15/5 DG. Lecture 1 (2h)

Plan. Smooth manifolds. What for?
Formal definition. Examples.

1. Example: $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 = 1\}$
this is clearly a 2-dimensional object; would like to define it with 2 coordinates.
Possible only locally.
Atlas of charts – a possible way of looking at such geometric object.

2. What for? For instance, describe a mechanical system: movements of a solid bar = Pair of points with distance $= d$

$$X = \{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 \mid \sum_i (x_i - y_i)^2 = d^2\}$$

Note: in other coordinats $X = \mathbb{R}^3 \times S^2$.

3.* Riemannian geometry studies smooth manifolds in which one can define a length of a path. Keywords: geodesic, curvature.

4. A typical way of determining a smooth manifold X – by its embedding into another smooth manifold, e.g., $\mathbb{R}^N$. Implicit function thm will give a sufficient condition ensuring that (sometimes)

$$X = \{x \in \mathbb{R}^N \mid f(x) = 0\}$$ is a smooth manifold. Here $f: \mathbb{R}^N \to \mathbb{R}^n$ is a smooth mapping.

5. <u>Def</u> Let X be a topological space, $U \subseteq X$ an open subset. A chart is a homeomorphism $\varphi : D \to U$ where D is a disc in some R^n: $D \subseteq R^n$.

Two charts, $\varphi_1 : D_1 \to U_1 \subseteq X$, $\varphi_2 : D_2 \to U_2 \subseteq X$, are <u>compatible</u> if the composition

$$\varphi_2^{-1}\varphi_1 : \quad \varphi_1^{-1}(U_1 \cap U_2) \longrightarrow \varphi_2^{-1}(U_1 \cap U_2)$$

is a <u>diffeomorphism</u>.

6. <u>Recall</u>: $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ is given by a collection $(f_1, f_2, \dots, f_m)$ of functions $U \longrightarrow \mathbb{R}$. It is <u>smooth</u> if f_i are smooth (of class C^∞).
<u>Diffeomorphism</u> = smooth + invertible in the class of smooth maps.

7. Some obvious examples.
$X = \mathbb{R}^n$, $\forall x \in \mathbb{R}^n$ a chart
$\varphi_x : D \longrightarrow \mathbb{R}^n$ $\varphi_x(y) = x + y$.
All charts φ_x are compatible.

8. <u>Definition</u> A smooth manifold is a top. space X with a collection of charts, compatible and covering the whole X.
Extra conditions on X are usually assumed:
1) X is Hausdorff 2) X is countable at ∞
(that is, X is a countable union of compact sets)

9. Example (obvious)

An open subset of $\mathbb{R}^n$ has a structure of a smooth manifold.

Remark The following definition should have been given long ago:

Def Two atlases $A_1 = \{\varphi_i : D \to X\}$, $A_2 = \{\psi_j : D \to X\}$ are equivalent if $A_1 \cup A_2$ is an atlas.

10. Recall Implicit function theorem.

Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be smooth at a neighborhood of $(a, b) \in \mathbb{R}^{n+m}$, $F = (F_1, \dots, F_m)$, $F(a, b) = c \in \mathbb{R}^m$.

Assume that the matrix $\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,m \\ j=n+1,\dots,n+m}}$ is invertible. Then $\exists\, U \ni a$, $V \ni b$ open neighborhoods in $\mathbb{R}^n, \mathbb{R}^m$ respectively, and a unique smooth function $f : U \to V$ such that for $(x, y) \in U \times V$ one has

$$F(x, y) = c \iff y = f(x)$$

Def A subset $X \subseteq \mathbb{R}^N$ is called a submanifold of dimension n if $\forall x \in X$ $\exists\, U \subset \mathbb{R}^N$, $U \ni x$, $\exists\, F : U \to \mathbb{R}^m$, $m = N - n$, having the jacobian matrix of rank m, so that $X \cap U = \{y \in U \mid F(y) = 0\}$

Example $S^n = \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i = 1 \right\}$ is a submanifold of $\dim = n$.

11. <u>Thm – construction</u> A submanifold $X \subseteq \mathbb{R}^N$ has a canonical structure of a smooth manifold

<u>Proof</u> We will present the construction of an atlas.

<u>step 1</u> $\forall x \in X$ find a chart covering x.

<u>step 2</u> Prove the charts are compatible.

<u>step 1:</u> Let $x \in X$, $x \in W \subset \mathbb{R}^N$, $F: W \to \mathbb{R}^m$ of rank m at x, $X \cap W = F^{-1}(0)$. By Linear algebra, one can choose m columns in the Jacobi matrix of F at x which are linearly independent they correspond to m coordinates which we will denote $x_{n+1}, \dots, x_N$. By IFT, there exist neighborhoods $U \ni a$, $V \ni b$, where $x = (a, b)$, and a unique smooth function $f: U \to V$ such that

$(u, v) \in (U \times V) \cap X \iff v = f(u)$. Thus,

$\varphi: U \to X$ given by $\varphi(u) = (u, f(u))$

is a chart.

<u>Step 2</u> Let us prove compatibility of so constructed charts. Assume U_1, U_2 given, with

$\varphi_i : U_i \to X$ constructed as above

(another collection of independent columns could have been chosen).

The map $\varphi_1 : U_1 \to X$ is given by the formula $\varphi_1(u) = (u, f(u))$.

The map $\varphi_2^{-1} \varphi_1 : \varphi_1^{-1}(\overline{U}_1 \cap \overline{U}_2) \to \varphi_2^{-1}(\overline{U}_1 \cap \overline{U}_2)$

[Here $\overline{U}_i = \varphi_i(U_i) \subseteq X$]

can be described as the composition

$U_1 \longrightarrow \mathbb{R}^N \xrightarrow{\pi} U_2$

where the map π is a projection to a part of the coordinates. Therefore, it is automatically smooth. $\boxtimes$

Homework (due 22/5)

1. Check in the last theorem that X is Hausdorff and countable at ∞

2. Present an explicit atlas for S^2. For $T^2 = S^1 \times S^1$

3. Prove the following groups are smooth manifolds:

   a) $GL(n, \mathbb{R}) = \{ M \in Mat_n(\mathbb{R}) \mid \det M \neq 0 \}$

   b) $SL(n, \mathbb{R}) = \{ M \in Mat_n(\mathbb{R}) \mid \det M = 1 \}$

   c) $O(2, \mathbb{R}) = \{ M \in Mat_2(\mathbb{R}) \mid M \cdot M^t = I \}$