DIFFERENTIAL GEOMETRY, LECTURE 16-17, JULY 14-17

6. Geodesics

A parametrized line $\gamma : [a, b] \to \mathbb{R}^n$ in $\mathbb{R}^n$ is straight (and the parametrization is uniform) if the vector $\gamma'(t)$ does not depend on t. Thus, there are two concurrent description of straight lines in $\mathbb{R}^n$: an 'integral' description saying that it is a shortest path between any two points, and a 'dfferential' description saying basically that $\gamma'' = 0$. For general Riemannian manifolds one has a similar (though not identical) picture.

6.1. **Geodesics: existence and uniqueness.** Any path $\gamma : [a, b] \longrightarrow X$ determines a vector field $\frac{d\gamma}{dt}$ along γ.

6.1.1. **Definition.** A path $\gamma : [a, b] \longrightarrow X$ in a Riemannian manifold (X, g) is called geodesic if its derivative $\frac{d\gamma}{dt}$ is parallel along the Levi-Civita connection ∇, that is

$$\frac{\nabla}{dt}(\frac{d\gamma}{dt}) = 0.$$

6.1.2. Velocity

One has

$$\frac{d}{dt}\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = \langle\frac{\nabla}{dt}(\frac{d\gamma}{dt}), \frac{d\gamma}{dt}\rangle + \langle\frac{d\gamma}{dt}, \frac{\nabla}{dt}(\frac{d\gamma}{dt})\rangle = 0$$

if γ is geodesic. Therefore, the length of velocity $||\gamma'(t)||$ is constant along a geodesic. Define the path length by the usual formula

$$s(t) = \int_a^t ||\gamma'(s)||ds.$$

We deduce that the parametrization of a geodesic is proportional to the path length.

6.1.3. Geodesic in local coordinates

If $x_1, \ldots, x_n$ are local coordinates of X, then the path γ is given by n smooth functions $x_i(t)$. Let us rewrite in the local coordinates the geodesicity condition.

Note that $\frac{d}{dt} = \sum_i x_i'\partial_i$ (this is an equality of vector fields on γ that is sections of $\gamma^*(TX)$).

Recall $\nabla_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{i,j}\partial_k$, where $\Gamma^k_{i,j}$ are Christoffel symbols of the Levi-Civita connection on X. Therefore,

$$\frac{\nabla}{dt}\gamma' = \frac{\nabla}{dt}(\sum_j x'_j\partial_j) = \sum_j x''_j\partial_j + \sum_j x'_j\frac{\nabla}{dt}(\partial_j) = \sum x''_j\partial_j + \sum_{i,j,k} x'_j x'_i\Gamma^k_{i,j}\partial_k.$$

Therefore, γ is geodesic iff the following second order differential equation is fulfilled.

$$x''_k + \sum_{i,j}\Gamma^k_{i,j}x'_i x'_j = 0. \tag{1}$$

This allows us to use the theory of ODE to get an existence and uniqueness result for geodesics.

Note that the differential equaition (1) is linear and homogenic. This means in particular that if $\gamma : (a,b) \longrightarrow X$ is a geodesic, a curve $\tilde{\gamma}; (c^{-1}a, c^{-1}b) \longrightarrow X$ defined by the formula $\tilde{\gamma}(t) = \gamma(ct)$ is as well a geodesic.

6.1.4. **Theorem.** *For any $x \in X$ there exists a neighborhood $U \ni x$ and $\epsilon > 0$ such that for any $y \in U$ and any $v \in T_y(X)$ with $||v|| < \epsilon$ there exists a unique geodesic $\gamma : [-1,1] \longrightarrow X$ such that*

$$\gamma(0) = y, \quad \gamma'(0) = v.$$

Moreover, the geodesic γ depends smoothly on $(y,v) \in TU$.

Note. Homogeneity of the equation means in particular that one can "trade" the velocity norm for the length of the segment of definition of the geodesic.

Proof. This is a direct consequence of the theorem on existence and uniqueness of a solution of ODE. Recall that our system of second order differential equation can be rewritten as a system of first order differential equations, adding new variables which are the first derivatives of the original variables $x_1(t), \ldots, x_n(t)$. The initial condition for such a problem consists of $2n$ values $x_i(0),\ x'_i(0)$. By ODE, there exists a neighborhood of $(x,0) \in TX$ and $\epsilon > 0$ for which there exists a unique solution defined on the segment $(-\epsilon,\epsilon)$. Because of the homogeneity of the equation, we can trade "time for velocity" in order to get a solution defined on the segment $[-1,1]$. □

6.2. **Examples.**

6.2.1. $X = \mathbb{R}^n$

In the case $X = \mathbb{R}^n$ with the standard metric $g = \sum(dx_i)^2$ we have $g_{i,j}$ are constant, so the Christoffel symbols vanish.

The equation of geodesic has form $x''_k = 0,\ k = 1, \ldots, n$. This defines the straight lines.

6.2.2. $X \subset \mathbb{R}^N$

Assume X is a smooth submanifold of $\mathbb{R}^N$. Any curve γin X can be automatically considered as a curve in $\mathbb{R}^N$ (whose image belongs to X).

The first derivative γ' defines a vector field along γ which can be considered equivalently embedded into X or $\mathbb{R}^N$. Theorem in Section 5.3.2 from the previous lecture allows one to compare the second derivatives,

$$\frac{\nabla_0}{dt}(\gamma') \text{ and } \frac{\nabla}{dt}(\gamma') \tag{2}$$

where ∇_0 and ∇ are the canonical connections on $\mathbb{R}^N$ and on X respectively. The vector field $\frac{\nabla}{dt}(\gamma')$ is the orthogonal projection of $\frac{\nabla_0}{dt}(\gamma')$ to TX.

We get the following result.

Proposition. *A curve γ in a smooth submanfold X of $\mathbb{R}^N$ is geodesic iff its second derivative $(\gamma_1''(t), \ldots, \gamma_N''(t))$ is orthogonal to $T_{\gamma(t)}X$ for all t.*

6.2.3. $X = S^2$

Let $x \in S^2$ and let $v \in T_x(S^2)$. The reflection J of S^2 with respect to the twp-dimensional plain containing x and v, is an isometry preserving x and v. If γ is a geodesic satisfying the conditions $\gamma(0) = x,\ \gamma'(0) = v$, the image $J(\gamma)$ is also a geodesic satisfying the same properties. By the uniqueness of geodesics $J(\gamma) = \gamma$. Therefore, the image of γ belongs to the fixed-points of J. This is the big circle of S^2. Existence theorem implies that geodesics on S^2 are precisely the big circles parametried so that the velocity is constant.

The same reasoning generalizes to $X = S^n$.

6.2.4. The upper half-plane

The upper half-plane $H = \{z = a + bi \in \mathbb{C} | b > 0\}$ plays a very important role in two-dimensional Riemannian geometry. We equip H with the Riemannian metric defined by the formula

$$g(x + yi) = \frac{dx^2 + dy^2}{y^2} \tag{3}$$

It is an exercise to explicitly calculate the Christoffel symbols and write down the differential equation for geodesics. We, however, will find geodesics using the idea of symmetry — similarly to what has been done for the sphere.

Recall (from Complex variables) that the group $GL(2, \mathbb{R})$ acts on H by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d}.$$

One can check that Möbius transformations are isometries (see Exercise).

Moreover, the transformation $x+yi \mapsto 2c-x+yi$ (reflection with respect to the vertical line $x=c$) also preserves the metric. This immediately implies that the vertical lines $x=c$ (properly parametried) are geodesic. Now, any isometry carries geodesics to geodesics. The Möbius transformation F carries the vertical line $x=a$ to a semicircle lying in the upper half-plane intersecting the real line at points $F(c),\ F(\infty)$. This implies that such semicircles are all geodesic. One can easily check that there is a unique semicircle passing through a given point and having there a given tangent line. Therefore, we have got all geodesics in H.

6.3. **Exponent.**

6.3.1. **Definition.** Let γ be a geodesic defined (at least) on $[0,1]$, $\gamma(0)=x$, $\gamma'(0)=v$. Then we denote $\exp_x(v)=\gamma(1)$.

By the above theorem, exp is defined, as a function of two variables, at an open subset V of TX containing the zero section $X \subset TX$.

By the homogeneity, if γ is the geodesic satisfying the conditions $\gamma(0)=x,\quad \gamma'(0)=v$, one has $\exp_x(tv)=\gamma(t)$ for all t in the domain of definitnion of γ. This implies, in particular, that the map $\exp_x$ considered as a smooth map from a neighborhood of zero at T_xX to X, satisfies the property

$$T_0\exp_x = \mathrm{id}: T_xX \longrightarrow T_xX. \tag{4}$$

Look at the map

$$F: V \longrightarrow X \times X$$

defined by the formula

$$F(x,v)=(x,\exp_x(v)).$$

Let us calculate the Jacobi matrix of F at $(x,0)\in V$. Choose local coordinates $x_1,\ldots,x_n$ of X at x. Then the local coordinates for V at $(x,0)$ are $x_1,\ldots,x_n,v_1,\ldots,v_n$, where $v_i=dx_i$. The tangent map $T_{(x,0)}F$ sends $\frac{\partial}{\partial x_i}$ to the pair $(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_i})$ since $F(y,0)=(y,y)$, whereas $T_{(x,0)}F(\frac{\partial}{\partial v_i})=(0,\frac{\partial}{\partial v_i})$ by (4).

As a result, the matrix $T_{(x,0)}F$ is invertible. Therefore, by the Implicit function theorem, F induces a diffeomorphism from a neighborhood of $(x,0)$ in TX to a neighborhood of $(x,x)\in X\times X$. We can replace the neighborhood of $(x,0)$ in TX by a smaller neighborhood of form

$$\mathcal{U}_{U,\epsilon}=\{(y,v)|y\in U,\ ||v||<\epsilon\}. \tag{5}$$

Finally, we can replace the image of $\mathcal{U}_{U,\epsilon}$ under F with a neighborhood of (x,x) of form $W\times W$. This proves the following result.

6.3.2. **Theorem.** 1. *For any $x\in X$ there exists a neighborhood W of x in X and $\epsilon>0$ such that for any two points y,z in W there exists a unique geodesic γ in X of length $<\epsilon$ connecting y and z.*

2. *The vector v defining the geodesic $\gamma = \exp_y(v)$ connecting y with z, depends smoothly on y and z.*
3. *The map F carries any open ϵ-ball with center at $y \in W$ diffeomorphically to a set $U_{y,\epsilon}$ containing W.*

6.4. **Riemannian manifold as a metric space.** Let (X, g) be a Riemannian manifold. Define the function on X

$$d(x, y) = \inf \ell(\gamma)$$

where $\gamma : [a, b] \to X$ is a piecewise smooth curve with $\gamma(a) = x,\ \gamma(b) = y$, and $\ell(\gamma)$ denotes the length of the curve γ.

The function d is obviously symmetric and satisfies the triangle axiom. We will prove later that $d(x, y) = 0$ implies $x = y$, so that d is a distance function. Moreover, we will prove that locally the distance between points is the length of the shortest geodesic connecting them.

We keep the notation of Theorem 6.3.2.

6.4.1. **Lemma.** *Fix $y \in X$ and let $U_{y,\epsilon}$ be as above. Geodesic lines through y in $U_{y,\epsilon}$ are trajectories orthogonal to the hypersurfaces*

$$\{\exp_y(v) | v \in T_v X,\ ||v|| = const.\}.$$

Proof. Let $t \mapsto v(t)$ be a line in $T_y X$ with $||v(t)|| = 1$. We have to check that the line $t \mapsto \exp_y(r_0 v(t))$ is orthogonal to the geodesic $r \mapsto \exp_y(rv(t_0))$ at $r = r_0,\ t = t_0$.

Note that both lines belong to a parametrized surface

$$(t, r) \mapsto f(t, r) = \exp_y(rv(t)), \tag{6}$$

so our aim is to prove that the partial deriatives $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial r}$ are orthogonal for all (r, t). We will check this claim for $r = 0$ and then will verify that the inner product

$$\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle$$

is independent of r.

Let $r = 0$. Then $f(t, 0) = y$, therefore $\frac{\partial f}{\partial t}(t, 0) = 0$ and (6) vanishes at $r = 0$. Now

$$\frac{\partial}{\partial r} \langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = \langle \nabla_{\frac{\partial}{\partial r}} \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle + \langle \frac{\partial f}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial f}{\partial t} \rangle. \tag{7}$$

The first summand of the right-hand side vanishes since $f(r, t_0)$ is geodesic for each t_0. The second summand contains the expression $\nabla_{\frac{\partial}{\partial r}} \frac{\partial f}{\partial t}$ which, since ∇ is torsion-free, can be rewritten as

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial r} = \frac{1}{2} \frac{\partial}{\partial t} \langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \rangle = \frac{1}{2} \frac{\partial}{\partial t} \langle v(t), v(t) \rangle = 0.$$

The lemma is proven. $\square$

The above lemma allows one to prove the minimality of geodesics using "Pythagoras theorem".

Let $\delta : [a,b] \to U_{y,\epsilon}$ be a piecewise smooth curve. We assume for simplicity that y does not belong to the image of δ.

The one can uniquely write $\delta(t) = \exp_y(r(t)v(t))$ where $v(t)$ is a curve in T_yX such that $||v(t)|| = 1$ and $r(t)$ is a smooth real function with positive values.

6.4.2. **Lemma.** *One has*

$$\ell(\delta) = \int_a^b ||\delta'(t)||dt \geq |r(b) - r(a)|,$$

the equality holding only if $v(t)$ is constant and $r(t)$ is monotone.

Proof. Put, as above, $f(t,r) = \exp_y(rv(t))$ so that $\delta(t) = f(t, v(t))$. One has

$$\delta'(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} r'(t)$$

where the summands are orthogonal by Lemma 6.4.1. Since $||\frac{\partial f}{\partial r}|| = 1$, one has

$$||\delta'|| \geq |r'(t)|$$

with the equality taking place iff $\frac{\partial f}{\partial t} = 0$ that is if v is constant. Therefore,

$$\ell(\delta) = \int_a^b ||\delta'(t)||dt \geq \int_a^b |r'(t)|dt \geq |r(b) - r(a)|$$

with the equality taking place only of v is constant and $r(t)$ is monotone. $\square$

Now it is easy to prove

6.4.3. **Theorem.** *Let W and ϵ be chosen as above. Then any geodesic γ in W with $\ell(\gamma) < \epsilon$ is the shortest piecewise smooth path connecting the end points.*

Proof. Let $y = \gamma(a)$, $z = \gamma(b)$. Assume there exists a line η with the ends y and z so that $\ell(\eta) < \ell(\gamma)$.

For arbitrary small δ the curve η contains a segment connecting z with a point y' corresponding to $r = \delta$. The length of such segment has to be at least $\ell(\gamma) - \delta$. Thus, $\ell(\eta) \geq \ell(\gamma) - \delta$. Since the inequality holds for arbitrary δ, we are done. $\square$

6.4.4. **Corollary.** *Let $\gamma : [a,b] \to X$ be a piecewise smooth curve which is the shortest among such curves connecting $\gamma(a)$ with $\gamma(b)$. Then γ is geodesic (and smooth).*

Proof. The property of being the shortest persists once one replaces the whole segment with its part. This allows to use the local result 6.4.3. $\square$

6.4.5. **Corollary.** *The function $d(x,y)$ on $X \times X$ defined above is a distance.*

Proof. The only thing one has to check is that $d(x,y) \neq 0$ for $x \neq y$. This follows from Lemma 6.4.2: if y belongs to the neighborhood $U_{x,\epsilon}$ of x, the shortest path between the points is geodesic. Otherwise the length of the path conecting x with y is bounded below by ϵ since any path passes through a point at the ϵ-sphere with the center at x. $\square$

6.5. **Geodesic completeness.** A manifold X is called *geodesically complete* if any geodesic can be extended to the whole real line $\gamma : \mathbb{R} \to X$.

The following important result belongs to Hopf and Rinow.

6.5.1. **Theorem.** *Let X be geodesically complete. Then any thwo points of X can be connected by a minimal (=shortest) geodesic.*

Proof. Let $x, y \in X$. Put $d = d(x,y)$. Choose a neighborhood $U_{x,\epsilon}$ as in Theorem 6.3.2. Choose a point z at the ϵ-sphere with the center at x which is the closest to y (this is possible since the sphere is compact). One has $z = \exp_x(\epsilon v)$ for some $v \in T_xX$ of norm one. We claim that $y = \exp_x(dv)$. This will result of a (more) general statement that the distance between y and $\exp_x(cv)$ is $d - c$ for any $c \in [\epsilon, d]$. Let us prove the assertion for $c = \epsilon$. In fact, for any small $\delta > 0$ there exists a path between x and y whose length is less than $d - \delta$. This path has to pass through the ϵ-sphere, so the distance between an intersection point and y is less than $d - \delta - \epsilon$. Since this is true for any δ, the minimal distance should be at least $d - \epsilon$; obviously, it can not be less than that.

Let A be the set of numbers a satisfying the condition

$$\forall (c < a) d(\exp_x(cv), y) = d - c.$$

By continuity, A is a closed segment. Assume $A = [\epsilon, b]$. Then we can replace the point x with the point $\exp_x(bv)$ and repeat the reasoning. This proves the theorem. $\square$

6.5.2. **Corollary.** *crl:completeness For a Riemannian manifold X geodesic and metric completeness are equivalent.*

Proof. Let X be complete with respect to the metric d defined via the Riemann structure. We will prove that the domain D of definition of any geodesic γ in X is the whole $\mathbb{R}$. By the theorem of existence and uniqueness of geodesics D is open. Let us prove it is closed. Let $t_i \in D$ converge to $t \in \mathbb{R}$. One has

$$d(\gamma(t_i), \gamma(t_j)) \leq c|t_i - t_j|$$

since the parametriation of γ is proportional to the arc length. Thus, $\gamma(t_i)$ form a Cauchy sequence converging to a limit $x \in X$. Now by Theorem 6.3.2 the point x admits a neighborhood W and a number $\epsilon > 0$ so that for any $y \in W$ the map $\exp_y$ sends diffeomorphically the open ϵ-ball with the center at 0 in T_yX to its image in X. We can choose y to be a point $\gamma(t_i)$ having distance less than $\frac{\epsilon}{2}$ from x — and this will assure our geodesic can be extended further.

In the opposite direction, let X be geodesically compete. We will prove that any closed bounded subset K of X is compact. Since Cauchy sequences are bounded, this will imply the completeness of X. Choose any point $x \in X$. Since K is bounded, the distance between x and the points of K is bounded. This means that there exists a closed ball in T_xX whose image under $\exp_x$ covers K. Since an image of a compact is compact, K is a closed subset of a compact set, hence it is compact. $\square$

Recall that two continuous paths $\gamma_0, \gamma_1 : [0,1] \to X$ with $\gamma_i(0) = x,\ \gamma_i(1) = y$ are called *homotopic* if there exists a continuous family $\gamma : [0,1] \times [0,1] \to X$ of paths such that

$$\gamma(0,t) = \gamma_0(t),\ \gamma(1,t) = \gamma_1(t),\ \gamma(s,0) = x,\ \gamma(s,1) = y.$$

Homotopy is an equivalence relation between the paths connecting x with y. The set of homotopy classes of paths between x and y will be denoted $\Pi_X(x,y)$.

Recall the construction of a universal covering of X. Fix a point $x \in X$ and denote

$$\widetilde{X} = \coprod_{y \in X} \Pi_X(x,y).$$

Define the map $p : \widetilde{X} \to X$ sending $\Pi_X(x,y)$ to y.

For $\widetilde{y} \in \Pi_X(x,y) \subset \widetilde{X}$ define a base of neighborhoods as follows: choose any open contractible neighborhood U of $y \in X$; Then define $\widetilde{U}$ as the set of points of $\widetilde{X}$ which can be connected with $\widetilde{y}$ by a path whose image lies in U. The open sets $\widetilde{U}$ so constructed form a base of topology of $\widetilde{X}$. Recall

6.5.3. **Theorem.** *The projection $p : \widetilde{X} \to X$ is a covering, that is for each $x \in X$ there exists a neighborhood U whose preimage in $\widetilde{X}$ is isomorphic to $U \times F$ where F is discrete. Furthermore, $\widetilde{X}$ is simply connected.*

Since X and $\widetilde{X}$ are locally isomorphic, the smooth structures, as well as the Riemann structures move automatically from X to $\widetilde{X}$ and back.

Moreover, if X is complete, $\widetilde{X}$ is as well complete. This proves the following

6.5.4. **Corollary.** *Let X be a complete Riemann manifold. Then in each homotopy class of paths connecting x to y there exists a geodesic path.*

Proof. Construct $\widetilde{X}$ using x; Choose $\widetilde{x}$ to correspond to the constant path and $\widetilde{y}$ to the homotopy class in question. The points $\widetilde{x}$ and $\widetilde{y}$ can be connected by a geodesic according to Hopf-Rinow theorem. The image of this path will give a geodesic connecting x to y and lying in the homotopy class of paths we wanted. $\square$