DIFFERENTIAL GEOMETRY. LECTURE 2, 19.05.08

1.12. **Remarks to the definition.**

1.12.1. Dimension If a smooth manifold X is connected, all its charts ($\phi : \mathbb{R}^n \supset D \longrightarrow U \subseteq X$) have the same dimension n. This number is called *the dimension of* X.

1.12.2. "Generalized" charts. Assume D is an arbitrary open subset in $\mathbb{R}^n$ (not necessarily a disc). A homeomorphism $\phi : D \to U \subset X$ to an open subset of X can be called a generalized chart of X. The notion of compatibility of the charts has a perfect meaning for generalized charts. Moreover, since any open subset of $\mathbb{R}^n$ is a (may be, infinite) union of discs, any generalized chart of X can be replaced by a collection of compatible "conventional" charts. Thus, there is no real difference between two notions of chart.

1.12.3. Separatedness of X. Why is it important for X to be Hausdorff? Here is an example of what we want to avoid. Take two copies of the real line $\mathbb{R}$ (with the coordinate x and y respectively) and let X be the result of gluing of these lines along the equivalence

$$\{x = a\} \sim \{y = a\} \text{ for all } a \neq 0.$$

We get a real line with "two copies of zero": the points $\{x = 0\}$ and $\{y = 0\}$ have no disjoint neighborhoods. This has some nasty complications we would like to avoid: any continuous function on X has the same values at these two points. If we try to define a distance between them, we will have a problem.

1.12.4. Compactness at ∞. Here is an example of "very long line" we would like to avoid (sketch).

Let Ω be an ordinal (a well-ordered set). Define $X = \Omega \times [0, 1)$ as a set with the lexicographic order. Define a topology on X choosing as the basis the collection of open segments in X.

First of all, one can prove (by induction) that if Ω is countable, the result will be homeomorphic to $[0, 1)$ (or, what is the same, to $[0, \infty)$).

cNow, if Ω is the first non-countable ordinal, any open segment will still be homeomorphic to $(0, 1)$, but the whole line will not be homeomorphic to $[0, \infty)$ — it will be "much longer".

1.13. **Example:** $\mathbb{RP}^n$**.** The real projective space is different from what we saw until now: it has no obvious embedding into affine space.

As a set, $\mathbb{RP}^n$ is the set of lines in $\mathbb{R}^{n+1}$ passing through the origin. It can be otherwise described as the set of collections $(x_0, x_1, \dots, x_n)$ *not all x_i being equal to zero*, modulo relation

$$(x_0, x_1, \dots, x_n) \sim (\lambda x_0, \lambda x_1, \dots, \lambda x_n),\ \ \lambda \in \mathbb{R}^*.$$

Another presentation of $\mathbb{RP}^n$: this is the quotient of $S^n \subset \mathbb{R}^{n+1}$ by the (simpler) relation

$$(x_0, x_1, \dots, x_n) \sim (-x_0, -x_1, \dots, -x_n).$$

The topology on $\mathbb{RP}^n$ is defined by the projection from the sphere: a subset in $\mathbb{RP}^n$ is open iff its preimage in S^n is open.

Let $\pi : S^n \to \mathbb{RP}^n$ be the projection. If U is open in S^n, $\pi^{-1}\pi(U) = U \cup -U$ is open, so by definition $\pi(U)$ is open in $\mathbb{RP}^n$. Moreover, if $U \cap -U = \emptyset$, the restriction

$$\pi_U : U \longrightarrow \pi(U)$$

is a homeomorphism.

Thus, if one chooses a collection of charts for S^n small enough so that $U \cap -U = \emptyset$, this gives automatically a collection of charts for the quotient $\mathbb{RP}^n$.

1.14. **Smooth functions. Smooth maps.**

1.14.1**. Definition.** A function $f : X \longrightarrow \mathbb{R}$ is smooth if for each chart $(\phi : D \longrightarrow U)$ the composition $f \circ \phi : D \longrightarrow \mathbb{R}$ is smooth. Thus, f is smooth iff its restriction to any open covering is smooth. It is enough to check that a restriction of f to some open covering is smooth.

1.14.2**. Definition.** A map $f : X \longrightarrow Y$ is smooth iff for any pair of charts $\phi : D_1 \to U \subset X$ and $\psi : D_2 \longrightarrow V \subset Y$ the composition $\psi^{-1} \circ f \circ \phi$ defines a smooth map from $\phi^{-1}(U \cap f^{-1}(V))$ to D_2.

Smooth functions can be added and multiplied. The collection of smooth functions on X is denoted $C^\infty(X)$. This is a commutative ring.

Smooth functions on X are the same as smooth maps $X \to \mathbb{R}$.

Smooth maps can be composed (see below).

1.14.3**. Proposition.** *Composition of smooth maps is a smooth map.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. This means that for any choice of three charts, $\phi : D \to U \subset X$, $\psi : D \to V \subset Y$, $\chi : D \to W \subset Z$, the maps $\psi^{-1} \circ f \circ \phi$ and $\chi^{-1} \circ g \circ \psi$ are smooth in their respective domains. This implies that their composition

$$\chi^{-1} \circ g \circ f \circ \phi \tag{1}$$

is smooth in the intersection of the respective domains

$$\phi^{-1}(U \cap f^{-1}(V \cap g^{-1}(W))). \tag{2}$$

Since the previous claim holds for any chart $\psi : D \to V \subset Y$, the composition (**??**) is smooth in the union of all possible (**??**) which is the whole of

$$\phi^{-1}(U \cap (g \circ f)^{-1}(W)).$$

□

1.15. **Open submanifolds. Submanifolds.** The first example of a smooth manifold has been an open subset of $\mathbb{R}^n$. This can be generalized as follows.

Let X be a smooth manifold and let U be an open subset of X. The intersection of a chart in X with U is a (generalized) chart of U. In this way, U aquires a canonical structure of a smooth manifold.

1.15.1**. Definition.** An open submanifold of X is an open subset U endowed with the canonical structure of a smooth manifold.

The following notion of submanifold is more general.

1.15.2**. Definition.** Let Y be a subset of a manifold X for which there exists a collection of charts $\phi_i : D_i \to U_i$ covering X so that for each i the subset $\phi_i^{-1}(U \cap Y)$ is a submanifold of $\mathbb{R}^n$. Then Y is called a submanifold of X.

We claim that a submanifold Y of X has a canonical structure of a smooth manifold. In fact, choose a covering family of charts $\phi_i : D_i \to U_i$ as in the definition above. For each i choose an atlas for $\phi_i^{-1}(U \cap Y)$, and compose it with ϕ_i. This will give a required atlas for Y.

Of course, the notion of open submanifold is a (very) special case of the notion of submanifold.

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1.16. **Immersions.** If X is a submanifold of Y, the embedding of X into Y is a smooth map: if $\phi : D_1 \longrightarrow U$ is a chart of Y and $\psi : D_2 \to V$ is a chart of $\phi^{-1}(U \cap X)$, then the composition $\phi \circ \psi : D_2 \to \phi(V)$ is a chart of X and the embedding of X into Y is given in this pair of charts by

$$\phi^{-1} \circ \phi \circ \psi : D_2 \to V \subset D_1.$$

This map is of course smooth. But it also satisfies the following extra property

The rank of the Jacobi matrix of ψ coincides with the number of its columns. In other words, the Jacobi matrix defines an injective linear map.

The converse is not necessarily true. For example, the smooth map

$$f : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$$

given by the formlas $x = t^2 - 1,\ y = t(t^2 - 1)$ has nowhere vanishing Jacobi matrix, but the image of $\mathbb{R}$ in $\mathbb{R}^2$ has a self-intersection at 0 ($t = 0, 1$), so it is not a submanifold.

1.16.1**. Definition.** A smooth map $f : X \to Y$ is called *an immersion* if for each $x \in X,\ y = f(x) \in Y$ there are charts $\phi : D_1 \to U$ and $\psi : D_2 \to V$ such that $x \in U,\ y \in V$ and the rank of the Jacobian matrix of the map $\psi^{-1} \circ f \circ \phi : \phi^{-1}(U \cap f^{-1}(V)) \to D_2$ equals the number of columns (that is $\dim X$).

1.17. **Submersions.** There is another case a smooth map looks nicely.

1.17.1**. Definition.** A smooth map $f : X \to Y$ is called *a submersion* if for each $x \in X,\ y = f(x) \in Y$ there are charts $\phi : D_1 \to U$ and $\psi : D_2 \to V$ such that $x \in U,\ y \in V$ and the rank of the Jacobian matrix of the map $\psi^{-1} \circ f \circ \phi : \phi^{-1}(U \cap f^{-1}(V)) \to D_2$ equals the number of rows (that is $\dim Y$).

A typical example of a submersion appears in Theorem 11: a function $f : \mathbb{R}^N \longrightarrow \mathbb{R}^m$ whose Jacobian matrix has rank m gives rise to submanifolds — the level sets $f(x) = a \in \mathbb{R}^m$. This is a general pattern.

1.17.2**. Proposition.** *Let $f : X \longrightarrow Y$ be a submersion. Then for any $y \in Y$ the preimage $f^{-1}(y) = \{x \in X | f(x) = y\}$ is a smooth submanifold of X.*

Proof. This easily follows from Theorem 1.11. □

1.17.3**. Remark.** One can explain why immersions and submersions are much better than general smooth maps. In both cases the Jacobian matrix has a maximal rank. If this property is fulfilled at a given point, this implies an existence of a non-vanishing minor which implies that in a neighborhood of the given point the rank remains maximal. In general, the rank of the Jacobian matrix can be different (greater) in an arbitrarily small neighborhood of the given point.