DIFFERENTIAL GEOMETRY. LECTURE 12-13, 3.07.08

5. Riemannian metrics. Examples. Connections

5.1. **Length of a curve.** Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a parametried curve. Its length can be calculated as the limit of partial sums

$$\sum_{i=1}^{N} ||\gamma(t_i) - \gamma(t_{i-1})||$$

where $t_0 = a < t_1 < \ldots < t_N = b$ is a partition of the segment.

This easily gives the well-known expression

$$\ell(\gamma) = \int_a^b ||\gamma'(t)||dt. \tag{1}$$

We want to have a similar expression for the length of a parametrized curve on a smooth manifold. Given a curve $\gamma : [a,b] \longrightarrow X$, we know that $\gamma'(t)$ is a vector in $T_{\gamma(t)}X$. We do not know, however, how to calculate the length of a tangent vector. To have it, we need an inner product on the tangent spaces. Of course, it makes sense to require that this inner product on T_xX depends smoothly on x.

5.1.1**. Definition.** A Riemannian metric on X is a section g of the bundle S^2T^*X such that for each $x \in X$ the value $g_x : T_x \times T_x \longrightarrow \mathbb{R}$ is a positively definite symmetric bilinear form.

A smooth manifold endowed with a Riemannian metric is called a Riemannian manifold. The formula (1) defines now length of a curve on any Riemannian manifold.

Let us note that the length of a curve does not depend on a parametrization.

In fact, let t be a monotone function of $s \in [c,d]$ so that $t(c) = a,\ t(d) = b$. Define $\delta(s) = \gamma(t(s))$.

Then

$$\int_a^b ||\gamma'(t)||dt = \int_c^d ||\gamma'(t(s))||t'(s)ds = \int_c^d ||\gamma'(t(s))t'(s)||ds = \int_c^d ||\delta'(s)||ds.$$

5.2. **Riemannian manifolds. What can be studied?** If we know what a length of a curve is, we can ask what are the shortest curves connecting given points. Lines satisfying (locally) minimality condition are *geodesic lines.* They are defined as the lines satisfying a certain differential equation.

One can study existence of geodesic lines and connection to topological properties of X. One can wish to classify Riemannian structures on a given smooth manifold. Also, a local behavior of Riemannian manifolds is interesting (note that smooth manifolds are locally the same; this is not true for Riemannian manifolds).

5.3. **Riemannian metric in local coordinates.** Let X be an open subset in $\mathbb{R}^n$. Riemannian tensor g has form

$$g=\sum g_{i,j}dx_i dx_j.$$

Given two vector fields, $v=\sum v^i\frac{\partial}{\partial x_i}$ and $w=\sum w^i\frac{\partial}{\partial x_i}$ where $v^i,\ w^i\in C^\infty(X)$, a function

$$g(v,w)=\sum_{i,j}g_{i,j}v^iw^j$$

is defined.

In general any chart $\phi:D\longrightarrow U\subset X$ allows one to describe a Riemannian metric g in coordinates $g_{i,j}$ where

$$g_{i,j}=g(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_i}).$$

If one has another chart $\phi':D'\to U'$, and if $g_{i,j}$, $g'_{k,l}$ are the coordinates of g in these charts, one has

$$g'_{k,l}=\sum_{i,j}\frac{\partial x_i}{\partial x'_k}\frac{\partial x_j}{\partial x'_l}g_{i,j}.$$

5.3.1. Important: The above formulas mean, in particular, that Riemannian metric exists for any open subset of $\mathbb{R}^n$. We will see very soon that it exists for any smooth manifold.

5.3.2**. Example.** Let us write down the standard Riemannian metric on $\mathbb{R}^2$ in the polar coordinates (r,ϕ).

We have $g=dx^2+dy^2$. Since $x=r\cos\phi,\ y=r\sin\phi$, we can get after an easy calculation $g=dr^2+r^2d\phi^2$.

5.3.3**. Example.** Let $X\subset\mathbb{R}^N$ be a smooth submanifold of $\mathbb{R}^N$. For each point $x\in X$ one has a natural embedding $T_xX\longrightarrow\mathbb{R}^N$. This induces an inner product on each tangent space $T_x(X)$. An easy calculation (see Homework) shows that this inner product depends smoothly on the point x. Therefore, this defines a Riemannian metric on X which is called *the Riemannian metric on X induced from that on $\mathbb{R}^N$.*

5.4. **Existence.** We will now prove the existence of a Riemannian metric on any smooth manifold. The proof is (of course) based on partition of unity.

Let $\phi_i : D_i \to U_i$ be a locally finite atlas of X. Choose a partition of unity $i \mapsto \alpha_i$ so that $\sum \alpha_i = 1$ and $\operatorname{Supp} \alpha_i \subset U_i$. Choose any Riemannian metric on U_i, for instance g^i written in the coordinates of the chart ϕ_i as $\sum_j dx_j^2$. Define $g = \sum_i \alpha_i g^i$. We claim that g so defined is a Riemannian metric on X. The value of g on a nonzero vector $v \in T_x X$ is a sum of nonnegative summands. If $\alpha_i(x) \neq 0$, the i-th summand is nonzero. That proves the positive definitness of g.

5.5. **Riemannian embeddings.** Given two Riemannian manifolds, (X, g) and (Y, h), a Riemannian embedding is a morphism $f : X \longrightarrow Y$ such that for each $x \in X$ the map $T_x f : T_x X \to T_{f(x)} Y$ is an isometry. An invertible Riemannian embedding is called *isometry.*

The main object of classical Riemannian geometry is to study isometry invariants of (differently embedded) Riemannian manifolds.

For instance, a famous Gauss' *Theorema Egregium* claims that the Gauss curvature of a surface embedded into $\mathbb{R}^3$ is invariant under isometry.

5.5.1. **Example.** Let X and Y be two surfaces in $\mathbb{R}^3$ defined by the equations $y = 0$ and $y = \sin x$ respectively. Both X and Y have a Riemannian metric induced from $\mathbb{R}^3$. They are isometric: an isometry $X \to Y$ can be easily obtained from a natural parametrization of the curve $y = \sin x$.

5.6. **Connections.** In a general smooth manifold X tangent spaces at different points are not connected to one another. On the contrary, if X is an open subset of $\mathbb{R}^n$, tangent spaces at all points "are the same". Informally speaking, connection on TX is a law that assigns an isomorphism

$$T_x X \longrightarrow T_y X$$

to each path $\gamma : [a, b] \longrightarrow X$ with $\gamma(a) = x,\ \gamma(b) = y$, so that concatenation of paths corresponds to composition of isomorphisms.

The definition we present below is of "infinitesimal" nature. Instead of an isomorphism between different tangent spaces defined by a path connecting the base points, we have a structure defined by a tangent vector to a point.

Later on we will find out that a Riemann structure on X defines a very special connection (Levi-Civita connection).

Since the notion of connection makes sense for any vector bundle (not necessarily the tangent bundle), we develop the theory in this generality.

5.6.1. **Definition.** Let $\mathbb{V}$ be a vector bundle on X. A connection on V is an $\mathbb{R}$-bilinear map

$$\nabla : \mathcal{T} \times \Gamma(\mathbb{V}) \longrightarrow \Gamma(\mathbb{V}), \quad (\tau, v) \mapsto \nabla_\tau(v),$$

satisfying the following properties

- $\nabla_{f\tau}(s) = f\nabla_\tau(s)$.
- $\nabla_\tau(fs) = f\nabla_\tau(s) + \tau(f)s$.

The first condition means that, for a fixed section $s \in \Gamma(\mathbb{V})$ the map $\mathcal{T} \longrightarrow \mathbb{V}$ is linear over $\mathbb{C}^\infty$, that is as we saw above, is induced by a map of vector bundles $TX \longrightarrow \mathbb{V}$ which is, as we know, the same as a section of $T^*X \otimes \mathbb{V}$. This allows one to rewrite the definitnion as follows.

5.6.2. **Definition.** Let $\mathbb{V}$ be a vector bundle on X. A connection on V is an operator

$$\nabla : \Gamma(\mathbb{V}) \longrightarrow \Gamma \operatorname{Hom}(TX, \mathbb{V}) = \Gamma(T^*X \otimes \mathbb{V}),$$

satisfying the following property

- $\nabla(fs) = df \otimes s + f\nabla(s)$.

5.6.3. Connection in local coordinates

Let $x_1, \ldots, x_n$ be local coordinates in X corresponding to a chart $\phi : D \longrightarrow U \subset X$ and let $\mathbb{V}|_U = U \times \mathbb{R}^n$ be trivial. The constant sections e_i form a basis of $\Gamma(\mathbb{V}|_U)$ over $C^\infty(U)$. Then the connection ∇ is uniquely defined by $n \cdot m^2$ coefficients $\Gamma^k_{i,j}$ defined from the formulas

$$\nabla(e_j) = \sum \Gamma^k_{i,j} dx_i \otimes e_k. \tag{2}$$

The coefficients $\Gamma^k_{i,j}$ are called *Christoffel symbols*. These are functions in coordinates $x_1, \ldots, x_n$. They are not components of a tensor!

Let us write down the general formula for ∇ in local coordinates. We have

$$\nabla(\sum_j \alpha_j e_j) = \sum_j d(\alpha_j) \otimes e_j + \sum_{i,j,k} \Gamma^k_{i,j} \alpha_j dx_i \otimes e_k = \tag{3}$$

$$\sum_k (d\alpha_k + \sum_{i,j} \Gamma^k_{i,j} \alpha_j dx_i) \otimes e_k = \sum_{i,k} \left(\frac{\partial \alpha_k}{\partial x_i} + \sum_j \Gamma^k_{i,j} \alpha_j \right) dx_i \otimes e_k.$$

5.6.4. In order to înderstand how to "integrate" a connection along a curve, to get an isomorphism between the fibers $\mathbb{V}_x$ and $\mathbb{V}_y$ at the ends of the curve, it is worthwhile to divide this question into two separate problems.

First of all, we will understand how to define, given a vector bundle $\mathbb{V}$ on X with a connection ∇, and a smooth map $f : Y \to X$, a connection on the inverse image $f^*(V)$. In particular, this will give, for each smooth curve $\gamma : [a, b] \to X$ a connection on the restriction of $\mathbb{V}$ to the curve.

Then, to construct an isomorphism between the fibers $\mathbb{V}_x$ and $\mathbb{V}_y$ one can forget about X and work with a vector bundle with connection on a segment.

5.6.5. Inverse image of a connection

Let ∇ be a connection on a vector bundle $\mathbb{V}$ over X and let $f : Y \to X$ be a smooth map. We claim that there exists a unique connection (which will be denoted by the same letter)

$$\nabla : \mathcal{T}(Y) \times \Gamma(f^*\mathbb{V}) \longrightarrow \Gamma(f^*\mathbb{V})$$

compatible with the original connection of $\mathbb{V}$, that is making the diagram

$$\begin{array}{ccc} \Gamma(\mathbb{V}) & \xrightarrow{\nabla} & \Gamma\,\mathrm{Hom}(TX, \mathbb{V}) \\ \Big\downarrow & & \Big\downarrow \\ & & \Gamma\,\mathrm{Hom}(f^*TX, f^*\mathbb{V}) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle Tf} \\ \Gamma(f^*\mathbb{V}) & \xrightarrow{\nabla} & \Gamma\,\mathrm{Hom}(TY, f^*\mathbb{V}) \end{array} \tag{4}$$

commutative. Let us explain the vertical maps in the diagram. The leftmost map assigns to a section $s : X \to \mathbb{V}$ the composition $s \circ f : Y \to \mathbb{V}$ which automatically corresponds to $Y \to f^*\mathbb{V}$.

Any map of bundles $TX \longrightarrow \mathbb{V}$ defines canonically a map of inverse images $f^*TX \longrightarrow f^*\mathbb{V}$. This explains the arrow in the upper right corner. Finallt, the map marked Tf is defined by the composition with the canonical tangent map

$$TY \longrightarrow f^*TX.$$

As usual, we will prove uniqueness and existence of such connection on $f^*\mathbb{V}$ in local coordinates, and this will automatically imply that the local construction are compatible at the intersections.

The formulas (2), (3) show that on an open set U for which $\mathbb{V}|_U$ is trivial, a connection ∇ is uniquely defined by its value on the generating sections e_i. The vector bundle $f^*\mathbb{V}$ is trivial on $f^{-1}(U) \subset Y$ and is generated by the same sections e_i. The commutative diagram (4) prescribes the value of ∇ on e_i, so that the connection on $f^*\mathbb{V}|_{f^{-1}(U)}$ exists and is defined uniquely.

This concludes the construction of the connection on $f^*\mathbb{V}$.

5.6.6. Connections on a segment.

We want to apply the above construction to a curve $\gamma : [a, b] \to X$. This is formally not allowed since $[a, b]$ is not a manifold, but all definitions easily extend to this case.

Let $\mathbb{W}$ be a vector bundle on $[a,b]$ (we will apply this to $\mathbb{W} := \gamma^*(\mathbb{V})$). Since vector fields on $[a,b]$ have form $f\frac{d}{dt}$ for $f \in C^\infty([a,b])$, a connection on $\mathbb{W}$ is given uniquely by an operator

$$\nabla_{\frac{d}{dt}} : \Gamma(\mathbb{W}) \longrightarrow \Gamma(\mathbb{W})$$

which will be denoted from now on $\frac{\nabla}{dt}$.

The operator $\frac{\nabla}{dt}$ satisfies the property

$$\frac{\nabla}{dt}(fs) = \frac{df}{dt}s + f\frac{\nabla}{dt}(s).$$

A section $s \in \Gamma(\mathbb{W})$ will be called *parallel* if $\frac{\nabla}{dt}(s) = 0$.

The parallel sections enjoy the following properties.

5.6.7. **Proposition.** 1. *The parallel sections of* $\mathbb{W}$ *form a vector subspace of* $\Gamma(\mathbb{W})$ *of dimension* $\operatorname{rk}\mathbb{W}$.

2. *For any* $t_0 \in [a,b]$ *and for any* $s_0 \in \mathbb{V}_{t_0}$ *there exists a unique parallel section* s *with* $s(t_0) = s_0$.

Proof. The set of parallel section is a linear subspace as the kernel of a linear operator. Its dimension equals $\operatorname{rk}\mathbb{W}$ by the second claim. Let us prove it. Choose a partition of the segment so that $\mathbb{W}$ is trivial on each segment of the partition. It is sufficient to prove the assertion on each small segment separately. Thus, we are allowed to assume $\mathbb{W}$ s trivial.

According to the general formula (3), one has

$$\frac{\nabla}{dt}(\sum \alpha_j e_j) = \sum_k (\frac{d\alpha_k}{dt} + \sum_j \Gamma^k_{1,j}\alpha_j)e_k,$$

where $\Gamma^k_{1,j}$ are Christoffel symbols of our connection. Thus, a section $s = \sum \alpha_j e_j$ is parallel iff the coefficients α_j satisfy the system of linear differential equations

$$\frac{d\alpha_k}{dt} + \sum_j \Gamma^k_{1,j}\alpha_j = 0. \tag{5}$$

By the man theorem of ODE such system has a unique solution satisfying an initial condition. ☐

Let $c,d \in [a,b]$. We define an isomorphism $\Psi^d_c : \mathbb{W}_c \longrightarrow \mathbb{W}_d$ as follows: the value $\Psi^d_c(w)$ for $w \in \mathbb{W}_c$ is equal to $s(d)$ where s is the parallel section of $\mathbb{W}$ satisfying the initial condition $s(c) = w$. The isomorphism Ψ^d_c (parallel transport from c to d along the connection ∇) has very nice composition properties:

$$\Psi^e_d \circ \Psi^d_c = \Psi^e_c.$$

5.6.8. Parallel transport in general

Let now ∇ be a connection on a vector bundle $\mathbb{V}$ over X and let $\gamma : [a,b] \longrightarrow X$ be a smooth path in X. Applying the constructions of the previous subsection to the inverse image $\mathbb{W} = \gamma^*\mathbb{V}$ and the inverse image connection, we get a parallel transport map

$$\Psi_c^d : \mathbb{V}_{\gamma(c)} \longrightarrow \mathbb{V}_{\gamma(d)}.$$

Recall that in order to construct the map Ψ_c^d, we defined two important notions: the space $\Gamma(\gamma^*\mathbb{V})$ of section of $\mathbb{V}$ over γ which is the space of maps $\widetilde{\gamma} : [a,b] \longrightarrow \mathbb{V}$ satisfying the condition $\pi \circ \widetilde{\gamma} = \gamma$, and the operator

$$\frac{\nabla}{dt} : \Gamma(\gamma^*\mathbb{V}) \longrightarrow \Gamma(\gamma^*\mathbb{V}).$$

We will use these notions in the future.

Homework.

1. Let $X \subset \mathbb{R}^N$ be a smooth submanifold of $\mathbb{R}^N$. We consider $\mathbb{R}^n$ endowed with the standard inner product. This defines an inner product on each $T_x(X) \subset \mathbb{R}^N$ (see Lecture 1). Prove this collection of inner products depends smoothly on $x \in X$, that is defines a Riemannian structure on X. *Hint: use the charts for X constructed in Theorem 1.11*

2. Let $T = S^1 \times S^1 = \{(\phi, \psi) | \phi, \psi \in S^1\}$ be the two-dimensional torus with the product Riemannian metric. Find whether the following embeddings are isometries.

a) $i : T^2 \longrightarrow \mathbb{R}^3$ defined as

$$i(\phi, \psi) = ((2 + \cos\phi)\cos\psi, (2 + \cos\phi)\sin\psi, \sin\phi).$$

b) $j : T^2 \longrightarrow \mathbb{R}^4$ defined as

$$j(\phi, \psi) = (\cos\phi, \sin\psi, \cos\psi, \sin\psi).$$

3. Let $H = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ be the upper half-plane. Define the metric on H by the formula

$$g = \frac{dx^2 + dy^2}{y^2}.$$

a) calculate the length of the segment connecting the points $(0,1)$ with $(0,2)$.

b) Can H be isometric to the upper half-plane endowed with the usual Euclidean metric ($g = dx^2 + dy^2$)?