DIFFERENTIAL GEOMETRY, LECTURE 18-19, JULY 21-24

7. Curvature

If we look at a small neighborhood of a two-dimensional sphere, we see that even locally the sphere does not look as $\mathbb{R}^2$. Thus, ther should exist local invariants of Riemann manifolds which distinguish between, say, S^2 and $\mathbb{R}^2$. Curvature is the most famous one.

7.1. **Curvature of a connection.** Let X be a smooth manifold, $\mathbb{V}$ a vector bundle over X and let $\nabla : \mathcal{T} \times \Gamma(\mathbb{V}) \longrightarrow \Gamma(\mathbb{V})$ be a connection on $\mathbb{V}$.

Define $R^\nabla : \mathcal{T} \times \mathcal{T} \times \Gamma(\mathbb{V}) \longrightarrow \Gamma(\mathbb{V})$ by the formula

$$R^\nabla(s, t, v) = \nabla_s \circ \nabla_t(v) - \nabla_t \circ \nabla_s(v) - \nabla_{[s,t]}(v). \tag{1}$$

We remember that connections are C^∞-linear in the first (vector field) argument, but not completely linear in the second (section of $\mathbb{V}$) argument.

7.1.1. **Lemma.** *The operator R^∇ is C^∞-linear in all three arguments.*

Proof. R^∇ is trilinear over $\mathbb{R}$. Recall that $\nabla_{fs} = f\nabla_s$ and $\nabla_s(fv) = f\nabla_s(v) + s(f)v$, so

$$R^\nabla(fs, t, v) = f\nabla_s(\nabla_t(v)) - \nabla_t(f\nabla_s(v)) - \nabla_{[fs,t]}(v) = f\nabla_s(\nabla_t(v)) - f\nabla_t(\nabla_s(v)) - \\ -t(f)\nabla_s(v) - f\nabla_{[s,t]}(v) + t(f)\nabla_s(v) = fR^\nabla(s, t, v).$$

Note that the formula for R^∇ is skew-symmetric with respect to the first two arguments, so that C^∞-linearity in the second argument is automatic. Finally,

$$R^\nabla(s, t, fv) = \nabla_s(\nabla_t(fv)) - \nabla_t(\nabla_s(fv)) - \nabla_{[fs,t]}(fv) = \nabla_s(f\nabla_t(v) + t(f)v) - \\ \nabla_t(f\nabla_s(v) + s(f)v) - f\nabla_{[s,t]}(v) - [s, t](f)v.$$

We have

$$\nabla_s(f\nabla_t(v) + t(f)v) = f\nabla_s(\nabla_t(v)) + s(f)\nabla_t(v) + t(f)\nabla_s(v) + s(t(f))v.$$

Substracting from this expression the one obtained by replacing s and t, one finally gets

$$R^\nabla(s, t, fv) = fR^\nabla(s, t, v).$$

□

As we know, C^∞-linearity in all arguments implies that R^∇ comes from a map of vector bundles (which we denote by the same letter)

$$R^\nabla : TX \otimes TX \otimes \mathbb{V} \longrightarrow \mathbb{V}.$$

Because of skew-symmetricity, it can be written as

$$R^\nabla : \wedge^2(TX)\otimes\mathbb{V} \longrightarrow \mathbb{V}.$$

This latter map of vector bundles will be called *the curvature tensor* of the connection ∇.

7.2. **Geometric meaning.** We know that a connection ∇ allows one to construct parallel sections along curves: given $\gamma : [a,b]\to X$ and given $v\in\mathbb{V}_{\gamma(a)}$ one can, using a connection, construct a unique parallel section $v(t)$ with $v(a)=v$. One can ask, whether parallel sections along parametrized surfaces exist.

Given a parametrized surface $\sigma:\mathbb{R}^2\to X$ and a vector $v\in\mathbb{V}_{sigma(0,0)}$ we can try to construct a parallel section $v(s,t)$ along σ with $v(0,0)=v$ in two steps: first of all, over a line $t=0$, and then over the lines $s=s_0$ for each s_0 separately.

We will now show that this is possible if and only if the curvature $R^\nabla(\frac{\partial}{\partial s},\frac{\partial}{\partial t})$ vanishes.

In fact, $v(s,t)$ is parallel iff $\nabla_{\frac{\partial}{\partial s}}(v)=\nabla_{\frac{\partial}{\partial t}}(v)=0$. The section $v(s,t)$ constructed above obviously satisfies the condition $\nabla_{\frac{\partial}{\partial t}}(v)=0$. The condition $\nabla_{\frac{\partial}{\partial s}}(v)=0$ is by construction of $v(s,t)$ fulfilled at $t=0$. Therefore, $\nabla_{\frac{\partial}{\partial t}}$ vanishes tautologically at v if and only if its derivative along t vanishes, that is

$$\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}(v)=0.$$

Since, by definition of curvature,

$$R^\nabla(\frac{\partial}{\partial s},\frac{\partial}{\partial t})=\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}-\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}$$

and since $\frac{\partial}{\partial t}(v)=0$, we get the required result.

7.3. **Curvature tensor for a Riemannian manifold.** We are mostly interested with the case X is a riemannian manifold and ∇ is the Levi-Civita connection.

In what follows we will write for simplicity R isnstead of R^∇. This is an $(1,3)$-tensor, that is a function assigning a tangent vector $R(s,t,v)$ at $x\in X$ to a triple of tangent vectors $s,t,v\in T_xX$.

7.3.1. **Proposition.** *One has the following identities (Bianchi identities)*

1. $R(s,t,v)+R(t,s,v)=0.$
2. $R(s,t,v)+R(v,s,t)+R(t,v,s)=0.$
3. $\langle R(s,t,v),w\rangle+\langle R(s,t,w),v\rangle=0.$
4. $\langle R(s,t,v),w\rangle=\langle R(v,w,s),t\rangle.$

Proof. The first identity (skew-symmetricity for the first two arguments) is obvious. To prove the second identity, let us note that since R is a tensor, we can prove the claim locally, and choose constant vecot fields so that the brackets between them $[s,t]$, $[s,v]$, $[t,v]$ vanish. Then the left-hand side of (2) is

$$\nabla_s(\nabla_t(v)) - \nabla_t(\nabla_s(v)) + \nabla_v(\nabla_s(t)) - \nabla_s(\nabla_v(t)) + \nabla_t(\nabla_v(s)) - \nabla_v(\nabla_t(s)) = 0$$

since ∇ is torsion-free.

Property 3 means that the form $v, w \mapsto \langle R(s,t,v), w\rangle$ is skew-symmetric. It is therefore enuogh to prove that

$$\langle R(s,t,v), v\rangle = 0 \tag{2}$$

for all s,t,v. Once more assume that $[s,t]=0$. Then (2) means that the form $s,t \mapsto \langle \nabla_s(\nabla_t(v)), v\rangle$ is symmetric.

Since ∇ is compatible with the metric, one has

$$st\langle v,v\rangle = 2s(\langle \nabla_t(v), v\rangle) = 2\langle \nabla_s \circ \nabla_t(v), v\rangle + 2\langle \nabla_t(v), \nabla_s(v)\rangle. \tag{3}$$

Since $[s,t]=0$, the left-hand side is symmetric with respect to s,t. The right summand of the right-hand side is also symmetric. Therefore,

$$\langle \nabla_s \circ \nabla_t(v), v\rangle$$

is also symmetric as required.

Finally, Property 4 can be formally deduced from the rest of the properties. □

7.3.2. Sectional curvature

It is convenient, lowering the index, to consider R as a $(0,4)$ tensor. Thus, we will write $R(s,t,v,w)$ instead of $\langle R(s,t,v), w\rangle$. The properties 1, 3 and 4 of Proposition 7.3.1 implies that R is a symmetric bilinear form on $\wedge^2 TX$.

Let P be a two-dimensional subspace of T_xX. Choose an orthonormal basis $\{e,f\}$ of P; The vector $e\wedge f \in \wedge^2 T_xX$ is uniquely defined up to a sign. Therefore, a number $-R(e,f,e,f) \in \mathbb{R}$ is defined. This is *the sectional curvature* of X. This is a function on pairs (x,P) where $x \in X$ and $P \subset T_xX$ is a two-dimensional subspace.

In case $\sim\ X = 2$ we have no choice of two-dimansional subspace. Thus, the sectional curvature becomes just a function on X called the Gaussian curvature.

Definition. A riemannian manifold X is said to have a constant / a positive / a negative curvature if its sectional curvature is constant / positive / negative. A riemannian manifold is *flat* if its sectional curvature vanishes.

7.4. **Examples.**

7.4.1. The space $\mathbb{R}^n$

The Christoffel symbols vanish in the standard coordinates on $\mathbb{R}^n$, so the curvature tensor vanishes as well.

7.4.2. Covering map Assume $p : \widetilde{X} \to X$ is a covering map and assume $\widetilde{X}$ and X have compatible metrics. This means that the maps of the tangent spaces are isometries. Then, since all the definitions are local and p is a local isometry, the sectional curvatures of X and of $\widetilde{X}$ are the same. In particular, the torus which is the quotient of $\mathbb{R}^2$ by a lattice, is flat.

7.4.3. The standard sphere S^n

The sectional curvature of S^n is constant. This follows from the existence of a "very large group of isometries" of S^n. In fact, let $x_1, x_2 \in S^n$ and let P_1 and P_2 be two-dimensional subspaces in $T_{x_1}S^n$ and $T_{x_2}S^n$ respectively. we claim there exists an isometry of S^n sending x_1 to x_2 and P_1 to P_2.

We can first of all take care of the points, and only after that of the planes. The group of isomotries of the standard sphere S^n is the orthogonal group $O(n+1, \mathbb{R})$. We know that any norm one vector can be completed to an orthonormal basis, so any point of the sphere can be transferred to any other point of the sphere.

Now we want to see that, given two planes P_1, P_2 in $T_x(S^n)$, there is an orthogonal transformation of $\mathbb{R}^{n+1}$ preserving x and sending P_1 to P_2. Since $T_x(S^n)$ is just the orthogonal complement of the vector $x \in \mathbb{R}^{n+1}$, any orthogonal transformation of T_xS^n defines an orthogonal transformation of $\mathbb{R}^{n+1}$ preserving x. Thus, we have only to check that the orhogonal transformations of $\mathbb{R}^n$ act transitively on two-dimensional subspaces of R^n. This is obvious.

7.4.4. The hyperbolic space H^n

We have already studied $H^2 = \{z = a + bi | b > 0\}$. This example generalizes to all dimensions as follows.

Endow the vector space $\mathbb{R}^{m+1}$ with the quadratic form

$$\langle x, x \rangle = -x_0^2 + \sum_{i=1}^{n} x_i^2.$$

It is not positively definite but this will not spoil us the example. Define

$$H^n = \{x \in \mathbb{R}^{n+1} | x_0 > 0,\ \langle x, x \rangle = -1\}.$$

For any $x \in H^n$ the tangent space T_xH^n identifies with the orthogonal subspace $\langle x \rangle^\perp$. It is positively definite since $\langle x, x \rangle < 0$. This defines a riemannian manifold denoted H^n.

The group of linear transformations preserving the quadratic form $x \mapsto \langle x, x \rangle$ is denoted $O(1, n)$. Any such transformation preserves $\{x \in \mathbb{R}^{n+1} | \langle x, x \rangle = -1\}$ but does not necessarily preserve H^n since the sign of the zeroth coordinate may change.

We denote $O_0(1, n)$ the subgroup of transformations preserving the sign of x_0. It acts on H^n by isometries.

Now we claim that, similarly to the case of sphere, the group $O_0(1,n)$ acts transitively on the set of pairs (x,P) where $x\in H^n$ and P is a two-dimensional subspace of T_xH^n. This is also done in two steps.

Let us, first of all, check that $O_0(1,n)$ acts transitively on H^n. Denote $e_i,\ i=1,\dots,n$ the standard basis vectors of $\mathbb{R}^{n+1}$. In particular, $e_0\in H^n$. Let us show there is a linear transformation in $O(1,n)$ carrying e_0 to any vector x with $\langle x,x\rangle=-1$. In fact, to describe an element in $O(1,n)$ one has to find a collection $f_0,\dots,f_n$ of pairwise orthogonal vectors with

$$\langle f_0,f_0\rangle=-1,\ \langle f_i,f_i\rangle=1 \text{ for } i>0.$$

We put $f_0=x$ and we use the (already mentioned) fact that the restriction of the form to $\langle x\rangle^\perp$ is positively definite. Therefore, the orthogonal complement has an orthonormal basis. This proves the claim.

Now we can assume that $x=e_0$. The group of transformations in $O(1,n)$ preserving e_0 is just $O(n)$. This group is known to act transitively on the set of two-dimensional subspaces of $\mathbb{R}^n$. We are done.

7.4.5. Sign of the curvature

An explicit calculation shows that the sectional curvature of S^n is positive, and that of H^n is negative.

8. Jacobi vector fields. Conjugate points. Cartan-Hadamard theorem.

8.1. **Variation of a geodesic.** Assume a smooth map

$$\gamma:(-\epsilon,\epsilon)\times[0,1]\to X$$

is given so that for all $u\in(-\epsilon,\epsilon)$ the curve $\gamma_u(t):=\gamma(u,t)$ is geodesic. Such family of geodesics can be considered as a "variation of γ_0 in the class of geodesics". We have a vector field $J(t)=\frac{\partial\gamma}{\partial u}(0,t)$ along the geodesic γ_0.

Since for each u the curve γ_u is a geodesic, one has $\frac{\nabla}{dt}(\frac{\partial\gamma}{\partial t})=0$. Therefore,

$$0=\frac{\nabla}{\partial u}\frac{\nabla}{\partial t}\frac{\partial\gamma}{\partial t}=\frac{\nabla}{\partial t}\frac{\nabla}{\partial u}\frac{\partial\gamma}{\partial t}-R(\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial u},\frac{\partial\gamma}{\partial t})= \tag{4}$$

$$\frac{\nabla^2}{\partial t^2}\frac{\partial\gamma}{\partial u}-R(\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial u},\frac{\partial\gamma}{\partial t}). \tag{5}$$

8.1.1. **Definition.** Let γ be a geodesic line in X. A vector field J along γ is called a Jacobi vector field if

$$\frac{\nabla^2 J}{dt^2}-R(\gamma',J,\gamma')=0.$$

Remark. Note that sometimes the opposite sign for R is used. In this case the definition of Jacobi field is altered accordingly.

A calculation above shows that if γ_u is a family of geodesics, the derivative $\frac{\partial\gamma}{\partial u}$ is a Jacobi vector field.

The converse of this statement is also true: any Jacobi vector field along a geodesic is a tangent tector field to a family of geodesics. We will not prove this in these lectures.

8.2. **Existence of Jacobi fields. Conjugate points.** Jacobi fields are described by a linear second order differential equation. If $v_1, \ldots, v_n$ is a system of n orthonormal parallel vector fields along γ, a vector field $J = \sum f_i v_i$ is Jacobi iff

$$\frac{d^2 f_i}{dt^2} + \sum_{j=1}^{n} a_{i,j}(t) f_j(t) = 0,$$

where $a_{i,j}(t) = -R(\gamma', v_j, \gamma', v_i)$.

A Jacobi vector field J is uniquely defined by its initial conditions $J(0)$, $\frac{\nabla J}{dt}(0) \in T_{\gamma(0)}X$.

8.2.1. **Definition.** Two points $x = \gamma(a)$ and $y = \gamma(b)$ with $a \neq b$ on a geodesic γ are called *conjugate* if there exists a Jacobi vector field J along γ vanishing at both a and b.

8.2.2. **Example.** Opposite points x, y on a sphere S^2 are conjugate: there is a family of geodesics passing through x and y; the tangent vector field to this family is Jacobi.

8.3. **The study of the exponential map.** Assume that X is complete so that the exponential map

$$\exp_x : T_x X \longrightarrow X$$

is everywhere defined. We wish to understand whether $\exp_x$ is a local isomorphism. This is a smooth map of manifolds of the same dimension. So $\exp_x$ is a local isomorphism at $v \in T_x X$ iff the tangent map

$$T\exp_x(v) : T_x X \longrightarrow T_{\exp_x(v)} X \tag{6}$$

is injective.

8.3.1. **Lemma.** *If the tangent map (6) is not injective, then the points x and $\exp_x(v)$ are conjugate along the geodesic $t \mapsto \exp_x(tv)$.*

Proof. By the assumptions, there exists a nonzero vector $w \in T_x X$ belonging to the kernel of (6). This means that the derivative at 0 of the function

$$t \mapsto \exp_x(v + tw)$$

vanishes.

Consider the family of geodesics

$$\gamma(u, t) = \exp_x(t(v + uw)).$$

For each u $\gamma_u(t) = \exp_x(t(v+uw))$ is a geodesics. Therefore, The tangent vector field to this family $J(t) = \frac{\partial\gamma}{\partial u}(0,t)$ is a Jacobi field. We claim that $J(0) = 0,\ J(1) = 0$. The first equality is obvious since $\gamma(u,0) = x$. The second equality is just a reformulation of the fact that w belongs to the kernel of (6). Lemma is proven. $\square$

8.4. **Cartan-Hadamard theorem.** We are now ready to prove the following theorem connecting the topology with the curvature of a manifold.

8.4.1**. Theorem.** *Let X be a complete Riemannian manifold with nonpositive sectional curvature. Then the universal covering of X is diffeomorphism to $\mathbb{R}^n$.*

8.4.2**. Remark.** The theorem implies, for instance, that the higher homotopy groups of X, $\pi_i(X)$ $(i > 1)$, vanish.

The proof goes as follows. First of all we prove that a manifold with nonpositive sectional curvature has no conjugate points. This implies that an exponent map

$$\exp_x : T_xX \longrightarrow X$$

is a local diffeomorphism. Finally, using completeness one can deduce that $\exp_x$ is a covering.

8.4.3**. Lemma.** *Let X have a nonpositive sectional curvature:*

$$-R(s,t,s,t) \leq 0.$$

Then X has no conjugate points along any geodesic.

Proof. Ley γ be a geodesic in X and let J be a Jacobi field along γ. One has

$$\frac{\nabla^2 J}{dt^2} - R(\gamma', J, \gamma') = 0.$$

Then

$$\langle\frac{\nabla^2 J}{dt^2}, J\rangle = R(\gamma', J, \gamma' < J) \geq 0,$$

therefore

$$\frac{d}{dt}\langle\frac{\nabla J}{dt}, J\rangle = \langle\frac{\nabla^2 J}{dt^2}, J\rangle + \langle\frac{\nabla J}{dt}, \frac{\nabla J}{dt}\rangle \geq 0.$$

We have proven that the function $\langle\frac{\nabla J}{dt}, J\rangle$ is nondecreasing. If $J(0) = J(a)$ the function vanishes at both 0 and a, and thus at the whole segment $[0,a]$. Then $J(0) = 0,\ \frac{\nabla J}{dt}(0) = 0$ which implies that $J = 0$ identically. $\square$

Now Lemma 8.3.1 implies that the map $\exp_x$ is a local diffeomorphism. The theorem will be proven if we deduce that

$$\exp_x : T_xX \longrightarrow X$$

is a covering.

Let us first of all explain what is the difference between a local isomorphism and a covering. Of course, each covering is a local isomorphism. For instance, the map from $\mathbb{R}^2$ to the cylinder $\mathbb{R}\times S^1$ is a covering. If we replace $\mathbb{R}^2$ with an open subset U obtained by cutting a closed disc in $\mathbb{R}^2$, we will get a local isomorphism $U\to\mathbb{R}\times S^1$ which is not a covering.

Now we will use that X is complete. Since $\exp_x$ is a local isomorphism, the space T_xX is endowed with a (nonstandard) Riemannian structure so that $\exp_x$ becomes a local isometry. Let Y denote the space T_xX endowed with this new Riemannian structure. Even though Y has a nonstandard Riemannian structure, the straight lines connecting 0 with any point in Y is geodesic and has the standard length.

This implies that any closed bounded subset of Y is compact. Therefore, Y is complete. The rest follows from the lemma below.

8.4.4. **Lemma.** *lem:complete-cov Let $f:Y\longrightarrow X$ be a local isometry of Riemannian manifolds. Assume that Y is complete. Then f is a covering.*

Proof. We have to find for each $x\in X$ a neighborhood U such that the inverse image $f^{-1}(U)$ is isomorphic to $U\times F$ where $F=f^{-1}(x)$ is discrete. Let r be small enough so that the map $\exp_x$ is a diffeomorphism of the radius r open disc with the center at 0 to its image. We let U to be the image of this disc. Let for each $y\in f^{-1}(x)$ U_y be the image of the radius r open disc under the exponent map $\exp_y:T_yY\longrightarrow Y$. We claim that $f:U_y\longrightarrow U$ is a diffeomorphism for each y and that $f^{-1}(U)=\sqcup U_y$. First of all, $f(U_y)\subset U$ since any point in U_y can be connected by a geodesic of length $<r$ with y and since f preserves geodesics. Then, the restriction $f|_{U_y}$ is a diffeomorphism since the composition $\exp_y\circ Tf^{-1}\circ\exp_x^{-1}$ is inverse to it. It remains to check that $f^{-1}(U)\subset\cup U_y$. Assume $z\in U$ and $z'\in Y$ so that $f(z')=z$. Connect z with x by a geodesic γ of length $s<r$. Sice the geodesic is uniquely defined by its starting point and the tangent vector at this point, γ can be uniquely lifted to a geodesic γ' in Y passing through z'. Since Y is complete, we can go along γ' the distance s, and we will definitely arrive to a point y over x. Thus $z'\in U_y$. $\square$

Homework.

1. Let $\dim X=2$. Prove that for arbitrary vector fields $s,t,v,w\in\mathcal{T}(X)$ one has
$$R(s,t,v,w)=K(\langle s,v\rangle\langle t,w\rangle-\langle s,w\rangle\langle t,v\rangle),$$
where K is the Gaussian curvature of X.

2. Calculate the sectional curvature of S^n and H^n in the standard metric.