DIFFERENTIAL GEOMETRY. LECTURE 11, 30.06.08

4. Partition of unity and applications

Lemma 3.8.2 (see lecture 9-10) is a special case of a very important theorem claiming the existence of a partition of unity in the sense made precise below. We will not prove the theorem but we will deduce Lemma 3.8.2 from it.

4.1. **Existence of partition of unity.** Recall a few basic notions.

4.1.1. **Definition.** A cover $X = \cup V_i$ is called *locally finite* if for any $x \in X$ there exists a small neighborhood U of x having nonempty intersection with only finite number of V_i.

4.1.2. **Definition.** An open cover $X = \cup V_j$ is *subordinate* to a cover $\{U_i\}$ if for each j there exist i so that $V_j \subset U_i$.

4.1.3. **Definition.** Let $X = \cup V_j$ be a locally finite covering of a manifold X. A partition of unity corresponding to $\{V_j\}$ is a collection of smooth functions α_i satisfying the following properties

- $\operatorname{Supp} \alpha_j \subset V_j$.
- $\alpha_i(x) \geq 0$.
- For any $x \in X$ one has $\sum_j \alpha_j(x) = 1$.

4.1.4. **Theorem.** *Let X be a manifold (recall: it is assumed to be countable at infinity). Then*

1. *Any covering $X = \cup U_i$ admits a locally finite subordinate covering $\{V_j\}$.*
2. *Any locally finite covering $\{V_j\}$ admits apartition of unity.*

Let us explan how Lemma 3.8.2 can be deduced from 4.1.4. Let $\bar{U'} \subset U$. We have an open covering $X = U \cup (X$
$barU')$. It is finite, so we do not need a subordinate covering. By the theorem, there exist a partition of unity, that is a pair of smooth functions α, β on X such that $\operatorname{Supp} \alpha \subset U$ and $\operatorname{Supp} \beta \subset X - \bar{U'}$. Then the function α satisfies the required property.

4.2. **Orientation of a manifold.** A choice of orientation on a manifold X is, by definition,a choice of an atlas for which all transition functions $\phi_2^{-1} \circ \phi_1$ have positive jacobians.

This definition has the following explanation. Recall that for vector spaces over $\mathbb{R}$ the following notion of orientation is being used. Two bases B and B' of V are said to be equivalent if the transitionmatrix has a positive determinant. Thus, we have two equivalence classes of bases — these classes are called orientations of V.

Endow $\mathbb{R}^n$ with the standard orientation, the one defined by the standard basis of $\mathbb{R}^n$. Thus, any chart $\phi : D \to U$ induces orientation in all tangent spaces $T_x(X),\ x \in U$. If two charts $\phi_i : D_i \longrightarrow U_i,\ i = 1, 2$, are given so that the transition function has a positive jacobian, this implies that the orientations of $T_x(X)$, $x \in U_1 \cap U_2$ defined by ϕ_1 and ϕ_2 coincide.

The following criterion of orientability uses partition of unity.

4.2.1. **Theorem.** *A smooth manifold X of dimension n is orientable iff the vector bundle $\wedge^n T^*(X)$ of higher differential forms is trivial.*

Proof. Note that $\wedge^n T^* X$ is a vector bundle of rank 1. It is rivial if and only if it admits a nowhere zero section $\omega \in \Omega^n(X)$ (that is, $\omega(x) \neq 0$ for all $x \in X$. If there exists such a section (it is called *a volume form*ω, any element $\omega' \in \Omega^n$ can be uniquely presented in form $\omega' = f\omega,\ f \in C^\infty$.

If ω' is as well nowhere vanishing, the function f has no zeroes, therefore, has a constant sign (at each connected component).

Assume we have chosen a volume form ω. Let us show how to define an orientation of X. We say that a chart $\phi : D \to U$ is compatible with ω if $\omega = f dx_1 \wedge \ldots \wedge dx_n$ with a positive function f. Compatible charts cover the whole X since any chart can be always made compatible by making, if necessary, a coordinate change $x_1 \mapsto -x_1$. Finally, two charts compatible with ω are compatible with each other. In fact, if

$$\omega = f dx_1 \wedge \ldots \wedge dx_n = g dy_1 \wedge \ldots \wedge dy_n,$$

one has $f(x) = g(y(x)) \cdot J$ where J is the jacobian of the transition function. Therefore, since both f and g are positive, J is as well positive.

The other direction of the claim is more difficult. Now we assume that a manifold X has an orientation and we have to construct a volume form.

Choose a locally finite atlas $i \mapsto \phi_i : D_i \longrightarrow U_i$, compatible in the sense of orientations (that is, with positive jacobians of the transition functions). Choose a partition of unity $i \mapsto \alpha_i$ such that $\operatorname{Supp} \alpha_i \subset U_i$. On each U_i we have a perfectly defined nowhere vanishing form ω_i which in the local coordinates given by ϕ_i is written as $dx_1 \wedge \ldots \wedge dx_n$. The product $\alpha_i \omega_i$ is already globally defined and the sum

$$\omega := \sum \alpha_i \omega_i$$

(the sum makes sense since the covering is locally finite) is, as we will immediately check, a volume form compatible with the atlas.

Our aim is to prove that if $\phi_i^*(\omega|_{U_i}) = f dx_1 \wedge \ldots \wedge dx_n$, then f is positive. One has

$$f = \phi_i^*(\alpha_i) + \sum_{j \neq i} \phi_i^*(\alpha_j) \cdot \det T(\phi_j{-}1\phi_i).$$

Since all jacobians of the transition matrices $\det T(\phi_j{-}1\phi_i)$ are positive, and all α_i are nonnegative, this gives the positivity of f. □

4.3. **Integration.**

4.3.1. What to intergrate?

To answer this question, it is enough to analyse carefully what we know about integration of functions of one variable.

The formula

$$\int_{x(a)}^{x(b)} f(x)dx = \int_a^b f(x(t))x'(t)dt$$

teaches us that what we integrate are not functions but the expressions $f(x)dx$, that is differential forms. If we are talking about n-dimensional integration, these will be n-forms.

But this is not all one can deduce analysing Calculus I integration. We know that

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

This means that we cannot simply intergrate a differential form $f(x)dx$ along a segment $[a, b]$. We have to know its direction. In n-dimensional integration this means that we will have to integrate differential forms along oriented manifolds.

Now we are ready to define the integral of an n-form *with compact support* over an oriented manifold X.

Assume $\omega \in \Omega^n(X)$ has a compact support. Assume first of all that $\mathrm{Supp}\,\omega \subset U$ where
$phi : D \to U$ is a chart. Then we define

$$\int_X \omega = \int_{\mathbb{R}^n} \phi^*(\omega).$$

The expression makes sense since it behaves well with respect to a base change with a positive jacobian.

For a general form ω with compact support one uses a partition of unity. Let α_i be a partition of unity with respect to a locally finite covering of X with oriented charts U_i. Then the integral of ω over X is defined by the formula

$$\int_X \omega = \sum_i \int_X \alpha_i\omega = \sum_i \int_{\mathbb{R}^n} \phi_i^*(\alpha_i\omega),$$

where $\alpha_i\omega$ have already support in U_i.

Of course, one has to do some work to check that the result does not depend on the choice of partition of unity.

4.4. **Embedding of a compact manifold.** Any smooth manifold of dimension n can be embedded into $\mathbb{R}^{2n+1}$. In this subsection we will prove a much weaker statement.

4.4.1**. Theorem.** *Let X be a smooth* compact *manifold. Then X can be embedded into $\mathbb{R}^N$ for N sufficiently big.*

Proof. Let n be the dimension of X. Choose a finite atlas $\phi_i : D_i \to U_i,\ i = 1, \ldots, m$. Choose a partition of unity $i \mapsto \alpha_i$. Let $\phi_i : X \longrightarrow \mathbb{R}^n$ be defined by the formula

$$\psi_i(x) = \alpha_i(x)\phi_i^{-1}(x) \text{ if } x \in U_i,\ 0 \text{ otherwise.}$$

Define a map $\Psi : X \longrightarrow \mathbb{R}^{mn+m}$ by the formula

$$\Psi(x) = (\psi_1(x), \ldots, psi_m(x), \alpha_1(x), \ldots, \alpha_m(x)).$$

The map Ψ is injective: if $\Psi(x) = \Psi(y)$, then for some k $\alpha_k(x) = \alpha_k(y) \neq 0$ and this implies that $x, y \in U_k$ and that $\phi_k^{-1}(x) = \phi_k^{-1}(y)$, which, of course, imply that $x = y$.

To prove that Ψ defines an embedding of X into $\mathbb{R}^{mn+m}$, we have to check that $T\Psi_x$ is injective at all points x. Choose k for which $\alpha_k(x) \neq 0$. In a neighborhood of $\Psi(x) \in \mathbb{R}^{mn+m}$ a map $u : \mathbb{R}^{mn+m} \longrightarrow \mathbb{R}^m$ is defined sending a point $(\psi_1, \ldots, \psi_m, \alpha_1, \ldots, \alpha_m)$ to $\alpha_k^{-1}\psi_k$. the composition $u \circ \Psi$ is just ϕ_k^{-1} which has an invertible tangent map. This proves the theorem.

□