Definition.** A vector field on X is a smooth map $s : X \longrightarrow TX$ such that $\pi \circ s = \mathrm{id}_X$.

The meaning of the definition is clear: a vector field assigns to any point $x \in X$ a tangent vector $s(x) \in T_x(X)$ which "smoothly depends" on x.

The set of vector fields on X, denoted $\mathcal{T}(X)$, has two natural operations, addition and multiplication by a smooth function:

- $(s_1 + s_2)(t) = s_1(t) + s_2(t)$.
- For $a \in C^\infty(X),\ s \in \mathcal{T}(X)$, one defines $as(t) = a(t)s(t)$.

A vector field on an open subset X of $\mathbb{R}^n$, that is, a section of the projection $X \times \mathbb{R}^n \to X$, is given by a smooth function
$$s : X \longrightarrow \mathbb{R}^n$$
or, what is the same, by n smooth functions. In this case $\mathcal{T}(X)$ really "looks like" an n-dimensional vector space over C^∞.

Let e_i denote the vector field assigning to all points of X the same i-th coordinate vector e_i of $\mathbb{R}^n$. Then a general vector field has form
$$s = f_1 e_1 + \ldots + f_n e_n$$
where $f_i \in C^\infty(X)$.

The same holds for any manifold X whose tangent bundle is isomorphic to $X \times \mathbb{R}^n$.

Let us mention without proof

2.2.4. **Theorem.** *Let s be a vector field on $X = S^2$. Then there exists at least one point $x \in S^2$ such that $s(x) = 0$.*

This implies that the tangent bundle of the two-dimensioinal sphere is not isomorphic to $S^2 \times \mathbb{R}^2$.