DIFFERENTIAL GEOMETRY, LECTURE 15, JULY 10

5. Levi-Civita connection

From now on we are interested in connections on the tangent bundle TX of a Riemanninam manifold (X, g).

Out main result will be a construction of a canonical connection on TX depending of the Riemannian tensor g.

5.1. **Compatibility with the metric.**

Recall that any connection ∇ on TX defines operators of parallel transport $\Psi_\gamma : T_{\gamma(a)}X \longrightarrow T_{\gamma(b)}X$.

5.1.1. **Definition.** A connection ∇ on TX is called to be compatible with g if the parallel transport operators preserve the metric:

$$\langle v, w\rangle = \langle \Psi(v), \Psi(w)\rangle.$$

Let us find out when does this happen in terms of ∇.

Recall that a vector field s along γ is called parallel if $\frac{\nabla s}{dt} = 0$. If ∇ is compatible with the metric, each pair of parallel vector fields s, t along γ has a constant inner product. Recall that for any $v \in T_x(X)$ there exists a unique parallel vector field s along γ satisfying $s(x) = v$.

5.1.2. **Lemma.** *The connection ∇ on TX is compatible with the metric iff for any vector fields v, w along γ one has*

$$\frac{d}{dt}\langle v, w\rangle = \langle \frac{\nabla v}{dt}, w\rangle + \langle v, \frac{\nabla w}{dt}\rangle. \tag{1}$$

Proof. Assume ∇ is compatible with the metric. Let γ be a curve connecting x with y. Choose an orthogonal basis $e_1, \ldots, e_n$ in T_xX. Denote the the same letters the parallel vector fields along γ defined by the e_i. Since ∇ is compatible with the metric, the vector fields e_i along γ are orthogonal at all points. Now if $v = \sum a_i e_i$ and $w = \sum b_i e_i$ where $a_i,\ b_i$ are smooth functions on γ, we have $\langle v, w\rangle = \sum a_i b_i$, $\frac{\nabla v}{dt} = \sum \frac{da_i}{dt} e_i$ and $\frac{\nabla w}{dt} = \sum \frac{db_i}{dt} e_i$. The comparison yields one of the implications. The converse implication is obvious. □

5.1.3. **Theorem.** *A connection ∇ is compatible with the metric iff for all $\tau, v, w \in \mathcal{T}$ one has*

$$\tau(\langle v, w\rangle) = \langle \nabla_\tau(v), w\rangle + \langle v, \nabla_\tau(w)\rangle. \tag{2}$$

Proof. Let ∇ be compatible. Then, on order to check the equality of the functions at a given point x, we have to choose any curve passing through x with tangent vector at x equal to τ_x; restrict vector fields v and w to γ and apply Lemma 5.1.2.

To prove the opposite claim we have to deduce compatibility of a connection satisfying the property (2). Or, equivalently, this means that the connection satisfies (1) for any curve γ for all pairs of vector fields on γ induced by global vector fields on X. Now we note that it is enough to check the claim for curves γ lying completely inside an open set U such that TX is trivial on U. For such small curves any vector field along γ can be presented as a linear combination $\sum \alpha_i v_i$ where v_i age global vector fields and α_i are functions on γ. Now, if $v = \sum \alpha_i v_i$ and $w = \sum \beta_j w_j$ so that v_i and w_j are global vector fields, then

$$\frac{d}{dt}\langle v, w\rangle = \frac{d}{dt}(\sum_{i,j} \alpha_i\beta_j\langle v_i, w_j\rangle) = \sum_{i,j}\frac{d}{dt}(\alpha_i\beta_j)\langle v_i, w_j\rangle + \sum_{i,j}\alpha_i\beta_j\frac{d}{dt}\langle v_i, w_j\rangle$$

whereas the right-hand side is

$$\langle\frac{\nabla v}{dt}, w\rangle + \langle v, \frac{\nabla w}{dt}\rangle =$$

$$\sum_{i,j}\frac{d\alpha_i}{dt}\beta_j\langle v_i, w_j\rangle + \sum_{i,j}\alpha_i\beta_j\langle\frac{\nabla v_i}{dt}, w_j\rangle + \sum_{i,j}\alpha_i\frac{d\beta_j}{dt}\langle v_i, w_j\rangle + \sum_{i,j}\alpha_i\beta_j\langle v_i, \frac{\nabla w_j}{dt}\rangle$$

which is obviously the same. □

5.2. **Torsion.** Compatibility of a connection with the metric could be defined for any vector bundle $\mathbb{V}$ with a metric $g \in \Gamma(S^2\mathbb{V}^*)$.

The following notion makes sense for connections on TX only.

5.2.1**. Definition.** A connection ∇ on TX is called *torsion-free* if for any pair of vector fields τ, σ one has

$$\nabla_\sigma(\tau) - \nabla_\tau(\sigma) = [\sigma, \tau].$$

Choose a local chart with coordinates $x_1, \ldots, x_n$, so that TX is generated by $\frac{\partial}{\partial x_i}$. In what follows we will write ∂_i instead of $\frac{\partial}{\partial x_i}$ for simplicity. Then

$$\nabla_{\partial_i}(\partial_j) = \sum \Gamma^k_{i,j}\partial_k.$$

Since $[\partial_i, \partial_j] = 0$, symmetricity implies

$$\Gamma^k_{i,j} = \Gamma^k_{j,i}.$$

5.2.2**. Lemma.** *In local coordinates, a connection on TX is torsion-free if and only if the corresponding Christoffel symbols satisfy the condiition $\Gamma^k_{i,j} = \Gamma^k_{j,i}$.*

Proof. The only if part has already been checked. The if part is a result of an easy calculation. □

5.3. **Levi-Civita connection.** Let (X, g) be a Riemannian manifold. Levi-Civita connection is the only connection ∇ on TX which is compatible with the metric and torsion-free. This is the claim of the following theorem which is "the principal theorem of Differential Geometry".

5.3.1. **Theorem.** *There is a unique connection ∇ on the tangent bundle of a Riemannian manifold (X, g) which is torsion-free and compatible with g.*

Proof. It suffices to prove the existence and uniqueness in local coordinates: because of the uniqueness the connections on different charts will coincide at the intersections. We will denote as usual

$$g_{i,j} = \langle \partial_i, \partial_j \rangle, \quad \nabla_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{i,j} \partial_k.$$

By compatibility with the metric

$$\partial_i g_{j,k} = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle + \langle \partial_j, \nabla_{\partial_i} \partial_k \rangle. \tag{3}$$

Replacing the triple (i, j, k) with (j, i, k) and (k, i, j) we get two more equations,

$$\partial_j g_{i,k} = \langle \nabla_{\partial_j} \partial_i, \partial_k \rangle + \langle \partial_i, \nabla_{\partial_j} \partial_k \rangle. \tag{4}$$

and

$$\partial_k g_{i,j} = \langle \nabla_{\partial_k} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_k} \partial_j \rangle. \tag{5}$$

Since ∇ is to be torsion-free, threre are only three different expressions on the right-hand side of the equations. This allows to express each one of them through the left-hand side as follows.

$$\langle \nabla_{\partial_i}(\partial_j), \partial_k \rangle = \frac{1}{2}(\partial_i g_{j,k} + \partial_j g_{i,k} - \partial_k g_{i,j}). \tag{6}$$

The formula (6) uniquely defines $\nabla_{\partial_i}(\partial_j)$ since the inner product is nondegenerate. This proves uniqueness of the connection satisfying the listed above properties.

To prove the existence of such a connection, we can define a connection by the formulas (6) and then to check the properties. Torsion-freeness follows directly from the definition. The formulas (3)–(5) can be immediately deduced from (6). This implies compatibility of ∇ with g in the generators. We have already seen in the proof of Theorem 5.1.3 that this implies the compatibility in general. □

5.3.2. Comparison It is instructive to compare Levi-Civita connections on an embedded pair of Riemannian manifolds.

Let Y be a submanifold of a Riemannian manifold X. The tangent space $T_y Y$ is embedded into $T_y X$ and, therefore, it carries an induced inner product. This gives a Riemannian structure on Y.

Let $i : Y \to X$ be the embedding. One has a map of vector bundles

$$j : TY \longrightarrow i^*(TX)$$

and the connection on $i^*(TX)$ defined by the Levi-Civita connection $\widetilde{\nabla}$ on TX. Orthogonal projections $\pi_y : T_yX \to T_yY$ define a map $\pi : i^*(TX) \longrightarrow TY$ splitting j.

Now we are ready to express the Levi-Civita connection ∇ on TY through $\widetilde{\nabla}$.

Theorem. *Let σ, τ be vector fields on Y. Then*

$$\nabla_\tau(\sigma) = \pi(\widetilde{\nabla}_{j(\tau)}(j(\sigma))). \tag{7}$$

Proof. The right-hand side of the formula (7) defines a connection on TY. Let us check it is torsion-free and compatible with the metric. Compatibility with the metric follows from the fact that j is an isometry. Torsion freeness can be checked locally; one can use local coordinates $x_1, \ldots, x_n$ for which Y is given by the equations $x_1 = \ldots = x_i = 0$. In this case torsion freeness is immediate. □

5.4. **Connection on the standard tensor bundles.** There is a general way to extend a connection on $\mathbb{V}$ to a connection on any tensor power $T^p(\mathbb{V}) \otimes T^q(\mathbb{V}^*)$.

It is done very similarly to the definition of Lie derivative on the standard bundles. More precisely, we claim that, given a connection ∇ on $\mathbb{V}$, there is a unique collection of connections (denoted by the same letter ∇) on $\mathbb{V}^p_q := T^p(\mathbb{V}) \otimes T^q(\mathbb{V}^*)$ so that

- ∇_τ acts as τ on $\mathbb{V}^0_0 = \mathbf{1}$.
- $\nabla_\tau(s \otimes t) = \nabla_\tau(s) \otimes t + s \otimes \nabla_\tau(t)$.
- ∇_τ commutes with the contractions.

Instead of giving a proof (which is identical to the proof given for Lie dervativee) we will present some formulas - for $\mathbb{V}^p_0$ and $\mathbb{V}^0_q$.

$$\nabla_\tau(s_1 \otimes s_p) = \sum_i s_1 \otimes \ldots \otimes \nabla_\tau(s_i) \otimes \ldots \otimes s_p, \quad s_1 \otimes \ldots \otimes s_p \in \Gamma(V^p_0). \tag{8}$$

(9)

$$(\nabla_\tau r)(s_1, \ldots, s_q) = \tau(r(s_1, \ldots, s_q)) - \sum_i r(s_1, \ldots, \nabla_\tau(s_i), \ldots, s_q). \quad s \in \Gamma(\mathbb{V}^0_q).$$

Of course, all said above can be applied to $\mathbb{V} = TX$ and ∇ the Levi-Civita connection.

5.5. **Moving indices up and down.**

5.5.1. Linear algebra Let (V, g) be a vector space endowed with an inner product g. Any quadratic form on V can be interpreted as a linear map $V \to V^*$.

Since g is non-degenerate, the map $V \to V^*$ is an isomorphism. It is given by the formula

$$v \mapsto g(v,\ ).$$

5.5.2. Similarly to the above, the Riemannian metric $g \in \mathcal{T}^0_2$ induces an isomorphism $TX \longrightarrow T^*X$. We will write this map as an isomorphism $\mathcal{T}^1_0 \longrightarrow \mathcal{T}^0_1$ (converting vectors to covectors). This can be automatically extended to maps $\mathcal{T}^p_q \longrightarrow \mathcal{T}^{p-r}_{q+r}$ (moving down r indices. All these maps are isomorphisms, so that one can really move maps up and down.

Homework.

0. Check that the formula $\nabla_\tau(f) = \tau(f)$ defines a connection on the trivial bundle $\mathbf{1}$.

1. Let ∇_0 and ∇_1 be two connections on a vector bundle $\mathbb{V}$. Prove that the difference $\nabla_1 - \nabla_0$ is a tensor, that is is iven by a map $TX \otimes \mathbb{V} \longrightarrow \mathbb{V}$ of vector bundles.

2. Let ∇ be a connection on TX. Define $T : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ by the formula

$$T(\sigma, \tau) = \nabla_\sigma(\tau) - \nabla_\tau(\sigma) - [\sigma, \tau].$$

Prove that T is a $(1, 2)$-tensor, that is is defined by a map $TX \otimes TX \longrightarrow TX$ of vector bundles. (The tensor T is called the torsion of ∇).

3. Prove that if ∇ is the Levi-Civita connection on (X, g) then for any $\tau \in \mathcal{T}$ one has $\nabla_\tau(g) = 0$. Here g is considered as an element of $\mathcal{T}^0_2$. Deduce from this fact that Levi-Civita connection commutes with moving indices up and down.