GENERIC POWER CONVERGENCE OF NONLINEAR OPERATORS IN BANACH SPACES

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Abstract. Let K be a closed convex subset of a Banach space X. We consider complete metric spaces of self-mappings of K which are, in some sense, nonexpansive with respect to the Bregman distance induced by a given convex function f on X. In this survey we present (under certain assumptions on f) several recent results regarding the strong convergence of the iterates of a generic operator in these spaces. In some cases the limits do not depend on the initial points and are the unique fixed point of the operator, while in others the iterates converge to a retraction onto the fixed point set of the operator.

Introduction

The following problem often occurs in Functional Analysis and Optimization as well as in other fields of Pure and Applied Mathematics: Given a nonempty closed convex subset K of a Banach space X and an operator $T : K \to K$, do the sequences iteratively generated in K by the rule $x^{k+1} = Tx^k$ converge (weakly or strongly) to a fixed point of T? It is well known (see, for example, [1, 5, 18, 19]) that this indeed happens for "standard" classes of operators (e.g., certain nonexpansive operators and operators of contractive type). In a series of recent papers [6, 7, 9, 15, 20] it has been shown that sometimes, when the space X is reflexive, the question asked above has an affirmative answer even if the operator T is not contractive in any standard sense, but instead is assumed to be quasi-nonexpansive with respect to the Bregman distance induced by a given convex function on X. In these results one usually obtains only weak subsequential convergence, with strong convergence, and even weak sequential convergence, resulting only under quite restrictive conditions. However, more recently we have been able to show [11-13] that in such situations, strong convergence is not the exception, but rather the rule. More precisely, we have shown that in appropriate complete metric spaces of operators which are, in some sense, nonexpansive with respect to a given (strictly) convex function f, there exists a subset which is a countable intersection of open everywhere dense sets such that for any operator belonging to this subset all its orbits converge strongly. Results of this kind for nonexpansive (in the usual sense) operators were already established in [17, 21, 22]. Thus, instead of considering a certain convergence property for a single operator, we investigate it for a space of all such operators equipped with some natural complete metric, and show that this property holds for most of these

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operators in the sense of Baire's categories. This allows us to establish strong convergence without restrictive assumptions on the space and on the operators themselves. Thus we are able to obtain the convergence of the iterations of a generic operator in the norm topology of an arbitrary Banach space X, while in individual theorems we usually obtain only weak subsequential convergence of iterations and the space X is assumed to be at least reflexive.

Throughout this survey we assume that we are given a convex function $f : X \to R^1 \cup \{\infty\}$ such that K is a nonempty closed convex subset of the algebraic interior $\mathcal{D}^0$ of

$$\mathcal{D} = \mathrm{dom}(f) = \{x \in X : \ f(x) < \infty\}.$$

We also assume that f is lower semicontinuous on K.

Recall that the Bregman distance [3, 14] $D_f : \mathcal{D} \times \mathcal{D}^0 \to [0, \infty)$ induced by f is defined by

$$D_f(y,x) = f(y) - f(x) + f^0(x, x-y),$$

where

$$f^0(x,v) = \lim_{t \to 0^+} t^{-1}(f(x+tv) - f(x)).$$

In recent years Bregman distances have been playing an ever-increasing role in Optimization and Nonlinear Analysis. See, for instance, [2, 4, 6-10, 15, 16, 20] and the references mentioned therein.

In this paper we consider several complete metric spaces of self-mappings of K which are, in some sense, nonexpansive with respect to to the Bregman distance D_f induced by the given convex function f. We present (under certain assumptions on f) several recent results regarding the strong convergence of the iterates of a generic operator in these spaces. In some cases the limits do not depend on the initial points and are the unique fixed point of the operator, while in others the iterates converge to a retraction onto the fixed point set of the operator.

The practical meaning of our results is that whenever one applies iterative algorithms of the form $x^{k+1} = Tx^k$ to compute a fixed point of an operator T which belongs to one of our spaces of operators, then there is a good chance that the resulting sequence $\{x^k\}$ converges strongly. This conclusion is consistent with many computational experiments, despite the fact that the study of particular classes of such operators T has so far produced individual strong convergence theorems only under conditions which are considerably more restrictive than those we assume in the present paper.

1. Generic power convergence for a class of continuous mappings

Let $(X, ||\cdot||)$ be a Banach space, $K \subset X$ a nonempty closed convex subset of X and let $f : X \to R^1 \cup \{\infty\}$ be convex. Let $\mathcal{D}^0$ be the algebraic interior of the domain $\mathcal{D}$ of f and let $D_f : \mathcal{D} \times \mathcal{D}^0 \to [0, \infty)$ denote the Bregman distance induced by f. We assume in the sequel that $K \subset \mathcal{D}^0$.

Denote by $\mathcal{M} = \mathcal{M}(K)$ the set of all mappings $T : K \to K$ which are bounded on bounded subsets of K. For the set $\mathcal{M}$ we consider the uniformity determined by the following base:

$$E(N,\epsilon) = \{(T_1, T_2) \in \mathcal{M} \times \mathcal{M} : \ ||T_1 x - T_2 x|| \le \epsilon \quad \text{for all } x \in K \text{ satisfying } ||x|| \le N\}, \tag{1.1}$$

where $N, \epsilon > 0$. Clearly, this uniform space is metrizable and complete. We equip the space $\mathcal{M}$ with the topology induced by this uniformity. Denote by $\mathcal{M}_c$ the set of all continuous $T \in \mathcal{M}$. It is obvious that $\mathcal{M}_c$ is a closed subset of $\mathcal{M}$. We consider the topological subspace $\mathcal{M}_c \subset \mathcal{M}$ with the relative topology.

Denote by $\mathfrak{M}$ the set of all $T \in \mathcal{M}_c$ for which there is $z_T \in K$ such that the following assumptions hold:

A (i)

$$Tz_T = z_T, \ D_f(z_T, \cdot) : \ K \to R^1 \text{ is convex, } D_f(z_T, Tx) \le D_f(z_T, x) \text{ for all } x \in K;$$

A (ii)

$$\text{for any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } x \in K \text{ and}$$

$$D_f(z_T, x) \le \delta, \text{ then } ||z_T - x|| \le \epsilon;$$

A (iii)

$$D_f(z_T, \cdot) : K \to R^1 \text{ is Lipschitzian in a neighborhood of } z_T.$$

Denote by $\bar{\mathfrak{M}}$ the closure of $\mathfrak{M}$ in $\mathcal{M}$. We equip the topological subspace $\bar{\mathfrak{M}} \subset \mathcal{M}$ with the relative topology.

Remark. Observe that here and in the sequel, the inequality

$$D_f(z_T, Tx) \le D_f(z_T, x) \text{ for all } x \in K$$

in $\mathbf{A}$(i), when combined with $\mathbf{A}$(ii), implies that z_T is a fixed point of T.

Note that $\mathbf{A}$(iii) holds if the function $D_f(z_T, \cdot) : \mathcal{D}^0 \to R^1$ is convex. Note also that $\mathbf{A}$(ii) holds if the function f is uniformly convex. Examples of such functions f can be found in [7]. Let $\xi \in K$. Denote by $\mathfrak{M}_\xi$ the set of all $T \in \mathfrak{M}$ such that Assumption **A** holds with $z_T = \xi$ and denote by $\bar{\mathfrak{M}}_\xi$ the closure of $\mathfrak{M}_\xi$ in $\mathcal{M}$. We consider the topological subspace $\bar{\mathfrak{M}}_\xi \subset \mathcal{M}$ with the relative topology. We endow the topological spaces $K \times \mathcal{M}$, $K \times \bar{\mathfrak{M}}$ and $K \times \bar{\mathfrak{M}}_\xi$ with the appropriate product topologies.

The following three results were established in [11]. The first two describe the asymptotic behavior of orbits of a generic T in $\bar{\mathfrak{M}}$ and in $\bar{\mathfrak{M}}_\xi$. The third one yields strong convergence of $\{T^n x\}_{n=1}^{\infty}$ for a generic pair (x, T) in the product space $K \times \bar{\mathfrak{M}}$.

Theorem 1.1. *Let $x_j \in K$, $j = 1, \dots, p$, where p is a natural number. Then there exists a set $\mathcal{F} \subset \bar{\mathfrak{M}}$ which is a countable intersection of open everywhere dense sets in $\bar{\mathfrak{M}}$ such that for each $T \in \mathcal{F}$ the following assertions hold:*

1. There exists $z_* \in K$ such that $T^n x_j \to z_*$ as $n \to \infty$ for $j = 1, \dots, p$.

2. For each $\epsilon > 0$, there exist an integer $N \ge 1$, a neighborhood $\mathcal{U}$ of T in $\mathcal{M}$, and neighborhoods V_j of x_j in K for $j = 1, \dots, p$ such that for $j = 1, \dots, p$,

$$||S^n y - z_*|| \le \epsilon \textit{ for each } S \in \mathcal{U}, \textit{ each } y \in V_j \textit{ and each integer } n \ge N.$$

Theorem 1.2. *Let $\xi \in K$ and $x_j \in K$, $j = 1, \ldots, p$, where p is a natural number. Then there exists a set $\mathcal{F}_\xi \subset \bar{\mathfrak{M}}_\xi$ which is a countable intersection of open everywhere dense sets in $\bar{\mathfrak{M}}_\xi$ such that for each $T \in \mathcal{F}_\xi$,*

$$\lim_{n\to\infty} T^n x_j = \xi, \ j = 1, \ldots, p,$$

and the following assertion holds:

For each $\epsilon > 0$, there exist an integer $N \geq 1$, a neighborhood $\mathcal{U}$ of T in $\mathcal{M}$ and neighborhoods V_j of x_j in K for $j = 1, \ldots, p$ such that for $j = 1, \ldots, p$,

$$||S^n y - \xi|| \leq \epsilon \text{ for each } S \in \mathcal{U}, \text{ each } y \in V_j, \text{ and each integer } n \geq N.$$

Theorem 1.3. *There exists a set $\mathcal{F} \subset K \times \bar{\mathfrak{M}}$ which is a countable intersection of open everywhere dense sets in $K \times \bar{\mathfrak{M}}$ such that for each $(x, T) \in \mathcal{F}$ there exists $\lim_{n\to\infty} T^n x$ and the following assertion holds:*

For each $\epsilon > 0$, there exists an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of (x, T) in $K \times \mathcal{M}$ such that for each $(y, S) \in \mathcal{U}$ and each integer $i \geq N$,

$$||S^i y - \lim_{n\to\infty} T^n x|| \leq \epsilon.$$

Our next theorem is an analog of Theorem 1.3 for the space $\bar{\mathfrak{M}}_\xi \subset \bar{\mathfrak{M}}$. Similarly, Theorem 1.6 is an analog of Theorem 1.5.

Theorem 1.4. *Let $\xi \in K$. Then there exists a set $\mathcal{F}_\xi \subset K \times \bar{\mathfrak{M}}_\xi$ which is a countable intersection of open everywhere dense sets in $K \times \bar{\mathfrak{M}}_\xi$ such that for each $(x, T) \in \mathcal{F}_\xi$, $\lim_{n\to\infty} T^n x = \xi$ and the following assertion holds:*

For each $\epsilon > 0$, there exists an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of (x, T) in $K \times \mathcal{M}$ such that for each $(y, S) \in \mathcal{U}$ and each integer $i \geq N$,

$$||S^i y - \xi|| \leq \epsilon.$$

Theorem 1.5. *Let K_0 be a nonempty separable closed subset of K. Then there exists a set $\mathcal{F} \subset \bar{\mathfrak{M}}$ which is a countable intersection of open everywhere dense sets in $\bar{\mathfrak{M}}$ such that for each $B \in \mathcal{F}$, there exists $x_B \in K$ and a set $\mathcal{K}_B \subset K_0$ which is a countable intersection of open everywhere dense sets in K_0 with the relative topology such that the following assertions hold:*

1. $\lim_{n\to\infty} B^n x = x_B$ *for all $x \in \mathcal{K}_B$.*

2. For each $x \in \mathcal{K}_B$ and each $\epsilon > 0$, there exist an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of (x, B) in $K \times \mathcal{M}$ such that for each $(y, S) \in \mathcal{U}$ and each integer $i \geq N$, we have $||S^i y - x_B|| \leq \epsilon$.

Theorem 1.6. *Let K_0 be a nonempty separable closed subset of K and let $\xi \in K$. Then there exists a set $\mathcal{F}_\xi \subset \bar{\mathfrak{M}}_\xi$ which is a countable intersection of open everywhere dense sets in $\bar{\mathfrak{M}}_\xi$ such that for each $B \in \mathcal{F}_\xi$, there exists a set $\mathcal{K}_B \subset K_0$ which is a countable intersection of open everywhere dense sets in K_0 with the relative topology such that the following assertions hold:*

1. $\lim_{n\to\infty} B^n x = \xi$ *for all $x \in \mathcal{K}_B$.*

2. For each $x \in \mathcal{K}_B$ and each $\epsilon > 0$, there exist an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of (x, B) in $K \times \mathcal{M}$ such that for each $(y, S) \in \mathcal{U}$ and each integer $i \geq N$, we have $||S^i y - \xi|| \leq \epsilon$.

2. Generic power convergence for a class of uniformly continuous mappings

In this section we assume that the space X, its subset K and the function f are as introduced in Section 1.

Denote by $\mathcal{A} = \mathcal{A}(K)$ the set of all mappings $T: K \to K$ which are uniformly continuous on bounded subsets of K. Clearly, any $T \in \mathcal{A}$ is bounded on bounded subsets of K and $\mathcal{A}$ is a closed subset of the complete uniform space $\mathcal{M}$ defined in Section 1. We consider the topological subspace $\mathcal{A} \subset \mathcal{M}$ with the relative topology.

Denote by $\mathcal{A}_*$ the set of all $T \in \mathcal{A}$ for which there is $z_T \in K$ such that the following properties hold:

B (i)

$$Tz_T = z_T,\ D_f(z_T, \cdot): K \to R^1 \text{ is convex, } D_f(z_T, Tx) \leq D_f(z_T, x) \text{ for all } x \in K;$$

B (ii)

$$\text{the function } D_f(z_T, \cdot) \text{ is bounded from above on any bounded subset of } K;$$

B (iii)

$$\text{for any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } x \in K \text{ and}$$

$$D_f(z_T, x) \leq \delta, \text{ then } ||z_T - x|| \leq \epsilon.$$

Denote by $\bar{\mathcal{A}}_*$ the closure of $\mathcal{A}_*$ in the space $\mathcal{A}$. We consider the topological subspace $\bar{\mathcal{A}}_* \subset \mathcal{A}$ with the relative topology. We note that **B**(iii) holds if the function f is uniformly convex.

Theorem 2.1. *There exists a set $\mathcal{F} \subset \bar{\mathcal{A}}_*$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{A}}_*$ such that for each $T \in \mathcal{F}$ the following assertions hold:*

1. There exists $z_* \in K$ such that $T^n x \to z_*$ as $n \to \infty$ for all $x \in K$.

2. For each $\epsilon > 0$ and each bounded set $C \subset K$, there exists an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of T in $\mathcal{A}$ such that for each $S \in \mathcal{U}$ and each $x \in C$,

$$||S^n x - z_*|| \leq \epsilon \text{ for all integers } n \geq N.$$

Let $\xi \in K$. Denote by $\mathcal{A}_\xi$ the set of all $T \in \mathcal{A}$ which satisfy Property **B** with $z_T = \xi$ and denote by $\bar{\mathcal{A}}_\xi$ the closure of $\mathcal{A}_\xi$ in $\mathcal{A}$. We equip the topological subspace $\bar{\mathcal{A}}_\xi \subset \mathcal{A}$ with the relative topology.

Theorem 2.2. *There exists a set $\mathcal{F}_\xi \subset \bar{\mathcal{A}}_\xi$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{A}}_\xi$ such that for each $T \in \mathcal{F}_\xi$ the following assertions hold:*

1. $T^n x \to \xi$ as $n \to \infty$ for all $x \in K$.

2. For each $\epsilon > 0$ and each bounded set $C \subset K$, there exists an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of T in $\mathcal{A}$ such that for each $S \in \mathcal{U}$ and each $x \in C$,

$$||S^n x - \xi|| \leq \epsilon \text{ for all integers } n \geq N.$$

The proofs of these two theorems can also be found in [11].

3. Convergence of powers for a class of mappings with a uniformly continuous Bregman distance

In this section we use the definitions and notations from Section 1 and the complete uniform spaces $\mathcal{M}$ and $\mathcal{M}_c$ introduced there.

Denote by $\mathcal{M}_*$ the set of all $T \in \mathcal{M}$ for which there is $z_T \in K$ such that the following assumptions hold:

C (i)

$$Tz_T = z_T,\ D_f(z_T, \cdot):\ K \to R^1 \text{ is convex, } D_f(z_T, Tx) \le D_f(z_T, x) \text{ for all } x \in K;$$

C (ii)

$$\text{the function } D_f(z_T, \cdot) \text{ is uniformly continuous on any bounded subset of } K;$$

C (iii)

$$\text{for any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } x \in K \text{ and}$$

$$D_f(z_T, x) \le \delta, \text{ then } ||z_T - x|| \le \epsilon;$$

C (iv)

$$\text{for each } \alpha > 0, \text{ the level set } \{y \in K:\ D_f(z_T, y) \le \alpha\} \text{ is bounded.}$$

Set $\mathcal{M}_{*c} = \mathcal{M}_c \cap \mathcal{M}_*$. Denote by $\bar{\mathcal{M}}_*$ the closure of $\mathcal{M}_*$ in the space $\mathcal{M}$ and by $\bar{\mathcal{M}}_{*c}$ the closure of $\mathcal{M}_{*c}$ in the space $\mathcal{M}$. We consider the topological subspaces $\bar{\mathcal{M}}_{*c}$ and $\bar{\mathcal{M}}_* \subset \mathcal{M}$ with the relative topologies. Again we note that **C**(iii) holds if the function f is uniformly convex.

The next two theorems were established in [11]. Roughly speaking, these theorems show that we can replace the uniform continuity assumption imposed in Section 2 with a uniform continuity assumption on the Bregman distance D_f.

Theorem 3.1. *There exists a set $\mathcal{F} \subset \bar{\mathcal{M}}_*$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}_*$ and a set $\mathcal{F}_c \subset \mathcal{F} \cap \bar{\mathcal{M}}_{c*}$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}_{*c}$ such that for each $T \in \mathcal{F}$ the following assertions hold:*

1. There exists $z_* \in K$ such that $T^n x \to z_*$ as $n \to \infty$ for all $x \in K$.

2. For each $\epsilon > 0$ and each bounded set $C \subset K$, there exists an integer $N \ge 1$ and a neighborhood $\mathcal{U}$ of T in $\mathcal{M}$ such that for each $S \in \mathcal{U}$ and each $x \in C$,

$$||S^n x - z_*|| \le \epsilon \text{ for all integers } n \ge N.$$

Let $\xi \in K$. Denote by $\mathcal{M}_\xi$ the set of all $T \in \mathcal{M}$ which satisfy Assumption **C** with $z_T = \xi$. Set $\mathcal{M}_{\xi c} = \mathcal{M}_\xi \cap \mathcal{M}_c$. Denote by $\bar{\mathcal{M}}_\xi$ the closure of $\mathcal{M}_\xi$ in $\mathcal{M}$ and by $\bar{\mathcal{M}}_{\xi c}$ the closure of $\mathcal{M}_{\xi c}$ in $\mathcal{M}$. We equip the topological subspaces $\bar{\mathcal{M}}_\xi$ and $\bar{\mathcal{M}}_{\xi c} \subset \mathcal{M}$ with the relative topologies.

Theorem 3.2. *There exists a set $\mathcal{F}_\xi \subset \bar{\mathcal{M}}_\xi$ (respectively, $\mathcal{F}_{\xi c} \subset \bar{\mathcal{M}}_{\xi c} \cap \mathcal{F}_\xi$) which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}_\xi$ (respectively, in $\bar{\mathcal{M}}_{\xi c}$) such that for each $T \in \mathcal{F}_\xi$ the following assertions hold:*

1. $T^n x \to \xi$ as $n \to \infty$ for all $x \in K$.

2. For each $\epsilon > 0$ and each bounded set $C \subset K$, there exists an integer $N \ge 1$ and a neighborhood $\mathcal{U}$ of T in $\mathcal{M}$ such that for each $S \in \mathcal{U}$ and each $x \in C$,

$$||S^n x - \xi|| \le \epsilon \text{ for all integers } n \ge N.$$

4. Asymptotic behavior of quasi-nonexpansive mappings

In this section we again consider the space $\mathcal{M}$, introduced in Section 1, of all mappings $T:\ K \to K$ which are bounded on bounded subsets of K, with the uniformity determinied by the base (1.1). Recall that this uniform space is metrizable and complete.

Denote by $\mathcal{M}_c$ the set of all continuous $T \in \mathcal{M}$. Clearly, $\mathcal{M}_c$ is a closed subset of $\mathcal{M}$. We endow the topological subspace $\mathcal{M}_c \subset \mathcal{M}$ with the relative topology.

Denote by $\mathcal{M}_u$ the set of all those mappings $T:\ K \to K$ which are uniformly continuous on bounded subsets of K. Clearly, any $T \in \mathcal{M}_u$ is bounded on bounded subsets of K and $\mathcal{M}_u$ is a closed subset of the complete uniform space $\mathcal{M}$. We consider the topological subspace $\mathcal{M}_u \subset \mathcal{M}$ with the relative topology.

Let $F \subset K$ be a nonempty closed convex subset of K. Denote by $\mathcal{M}^{(F)}$ the set of all $T \in \mathcal{M}$ such that

$$Tx = x \text{ for all } x \in F$$

and

$$D_f(z, Tx) \le D_f(z, x) \text{ for all } z \in F \text{ and all } x \in K.$$

Set

$$\mathcal{M}_c^{(F)} = \mathcal{M}^{(F)} \cap \mathcal{M}_c \text{ and } \mathcal{M}_u^{(F)} = \mathcal{M}^{(F)} \cap \mathcal{M}_u.$$

We assume that

$$D_f(z, \cdot):\ K \to R^1 \text{ is convex and continuous for all } z \in F.$$

This assumption implies that $\mathcal{M}^{(F)}$, $\mathcal{M}_c^{(F)}$ and $\mathcal{M}_u^{(F)}$ are closed subsets of $\mathcal{M}$. We consider the topological subspaces $\mathcal{M}^{(F)}$, $\mathcal{M}_c^{(F)}, \mathcal{M}_u^{(F)} \subset \mathcal{M}$ with the relative topologies.

We also assume that for each $x \in K$, each $y, z \in F$, and each $\gamma \in (0,1)$,

$$D_f((1-\gamma)z + \gamma y, (1-\gamma)x + \gamma y) \le (1-\gamma) D_f(z, x). \tag{4.1}$$

Evidently, this last assumption is true if the function $D_f(\cdot,\cdot)\ :\ F \times K \to R^1$ is convex as a function of two variables. Although this convexity assumption is rather restrictive we note that there are quite a few examples of convex functions f for which the corresponding Bregman distance D_f is convex as a function of two variables. These examples include the square of the norm in Hilbert space, the negentropy in R^n [6] and the examples discussed in [2].

For each $x \in K$ and each nonempty $G \subset K$, set

$$\rho_f(x, G) = \inf\{D_f(z, x):\ z \in G\}.$$

In addition, we assume the following uniform continuity condition:

For any $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in K,\ z \in F,$ and

$$D_f(z, x) \le \delta, \text{ then } ||z - x|| \le \epsilon.$$

In [13] we have established, for appropriate complete metric spaces of mappings, the existence of an everywhere dense G_δ subset such that the iterates of any of its elements converge strongly to a retraction onto F. These results are presented in this section.

In our first theorem we also make the following assumption:

$$\text{For each bounded set } K_0 \subset K \text{ and each } \epsilon > 0, \text{ there is } \delta > 0$$

$$\text{such that if } x \in K_0,\ z \in F, \text{ and } ||z - x|| \le \delta, \text{ then } D_f(z,x) \le \epsilon.$$

Note that this assumption is true if the function f is Lipschitzian on each bounded subset of K.

We also assume that for each bounded set $K_0 \subset K$,

$$\sup\{\rho_f(x,F) :\ x \in K_0\} < \infty.$$

Theorem 4.1. *Assume that $P \in \mathcal{M}_u^{(F)}$ and $P(K) = F$. Then there exists a set $\mathcal{F} \subset \mathcal{M}_u^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_u^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

(i) There exists $P_B \in \mathcal{M}_u^{(F)}$ such that $P_B(K) = F$ and $B^n x \to P_B x$ as $n \to \infty$, uniformly on bounded subsets of K;

(ii) for each $\epsilon > 0$ and each bounded set $C \subset K$, there exist a neighborhood U of B in $\mathcal{M}_u^{(F)}$ and an integer $N \ge 1$ such that for each $S \in U$, each $x \in C$, and each integer $n \ge N$, we have $||S^n x - P_B x|| \le \epsilon$

In this theorem, as well as in our subsequent theorems, we assume that certain retractions belong to $\mathcal{M}^{(F)}$. Sufficient conditions for the existence of such retractions can be found in [12].

Whereas this theorem deals with a class of uniformly continuous mappings, our next two results are concerned with a uniformly continuous Bregman distance. More precisely, we assume that there exists $P \in \mathcal{M}^{(F)}$ for which $P(K) = F$ and that the function $D_f(\cdot,\cdot) : F \times K \to R^1$ is uniformly continuous on bounded subsets of $F \times K$.

Theorem 4.2. *There exists a set $\mathcal{F} \subset \mathcal{M}^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

1. There exists $P_B \in \mathcal{M}^{(F)}$ such that $P_B(K) = F$ and $B^n x \to P_B x$ as $n \to \infty$, uniformy on bounded subses of K; if $B \in \mathcal{M}_c^{(F)}$, then $P_B \in \mathcal{M}_c^{(F)}$.

2. For each $\epsilon > 0$ and each nonempty bounded set $C \subset K$, there exists a neighborhood U of B in $\mathcal{M}^{(F)}$ and a natural number N such that for each $S \in U$ and each $x \in C$, there is $z(S,x) \in F$ such that $||S^n x - z(S,x)|| \le \epsilon$ for all integers $n \ge N$.

Moreover, if $P \in \mathcal{M}_c^{(F)}$, then there exists a set $\mathcal{F}_c \subset \mathcal{F} \cap \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$.

Theorem 4.3. *Let the set $\mathcal{F} \subset \mathcal{M}^{(F)}$ be as guaranteed by Theorem 4.2, $B \in \mathcal{F} \cap \mathcal{M}_c^{(F)}$, $P_B z = \lim_{n\to\infty} B^n z$, $z \in K$, and let $x \in K$, $\epsilon > 0$. Then there exist a neighborhood U of B in $\mathcal{M}^{(F)}$, a number $\delta > 0$ and a natural number N such that for each $y \in K$ satisfying $||x - y|| \le \delta$, each $S \in U$, and each integer $n \ge N$, we have $||S^n y - P_B x|| \le \epsilon$.*

Finally, we mention three theorems, also established in [13], for the space $\mathcal{M}_c^{(F)}$. In all three theorems we assume that there is $P \in \mathcal{M}_c^{(F)}$ such that $P(K) = F$.

Theorem 4.4. *Let $x \in K$. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

1. There exists $\lim_{n\to\infty} B^n x \in F$.

2. For each $\epsilon > 0$, there exist a neighborhood U of B in $\mathcal{M}_c^{(F)}$, a natural number N, and a number $\delta > 0$ such that for each $S \in U$, each $y \in K$ satisfying $||y-x|| \leq \delta$, and each integer $n \geq N$, $||S^n y - \lim_{i\to\infty} B^i x|| \leq \epsilon$.

In the last two theorems of this section we use the space $K \times \mathcal{M}_c^{(F)}$ with the product topology.

Theorem 4.5. *There exists a set $\mathcal{F} \subset K \times \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $K \times \mathcal{M}_c^{(F)}$ such that for each $(z, B) \in \mathcal{F}$ the following two assertions hold:*

1. There exists $\lim_{n\to\infty} B^n z \in F$.

2. For each $\epsilon > 0$, there exist a neighborhood U of (z, B) in $K \times \mathcal{M}_c^{(F)}$ and a natural number N such that for each $(y, S) \in U$ and each integer $n \geq N$,

$$||S^n y - \lim_{i\to\infty} B^i z|| \leq \epsilon.$$

Theorem 4.6. *Assume that the set K_0 is a nonempty separable closed subset of K. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$ such that for each $T \in \mathcal{F}$, there exists a set $\mathcal{K}_T \subset K_0$ which is a countable intersection of open everywhere dense subsets of K_0 with the relative topology such that the following two assertions hold:*

1. For each $x \in \mathcal{K}_T$ there exists $\lim_{n\to\infty} T^n x \in F$.

2. For each $x \in \mathcal{K}_T$ and each $\epsilon > 0$, there exist an integer $N \geq 1$ and a neighborhood U of (x, T) in $\mathcal{K} \times \mathcal{M}_c^{(F)}$ such that for each $(y, S) \in U$ and each integer $i \geq N$, we have $||S^i y - \lim_{n\to\infty} T^n x|| \leq \epsilon$.

5. Convergence of powers to retractions

In this section we assume that the convex function f satisfies the following four conditions:

D(i) For any nonempty bounded set $E \subset K$ and any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } x \in E,\ z \in F, \text{ and } D_f(z, x) \leq \delta, \text{ then } ||z - x|| \leq \epsilon.$$

D(ii) There exists $\theta \in F$ such that the restriction to K of the function $g(\cdot) := D_f(\theta, \cdot)$ has the following property: For any subset $E \subset K$, $g(E)$ is bounded if and only if E is bounded.

D(iii) For any $z \in F$, the function $D_f(z, \cdot) : K \to R^1$ is convex and lower semicontinuous.

D(iv) For any $x \in K$, there exists a vector $Px \in F$ such that

$$D_f(Px, x) \leq D_f(z, x) \text{ for all } z \in F.$$

We continue to use the spaces $\mathcal{M}^{(F)}$, $\mathcal{M}_c^{(F)}$ and $\mathcal{M}_u^{(F)}$ introduced in Section 4.

In practical situations, the following stronger version of **D**(i) turns out to be useful:

For any nonempty bounded set $E \subset K$,

$$\nu_f(E,t) = \inf\{\nu_f(x,t) : \; x \in E\}$$

is positive for all $t > 0$, where

$$\nu_f(x,t) = \inf\{D_f(y,x) : \; y \in X \text{ and } ||y-x|| = t\}.$$

In [8] this condition is termed *sequential compatibility* of the function f with the relative topology of the set K. In [12, Lemma 1.1] we show that sequential compatibility implies **D**(i). In its turn, condition **D**(i) implies that all $z \in F$ are common fixed points of the operators in $\mathcal{M}^{(F)}$.

Conditions **D**(ii) and **D**(i) are satisfied by many useful functions and, among them, by many functions which are sequentially compatible with the relative topology of K. In contrast, condition **D**(iii) is quite restrictive. However, it does hold for many functions f which are of interest in applications (see the examples below). The vector Px in **D**(iv) was termed the *Bregman projection* with respect to f of x onto F in [14]. Condition **D**(iv) is automatically satisfied when X is reflexive and f is totally convex on K (in particular, when f is sequentially compatible with the relative topology of K) as follows from [8, Proposition 2.1.5(i)]. We now mention four typical situations in which all the conditions **D**(i)-**D**(iv) are satisfied simultaneously.

(i) (cf. [8]) X is a Hilbert space, K and F are nonempty closed convex subsets of X such that $F \subset K$, and $f(x) = ||x||^2$;

(ii) (cf. [6]) $F \subset K \subset R^n_{++}$ and f is the negentropy;

(iii) (cf. [9]) X is a Lebesgue space L^p or l^p, $1 < p \leq 2$, $f(x) = ||x||^p$ and K consists of either nonnegative or nonpositive functions;

(iv) (cf. [10]) X is smooth and uniformly convex, F is a singleton $\{z\}$, and $f(x) = ||x-z||^r$ with $r > 1$.

In this section we present the results on power convergence of mappings in $\mathcal{M}^{(F)}$ which have been obtained in [12]. These results are extensions of the results discussed in Section 4. The main difference is that now we do not use the rather restrictive assumption (4.1). Instead we use assumption **D**(iv) which is easier to handle in applications.

We begin with a result for the space $\mathcal{M}_u^{(F)}$. In this result we assume that the operator P, the existence of which is stipulated in condition **D**(iv), belongs to $\mathcal{M}_u^{(F)}$, and that the following condition is satisfied:

$$\text{For each bounded set } K_0 \subset K \text{ and each } \epsilon > 0, \text{ there is } \delta > 0$$

$$\text{such that if } x \in K_0, \; z \in F, \text{ and } ||z-x|| \leq \delta, \text{ then } D_f(z,x) \leq \epsilon.$$

Note that this condition holds if the function f is Lipschitzian on each bounded subset of K.

Theorem 5.1. *There exists a set $\mathcal{F} \subset \mathcal{M}_u^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_u^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

(i) There exists $P_B \in \mathcal{M}_u^{(F)}$ such that $P_B(K) = F$ and $B^n x \to P_B x$ as $n \to \infty$, uniformly on bounded subsets of K;

(ii) for each $\epsilon > 0$ and each bounded set $C \subset K$, there exist a neighborhood U of B in $\mathcal{M}_u^{(F)}$ and an integer $N \geq 1$ such that for each $S \in U$, each $x \in C$, and each integer $n \geq N$, we have $||S^n x - P_B x|| \leq \epsilon$.

In our next two results we assume that the function $D_f(\cdot, \cdot) : F \times K \to R^1$ is uniformly continuous on bounded subsets of $F \times K$.

Theorem 5.2. *There exists a set $\mathcal{F} \subset \mathcal{M}^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

1. There exists $P_B \in \mathcal{M}^{(F)}$ such that $P_B(K) = F$ and $B^n x \to P_B x$ as $n \to \infty$, uniformly on bounded subsets of K; if $B \in \mathcal{M}_c^{(F)}$, then $P_B \in \mathcal{M}_c^{(F)}$.

2. For each $\epsilon > 0$ and each nonempty bounded set $C \subset K$, there exists a neighborhood U of B in $\mathcal{M}^{(F)}$ and a natural number N such that for each $S \in U$ and each $x \in C$, there is $z(S, x) \in F$ such that $||S^n x - z(S, x)|| \leq \epsilon$ for all integers $n \geq N$.

Moreover, if $P \in \mathcal{M}_c^{(F)}$, then there exists a set $\mathcal{F}_c \subset \mathcal{F} \cap \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$.

Theorem 5.3. *Let the set $\mathcal{F} \subset \mathcal{M}^{(F)}$ be as guaranteed in Theorem 5.2, $B \in \mathcal{F} \cap \mathcal{M}_c^{(F)}$, $P_B z = \lim_{n\to\infty} B^n z$, $z \in K$, and let $x \in K$, $\epsilon > 0$. Then there exists a neighborhood U of B in $\mathcal{M}^{(F)}$, a number $\delta > 0$, and a natural number N such that for each $y \in K$ satisfying $||x - y|| \leq \delta$, each $S \in U$, and each integer $n \geq N$, $||S^n y - P_B x|| \leq \epsilon$.*

Finally, we present three results on convergence of powers for a class of continuous operators. We assume that $P \in \mathcal{M}_c^{(F)}$ and that the function $D_f(z, \cdot) : K \to R^1$ is continuous for all $z \in F$. We equip the space $K \times \mathcal{M}_c^{(F)}$ with the product topology.

Theorem 5.4. *Let $x \in K$. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

1. There exists $\lim_{n\to\infty} B^n x \in F$.

2. For each $\epsilon > 0$, there exist a neighborhood U of B in $\mathcal{M}_c^{(F)}$, a natural number N and a number $\delta > 0$ such that for each $S \in U$, each $y \in K$ satisfying $||y - x|| \leq \delta$, and each integer $n \geq N$, we have $||S^n y - \lim_{i\to\infty} B^i x|| \leq \epsilon$.

Theorem 5.5. *There exists a set $\mathcal{F} \subset K \times \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $K \times \mathcal{M}_c^{(F)}$ such that for each $(z, B) \in \mathcal{F}$ the following assertions hold:*

1. There exists $\lim_{n\to\infty} B^n z \in F$.

2. For each $\epsilon > 0$, there exist a neighborhood U of (z, B) in $K \times \mathcal{M}_c^{(F)}$ and a natural number N such that for each $(y, S) \in U$ and each integer $n \geq N$,

$$||S^n y - \lim_{i\to\infty} B^i z|| \leq \epsilon.$$

Theorem 5.6. *Assume that the set K_0 is a nonempty separable closed subset of K. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$ such that for each $T \in \mathcal{F}$, there exists a set*

$\mathcal{K}_T \subset K_0$ which is a countable intersection of open everywhere dense subsets of K_0 with the relative topology such that the following assertions hold:

1. For each $x \in \mathcal{K}_T$, there exists $\lim_{n\to\infty} T^n x \in F$.

2. For each $x \in \mathcal{K}_T$ and each $\epsilon > 0$, there exist an integer $N \geq 1$ and a neighborhood U of (x, T) in $\mathcal{K} \times \mathcal{M}_c^{(F)}$ such that for each $(y, S) \in U$ and each integer $i \geq N$, we have $\|S^i y - \lim_{n\to\infty} T^n x\| \leq \epsilon$.

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