DIFFERENTIAL GEOMETRY. LECTURE 3, 22.05.08

2. Tangent space. Tangent bundle

We know from the Elementary Calculus course that the graph of a function $y = f(x)$ has a tangent line at a point (a, b), $b = f(a)$ if f is differentiable at a; this line has a slope $f'(a)$.

This has a far-reaching generalization in the theory of smooth manifolds. We will first of all consider the case of a submanifold of $\mathbb{R}^N$, and then will try to formulate a coordinate-independent notion of a tangent space.

Here is a short plan of what we intend to do.

- Define what is a tangent vector at a point of a submanifold of $\mathbb{R}^N$.
- Define the tangent space at a point of a submanifold of $\mathbb{R}^N$ — this is the vector space of all tangent vectors.
- Define the tangent space for general smooth manifolds.
- Study the collection of all tangent spaces at different points. It turns out they can be assembled in a new smooth manifold called *the tangent bundle*.

2.1. **Tangent space to a submanifold of $\mathbb{R}^N$.** Let $X \in \mathbb{R}^N$ be a smooth submanifold and let $x \in X$. We say that a vector $v \in \mathbb{R}^N$ is a tangent vector to X at x if there is a smooth curve $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^N$ such that $\gamma(0) = x$, the image of γ lies in X, and $\gamma'(0) = v$. Note that by definition $\gamma = (\gamma_1, \ldots, \gamma_N)$, so the vector γ' has by definition the components $(\gamma'_1, \ldots, \gamma'_N)$.

2.1.1. **Proposition.** 1. *The set of tangent vectors $T_x(X)$ to X at x is a vector subspace of $\mathbb{R}^N$.*

2. *If X is given locally, in a neighborhood of x, by an equation*

$$F(y) = 0,$$

where $F : W \to \mathbb{R}^m$ is a smooth function, the set $T_x(X)$ of tangent vectors at x can be described as

$$\{v \in \mathbb{R}^N | DF_x(v) = 0\} = \mathrm{Ker}(DF_x), \tag{1}$$

where $DF_x : \mathbb{R}^N \longrightarrow \mathbb{R}^m$ is the linear map described by the Jacobi matrix of F at x.

Proof. Let $F : W \longrightarrow \mathbb{R}^m$ be a local equation of X near X, that is a smooth function such that $X \cap W = \{y \in W | F(y) = 0\}$ and DF has rank m at x. Recall that, using Implicit Function theorem, we constructed in 1.11 a chart

$\phi : D \longrightarrow U \subseteq W$ where D is an open subset of $\mathbb{R}^n$, $n = N - m$, and $\phi(u) = (u, f(z)) \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^M$.

Since the chart $\phi : D \longrightarrow U$ is a diffeomorphism, any curve $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^N$ with image in U can be presented as the composition $\gamma = \phi \circ \delta$ for a unique curve $\delta : (-\epsilon, \epsilon) \longrightarrow D$. By the Chain Rule,

$$\gamma'(0) = D\phi(\delta'(0)),$$

where, as usual, $D\phi$ is given by the Jacobi matrix of the map $\phi : D \longrightarrow \mathbb{R}^N$. Since D is open in $\mathbb{R}^n$, $\delta'(0)$ can be any vector in $\mathbb{R}^n$, so $T_x(X)$ is the image of the map $D\phi : \mathbb{R}^n \to \mathbb{R}^N$.

This proves that $T_x(X)$ is a vector subspace of $\mathbb{R}^N$.

Let us deduce the second claim. Once more by the Chain Rule, $DF \circ D\phi = D(F \circ \phi) = 0$ since F vanishes on $\phi(D) = U$. Since DF has rank m, its kernel has dimension $N - m = n$. Since it contains $T_x(X) = D\phi(\mathbb{R}^n)$, the spaces coincide: $T_x(X) = \mathrm{Ker}(DF)$. □

2.2. **Tangent space for abstract manifolds.** For $X \subset \mathbb{R}^N$ we defined the tangent space $T_x(X)$ as a subspace of $\mathbb{R}^N$. For a general manifold we have a priori no big vector space to embed $T_x(X)$. Thus, one should look for another way for defining $T_x(X)$.

As we have seen, any vector $v \in T_x(X)$ is defined by a curve γ on X. Of course, some curves define the same tangent vector. This leads to the following general definition.

In what follows *a smooth curve* on X is a smooth map from an open segment (a, b) to X.

2.2.1. **Definition.** Let X be a smooth manifold, $x \in X$ a point. Two (smooth) curves $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \longrightarrow X$ with $\gamma_1(0) = \gamma_2(0)$ are called *equivalent* if for any chart $\phi : D \longrightarrow U \subset X$ containing x the curves $\phi^{-1} \circ \gamma_1$ and $\phi^{-1} \circ \gamma_2$ have the same tangent vector at 0:

$$(\phi^{-1} \circ \gamma_1)'(0) = (\phi^{-1} \circ \gamma_2)'(0).$$

Note that in order to check that two curves are equivalent at a point, it is sufficient to check the condition for only one chart containing the point.

2.2.2. **Definition.**

1. A tangent vector of a manifold X at a point x is an equivalence class of curves $\gamma : (-\epsilon, \epsilon) \longrightarrow X$ satisfying the condition $\gamma(0) = x$.
2. The tangent space $T_x(X)$ is the set of equivalence classes of curves.

The above definition gives immediately what we expect in the case of submanifolds of $\mathbb{R}^N$. Note that we do not see from the definition that the tangent space has a structure of a vector space. As usual, this will be done using the charts. Note first of all

2.2.3. **Lemma.** *Let $x \in X$ and let $\phi : D \longrightarrow U \subset X$ be a chart containing x. Define a map $D\phi : \mathbb{R}^n \longrightarrow T_x(X)$ by assigning for each curve*

$$\gamma : (-\epsilon, \epsilon) \longrightarrow X, \ \gamma(0) = x$$

to the vector $(\phi^{-1} \circ \gamma)'(0) \in \mathbb{R}^n$ of the class $[\gamma] \in T_x(X)$. Then $D\phi$ is a bijection.

Proof. This is a tautology. □

The map $D\phi$ allows one to define on $T_x(X)$ a structure of a vector space: one defines $v + w = D\phi(D\phi^{-1}v + D\phi^{-1}w)$ and similarly for the multiplication by a number. One has, however, to check that this linear structure will be the same if we replace a chart with another one.

This results from the following

2.2.4. **Lemma.** *Let $\phi_i : D_i \longrightarrow U_i$, $i = 1, 2$, be two compatible charts of X containing x. Then one has*

$$D\phi_1 = D\phi_2 \circ A \tag{2}$$

where A is the Jacobi matrix of the diffeomorphism $\phi_2^{-1} \circ \phi_1$ at the point $\phi_1^{-1}(x) \in D_1$.

Proof. Immediate consequence of the Chain Rule. □

Now it is easy to see that usage of two different charts gives the same vector space structure on $T_x(X)$. In fact, if $v, w \in T_x(X)$ than the first formula gives

$$v + w = D\phi_1(D\phi_1^{-1}(v) + D\phi_1^{-1}(w)).$$

By (2) $D\phi_1 = D\phi_2 \circ A$, so

$$v + w = D\phi_2 \circ A(A^{-1} \circ D\phi_2^{-1}(v) + A^{-1} \circ D\phi_2^{-1}(w)).$$

Since A is linear, this implies that

$$v + w = D\phi_2(D\phi_2^{-1}(v) + D\phi_2^{-1}(w)).$$

2.2.5. Tangent vector as a derivative

Let $v \in T_x(X)$ and let $f \in C^\infty(X)$. We will assign now a number $\langle f, v\rangle$ which can be interpreted as the derivative of f in direction v. Here is the definition.

Let v be the class of a curve $\gamma : (-\epsilon, \epsilon) \longrightarrow X$ such that $\gamma(0) = x$. The composition $f \circ \gamma$ is a function defined at $(-\epsilon, \epsilon)$, with valies in $\mathbb{R}$. Its derivative at 0 is what we need:

Definition.

$$\langle f, v\rangle = (f \circ \gamma)'(0).$$

Formally, one has to check that the result does not depend on the choice of γ. Instead of doing this, we will give an equivalent definition which does not depend on the choice of γ (but formally depends on the choice of a chart).

Let $\phi : D \to U \subseteq X$ be a chart containing $x \in X$.

Definition.

$$\langle f, v\rangle = (f \circ \phi)'_w,$$

where $w = D\phi^{-1}(v) \in \mathbb{R}^n$ and the right-hand side is the directional derivative of the function $f \circ \phi$ along w.

It is a good exercise to check that both definitions agree (and are therefore independent of the choices).

Homework.

1. Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^m$ be a smooth function. Assume that the set

$$X = \{x \in \mathbb{R}^N | F(x) = 0\}$$

is a smooth submanifold (eventhough we do not assume that the rank of DF equals m). Is this still true that

$$T_x(X) = \mathrm{Ker}(DF_x)?$$

2. Prove the equivalence of two definitions of $\langle f, v\rangle$ mentioned above.
3. Prove Leibniz rule: for $x \in X,\ v \in T_x(X),\ f, g \in C^\infty(X)$

$$\langle fg, v\rangle = \langle f, v\rangle g(x) + f(x)\langle g, v\rangle.$$