A Mosco Stability Theorem for the Generalized Proximal Mapping

Dan Butnariu, Elena Resmerita, and Shoham Sabach

Abstract. We consider the generalized proximal mapping $\mathrm{Prox}_{\varphi}^{f} := (\nabla f + \partial\varphi)^{-1}$ in which f is a Legendre function and φ is a proper, lower semicontinuous, convex function on a reflexive Banach space X. Does the sequence $\mathrm{Prox}_{\varphi_n}^{f}(\xi_n)$ converge weakly or strongly to $\mathrm{Prox}_{\varphi}^{f}(\xi)$ as the functions φ_n Mosco-converge to φ and the vectors ξ_n converge to $\xi \in \mathrm{int\,dom}\, f^*$? Previous results show that, if the functions φ_n are uniformly bounded from below, then weak convergence holds when f is strongly coercive or uniformly convex on bounded sets, with strong convergence resulting from weak convergence whenever f is totally convex. We prove that the same is true when f is only coercive and the sequence $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above. In the context, we establish some continuity type properties of $\mathrm{Prox}_{\varphi}^{f}$.

1. Introduction

In this paper X denotes a real reflexive Banach space with the norm $\|\cdot\|$ and X^* represents the (topological) dual of X whose norm is denoted $\|\cdot\|_*$. Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous, convex function and let $f^* : X^* \to (-\infty, +\infty]$ be the Fenchel conjugate of f. We assume that f is a Legendre function (see [**7**, Definition 5.2]). This implies that both functions f and f^* are Legendre, have domains with nonempty interior, are differentiable on the interiors of their respective domains,

$$\mathrm{dom}\,\nabla f = \mathrm{int\,dom}\, f = \mathrm{dom}\,\partial f, \tag{1.1}$$

$$\mathrm{ran}\,\nabla f = \mathrm{dom}\ \nabla f^* = \mathrm{int\,dom}\, f^*, \tag{1.2}$$

$$\mathrm{ran}\,\nabla f^* = \mathrm{dom}\ \nabla f = \mathrm{int\,dom}\, f, \tag{1.3}$$

and

$$\nabla f = (\nabla f^*)^{-1}. \tag{1.4}$$

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We denote by $\mathcal{F}_f$ the set of proper, lower semicontinuous, convex functions $\varphi : X \to (-\infty, +\infty]$ which satisfy the conditions that

$$\operatorname{dom}\varphi \cap \operatorname{int}\operatorname{dom} f \neq \varnothing, \tag{1.5}$$

and

$$\varphi_f := \inf\left\{\varphi(x) : x \in \operatorname{dom}\varphi \cap \operatorname{dom} f\right\} > -\infty. \tag{1.6}$$

According to [**8**, Propositions 3.22 and 3.23] (see [**13**, Lemma 2.1] for another proof of the same result), for any $\varphi \in \mathcal{F}_f$, the operator $\operatorname{Prox}_\varphi^f : X^* \to 2^X$ given by

$$\operatorname{Prox}_\varphi^f(\xi) := \arg\min\left\{\varphi(x) + W^f(\xi, x) : x \in X\right\}, \tag{1.7}$$

where

$$W^f(\xi, x) := f(x) - \langle \xi, x\rangle + f^*(\xi), \tag{1.8}$$

is single valued on $\operatorname{int}\operatorname{dom} f^*$ and, for any $\xi \in \operatorname{int}\operatorname{dom} f^*$, it has

$$\operatorname{Prox}_\varphi^f(\xi) = \left[\partial\left(\varphi + f\right)\right]^{-1}(\xi) = \left(\partial\varphi + \nabla f\right)^{-1}(\xi), \tag{1.9}$$

and

$$\operatorname{Prox}_\varphi^f(\xi) \in \operatorname{dom}\partial\varphi \cap \operatorname{int}\operatorname{dom} f. \tag{1.10}$$

We call $\operatorname{Prox}_\varphi^f$ *the (generalized) proximal mapping relative to f associated to φ*. It is a natural generalization of the classical concept of proximal mapping originally introduced in Hilbert spaces for $f = \frac{1}{2}\left\|\cdot\right\|^2$ (see [**1**], [**2**], [**3**], [**8**], [**9**], [**24**] for more details concerning the history, properties and applications of proximal mappings). We denote

$$\operatorname{Env}_\varphi^f(\xi) = \inf\{\varphi(x) + W^f(\xi, x) : x \in X\}, \tag{1.11}$$

and then, for $\xi \in \operatorname{int}\operatorname{dom} f^*$, the vector $\operatorname{Prox}_\varphi^f(\xi)$ is the only vector in X such that

$$\operatorname{Env}_\varphi^f(\xi) = \varphi(\operatorname{Prox}_\varphi^f(\xi)) + W^f(\xi, \operatorname{Prox}_\varphi^f(\xi)). \tag{1.12}$$

In this paper we are concerned with the question whether the operator $\operatorname{Prox}_\varphi^f$ is *stable* with respect to the Mosco convergence. Precisely, we would like to know whether, and under which conditions, given the functions $\varphi_n, \varphi : X \to (-\infty, +\infty]$, $(n \in \mathbb{N})$, contained in $\mathcal{F}_f$ and such that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to φ, and given a convergent sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in $\operatorname{int}\operatorname{dom} f^*$ with $\lim_{n\to\infty}\xi_n = \xi \in \operatorname{int}\operatorname{dom}\, f^*$, does the sequence $\left\{\operatorname{Prox}_{\varphi_n}^f(\xi_n)\right\}_{n\in\mathbb{N}}$ converges (weakly or strongly) to $\operatorname{Prox}_\varphi^f(\xi)$? Recall (see [**20**, Definition 1.1 and Lemma 1.10]) that the sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ is said to be *convergent in the sense of Mosco to* φ (and we write $\text{M-}\lim_{n\to\infty}\varphi_n = \varphi$) if the following conditions are satisfied:

(M1) If $\{x_n\}_{n\in\mathbb{N}}$ is a weakly convergent sequence in X such that $\text{w-}\lim_{n\to\infty} x_n = x$, and if $\left\{\varphi_{i_n}\right\}_{n\in\mathbb{N}}$ is a subsequence $\{\varphi_n\}_{n\in\mathbb{N}}$, then $\liminf_{n\to\infty}\varphi_{i_n}(x_n) \geq \varphi(x)$;

(M2) For every $u \in X$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\lim_{n\to\infty} u_n = u \text{ and } \lim_{n\to\infty}\varphi_n(u_n) = \varphi(u). \tag{1.13}$$

Stability properties with respect to Mosco convergence of the proximal mapping $\operatorname{Prox}_\varphi^f$ are already known to hold in various circumstances similar to those described here. For instance, Theorem 3.26 in [**5**] implies that if X is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$, then $\mathrm{Prox}^f_{\varphi_n}(\xi)$ converges strongly to $\mathrm{Prox}^f_{\varphi}(\xi)$ whenever M-$\lim_{n\to\infty}\varphi_n = \varphi$ and $\xi \in X^*$. Generalizations of this result occur in [**14**], [**17**], [**18**], [**22**]. These generalizations were summarized in [**15**] where weak stability of $\mathrm{Prox}^f_{\varphi}$ (i.e., weak convergence of $\mathrm{Prox}^f_{\varphi_n}(\xi_n)$ to $\mathrm{Prox}^f_{\varphi}(\xi)$) was established in arbitrary reflexive Banach spaces provided with a Legendre function f which is either strongly coercive (i.e., $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$) or uniformly convex on bounded sets while requiring, in addition to the Mosco convergence of $\{\varphi_n\}_{n\in\mathbb{N}}$ to φ, that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is uniformly bounded from below. Moreover, Theorem 2.1 in [**15**] shows that whenever weak stability of $\mathrm{Prox}^f_{\varphi}$ can be ensured and the Legendre function f is totally convex, then the convergence of $\mathrm{Prox}^f_{\varphi_n}(\xi_n)$ to $\mathrm{Prox}^f_{\varphi}(\xi)$ is strong.

The relevance of the results concerning the Mosco stability of the proximal mapping with functions f which are not necessarily the square of the norm should be seen in the larger context of the analysis of generalized variational inequalities requiring to find $x \in \mathrm{int}\,\mathrm{dom}\, f$ such that

$$\exists\, \xi \in Bx : [\langle \xi, y - x\rangle \geq \varphi(x) - \varphi(y),\ \forall y \in \mathrm{dom}\, f]\,, \tag{1.14}$$

where $\varphi \in \mathcal{F}_f$ and $B : X \to 2^{X^*}$ is an operator which satisfies some conditions (see, for instance, [**4**], [**10**], [**13**], [**16**] and [**20**] for more details on this topic). Mosco stability is a tool of ensuring that, in some circumstances, "small" data perturbations in (1.14) do not essentially alter its solution. The main result in [**15**], described above, involving the requirement of uniform boundedness from below of $\{\varphi_n\}_{n\in\mathbb{N}}$, naturally applies to classical variational inequalities where the function φ and its perturbations φ_n usually are indicator functions of closed convex sets. However, the uniform boundedness from below of $\{\varphi_n\}_{n\in\mathbb{N}}$ happens to be a restrictive condition for the study of some non-classical generalized variational inequalities.

This leads us to the topic of the current paper. Can stability with respect to Mosco convergence of the proximal mapping be established in conditions which are different and, hopefully, less demanding than those mentioned above? That uniform boundedness from below of $\{\varphi_n\}_{n\in\mathbb{N}}$ (as presumed in [**15**]) is not a necessary condition for the weak/strong convergence of $\mathrm{Prox}^f_{\varphi_n}(\xi)$ to $\mathrm{Prox}^f_{\varphi}(\xi)$ can be observed from [**5**, Theorem 3.26] which applies in our setting when X is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$. Our main result, Theorem 1 of Section 2 below, proves that weak – and if f is totally convex (see [**11**] or [**14**]) then strong – convergence of $\mathrm{Prox}^f_{\varphi_n}(\xi_n)$ to $\mathrm{Prox}^f_{\varphi}(\xi)$ as M-$\lim_{n\to\infty}\varphi_n = \varphi$ and $\lim_{n\to\infty}\xi_n = \xi$ can be ensured when $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above for Legendre functions f which are coercive (i.e., $\lim_{\|x\|\to\infty} f(x) = \infty$) and have the property that $\{f + \varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to $f + \varphi$. Note that the requirement that $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above is equivalent to the condition that there exists a real number q such that for all $x \in X$

$$\varphi_n(x) \geq \langle \xi_n, x\rangle - q,\ \forall n \in \mathbb{N}. \tag{1.15}$$

This requirement does not imply uniform boundedness from below of the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ unless $\xi_n = 0^*$ for all $n \in \mathbb{N}$. However, if $\{\varphi_n^*(0^*)\}_{n\in\mathbb{N}}$ is bounded from above (i.e., if $\{\varphi_n\}_{n\in\mathbb{N}}$ is uniformly bonded from below), then the main result in [**15**] guarantees the conclusion of Theorem 1 in our current paper without the additional requirement that $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ should be bounded from above, but provided that f is better conditioned than we require here.

In view of [**5**, Theorem 3.66] combined with (1.9), the problem of Mosco stability for the proximal mapping can be seen as an instance of the more general problem of convergence of resolvents $(\nabla f + A_n)^{-1}$ associated to maximal monotone operators $A_n : X \to X^*$ approximating graphically (in the sense of [**5**, Definition 3.58]) some maximal monotone operator A to $(\nabla f + A)^{-1}$. In the case of a Hilbert space X provided with the functions $f = \frac{1}{2}\|\cdot\|^2$, strong pointwise convergence of $(\nabla f + A_n)^{-1}$ to $(\nabla f + A)^{-1}$ results from [**5**, Theorem 3.60]. Does this also happen in not necessarily hilbertian Banach spaces X provided with a Legendre function f as described above? Theorem 1 proved in this paper, as well as the main result in [**15**], give sufficient conditions in this sense in the case that the operators A_n and A are maximal cyclically monotone (i.e., subgradients of lower semicontinuous convex functions). Whether it is possible to extrapolate those results in our more general setting, to maximal monotone operators A_n and A (which are not necessarily cyclically monotone), is an interesting question whose answer we do not know.

2. A stability theorem for the proximal mapping

In this section we establish a set of sufficient conditions for Mosco stability of the proximal mapping Prox^f_φ. Analyzing our Mosco stability theorem for Prox^f_φ, given below, one should observe that conditions (A) and (B) are only needed for ensuring that $\{f + \varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to $f + \varphi$ when M-$\lim_{n\to\infty} \varphi_n = \varphi$. Alternative conditions for this to happen can be derived from [**19**, Theorem 5] and [**21**, Theorem 30(h)] and they can be used as replacements of (A) and (B) (see also Corollary 2 in the next section).

Theorem 1. *Suppose that the Legendre function f is coercive and $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ are functions contained in $\mathcal{F}_f$ such that M-$\lim_{n\to\infty} \varphi_n = \varphi$. If any of the following conditions is satisfied*

(A) The function f has open domain;

(B) The function $f\,|_{\mathrm{dom}\, f}$, the restriction of f to its domain, is continuous and $\mathrm{dom}\,\varphi_n \subseteq \mathrm{dom}\, f$, $(n \in \mathbb{N})$;

and if $\{\xi_n\}_{n\in\mathbb{N}}$ is a convergent sequence contained in $\mathrm{int\,dom}\, f^*$ such that $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above and $\xi := \lim_{n\to\infty} \xi_n \in \mathrm{int\,dom}\, f^*$, then

$$\text{w-}\lim_{n\to\infty} \mathrm{Prox}^f_{\varphi_n}(\xi_n) = \mathrm{Prox}^f_\varphi(\xi) \tag{2.1}$$

and

$$\lim_{n\to\infty} \mathrm{Env}^f_{\varphi_n}(\xi_n) = \mathrm{Env}^f_\varphi(\xi). \tag{2.2}$$

Moreover, if the function f is also totally convex, then the convergence in (2.1) *is strong, that is,*

$$\lim_{n\to\infty} \mathrm{Prox}^f_{\varphi_n}(\xi_n) = \mathrm{Prox}^f_\varphi(\xi). \tag{2.3}$$

Proof. Denote

$$\hat{x} = \mathrm{Prox}^f_\varphi(\xi) \text{ and } \hat{x}_n = \mathrm{Prox}^f_{\varphi_n}(\xi_n). \tag{2.4}$$

By (1.12) we have that, for each $x \in X$,

$$\varphi_n(\hat{x}_n) + W^f(\xi_n, \hat{x}_n) \le \varphi_n(x) + W^f(\xi_n, x),\ \forall n \in \mathbb{N}. \tag{2.5}$$

Hence, by (1.8) we have

$$\langle \xi_n, \hat{x}_n \rangle - (\varphi_n(\hat{x}_n) + f(\hat{x}_n)) \geq \langle \xi_n, x \rangle - (\varphi_n(x) + f(x)), \ \forall n \in \mathbb{N},$$

whenever $x \in X$ and this shows that

$$\langle \xi_n, \hat{x}_n \rangle - (\varphi_n(\hat{x}_n) + f(\hat{x}_n)) \geq (\varphi_n + f)^* (\xi_n), \ \forall n \in \mathbb{N}.$$

Clearly, this implies that

$$\langle \xi_n, \hat{x}_n \rangle - (\varphi_n(\hat{x}_n) + f(\hat{x}_n)) = (\varphi_n + f)^* (\xi_n), \ \forall n \in \mathbb{N}. \tag{2.6}$$

Now we are going to establish the following fact which may be well-known but we do not have a specific reference for it:

Claim 1: *The sequence $\{f + \varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to $f+\varphi$.*

In order to prove this claim we verify conditions $(M1)$ and $(M2)$ given above. To this end, let $\{x_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in X and let x be its weak limit. Then

$$\liminf_{n\to\infty} (f + \varphi_n)(x_n) \geq \liminf_{n\to\infty} f(x_n) + \liminf_{n\to\infty} \varphi_n(x_n) \geq f(x) + \varphi(x).$$

where the last inequality holds because f is convex and lower semicontinuous (and, hence, weakly lower semicontinuous) and because, by hypothesis, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to φ (and, hence, it satisfies $(M1)$). Consequently, the sequence $\{f + \varphi_n\}_{n\in\mathbb{N}}$ and the function $f + \varphi$ satisfy $(M1)$. Now, in order to verify $(M2)$, let $u \in X$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in X such that (1.13) holds (a sequence like that exists because M-$\lim_{n\to\infty} \varphi_n = \varphi$). In view of the validity of $(M1)$, it is sufficient to prove that

$$\limsup_{n\to\infty} (f(u_n) + \varphi_n(u_n)) \leq f(u) + \varphi(u). \tag{2.7}$$

We distinguish the following possible situations.

Case 1: If $u \notin \operatorname{dom} f$, then

$$\begin{aligned} \limsup_{n\to\infty} (f(u_n) + \varphi_n(u_n)) &\leq \limsup_{n\to\infty} f(u_n) + \limsup_{n\to\infty} \varphi_n(u_n) \\ &\leq \infty = f(u) + \varphi(u), \end{aligned} \tag{2.8}$$

that is, (2.7) holds.

Case 2: Suppose that $u \in \operatorname{dom} f$. If $u \in \operatorname{int} \operatorname{dom} f$, then there exists a positive integer n_0 such that $u_n \in \operatorname{int} \operatorname{dom} f$ for all $n \geq n_0$. Taking into account that, being lower semicontinuous, f is continuous on $\operatorname{int} \operatorname{dom} f$, this implies

$$\begin{aligned} \limsup_{n\to\infty} (f + \varphi_n)(u_n) &\leq \limsup_{n\to\infty} f(u_n) + \limsup_{n\to\infty} \varphi_n(u_n) \\ &= \lim_{n\to\infty} f(u_n) + \lim_{n\to\infty} \varphi_n(u_n) = f(u) + \varphi(u), \end{aligned} \tag{2.9}$$

showing that $(M2)$ holds in this situation. Hence, if condition (A) is satisfied, then (2.7) is true in all possible cases. Also, if (B) is satisfied, then (2.7) is true whenever u is not an element of the boundary of $\operatorname{dom} f$. Now, assume that condition (B) is satisfied and u is a boundary point of $\operatorname{dom} f$. In this situation, if there are infinitely many vectors u_n such that $u_n \notin \operatorname{dom} f$, then

$$\limsup_{n\to\infty} f(u_n) = \infty = \limsup_{n\to\infty} \varphi_n(u_n),$$

because, by (B), if $u_n \notin \operatorname{dom} f$, then $u_n \notin \operatorname{dom} \varphi_n$. Hence, according to (1.13), we deduce that $\varphi(u) = \limsup_{n\to\infty} \varphi_n(u_n) = \infty$ and, thus, (2.8) is true and, by consequence, (2.7) is also true. If all but finitely many vectors u_n are contained in $\operatorname{dom} f$, then

$$\limsup_{n\to\infty} f(u_n) = \lim_{n\to\infty} f(u_n) = f(u),$$

because of the continuity of $f\mid_{\operatorname{dom} f}$. By (1.13) this implies (2.9) and, thus, (2.7) is true in this situation too. Hence, when (B) holds, condition $(M2)$ is satisfied in all possible situations. This proves Claim 1.

Now we are going to establish the following fact:

Claim 2: *The sequence* $\{\hat{x}_n\}_{n\in\mathbb{N}}$ *defined by* (2.4) *is bounded.*

In order to prove this claim, suppose by contradiction that the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is not bounded. Then there exists a subsequence $\{\hat{x}_{k_n}\}_{n\in\mathbb{N}}$ of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|\hat{x}_{k_n}\| = +\infty$. Since, by hypothesis, the function f is coercive, we deduce that

$$\lim_{n\to\infty} f(\hat{x}_{k_n}) = +\infty. \tag{2.10}$$

According to (2.6), we have

$$f(\hat{x}_{k_n}) + \left(\varphi_{k_n} + f\right)^*(\xi_{k_n}) = \langle \xi, \hat{x}_{k_n}\rangle - \varphi_{k_n}(\hat{x}_{k_n}) \le \varphi_{k_n}^*(\xi_{k_n}),\ \forall n\in\mathbb{N}. \tag{2.11}$$

By [**5**, Theorem 3.18, p. 295] combined with Claim 1, we have that

$$M - \lim_{n\to\infty} (\varphi_n + f)^* = (\varphi + f)^*.$$

Therefore,

$$M - \lim_{n\to\infty} \left(\varphi_{k_n} + f\right)^* = (\varphi + f)^*.$$

This implies (using $(M1)$ applied to the convergent sequence $\left\{\xi_{k_n}\right\}_{n\in\mathbb{N}}$ in X^*) that

$$\liminf_{n\to\infty} \left(\varphi_{k_n} + f\right)^*(\xi_{k_n}) \ge (\varphi + f)^*(\xi). \tag{2.12}$$

An argument similar to that leading to (2.6) shows that

$$(\varphi + f)^*(\xi) = \langle \xi, \hat{x}\rangle - (\varphi(\hat{x}) + f(\hat{x})). \tag{2.13}$$

By (1.10) we have that

$$\hat{x} \in \operatorname{dom}\partial\varphi \cap \operatorname{dom} f \subseteq \operatorname{dom}\varphi \cap \operatorname{dom} f,$$

showing that $\varphi(\hat{x}) + f(\hat{x})$ is finite. Hence, by (2.13), $(\varphi + f)^*(\xi)$ is finite too. Thus, by (2.12),

$$\liminf_{n\to\infty} \left(\varphi_{k_n} + f\right)^*(\xi_{k_n}) > -\infty. \tag{2.14}$$

Taking lim inf as $n\to\infty$ on both sides of (2.11) we deduce that

$$\lim_{n\to\infty} f(\hat{x}_{k_n}) + \liminf_{n\to\infty} \left(\varphi_{k_n} + f\right)^*(\xi_{k_n}) \le \liminf_{n\to\infty} \varphi_{k_n}^*(\xi_{k_n}).$$

This, (2.10) and (2.14) imply that $\liminf_{n\to\infty} \varphi_{k_n}^*(\xi_{k_n}) = +\infty$, that is, $\lim_{n\to\infty}\varphi_{k_n}^*(\xi_{k_n}) = +\infty$, which contradicts the boundedness of $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$. So, the proof of Claim 2 is complete.

The sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ being bounded in the reflexive space X, has weak cluster points. The claim we prove below shows that $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is weakly convergent to $\hat{x}$ and, consequently, formula (2.1) holds.

Claim 3: *The only weak cluster point of* $\{\hat{x}_n\}_{n\in\mathbb{N}}$ *is* $\hat{x}$.

In order to prove Claim 3 let v be a weak cluster point of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ and let $\{\hat{x}_{i_n}\}_{n\in\mathbb{N}}$ be a subsequence of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ such that $\text{w-}\lim_{n\to\infty}\hat{x}_{i_n} = v$. Let u be any vector in $\operatorname{dom} f \cap \operatorname{dom} \varphi$. Since M-$\lim_{n\to\infty} \varphi_n = \varphi$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

$$\lim_{n\to\infty} u_n = u \text{ and } \lim_{n\to\infty} \varphi_n(u_n) = \varphi(u). \tag{2.15}$$

The function f being convex and lower semicontinuous is also weakly lower semicontinuous. The sequences $\left\{f^*(\xi_{i_n})\right\}_{n\in\mathbb{N}}$ and $\left\{\left\langle \xi_{i_n}, \hat{x}_{i_n}\right\rangle\right\}_{n\in\mathbb{N}}$ converge to $f^*(\xi)$ and $\langle \xi, v\rangle$, respectively. Consequently, we have

$$\begin{aligned}\liminf_{n\to\infty} W^f(\xi_{i_n}, \hat{x}_{i_n}) &\geq \liminf_{n\to\infty} f(\hat{x}_{i_n}) + \liminf_{n\to\infty} \left[f^*(\xi_{i_n}) - \left\langle \xi_{i_n}, \hat{x}_{i_n}\right\rangle\right] \\ &\geq f(v) - \langle \xi, v\rangle + f^*(\xi) = W^f(\xi, v).\end{aligned} \tag{2.16}$$

Due to the Mosco convergence of $\{\varphi_n\}_{n\in\mathbb{N}}$ (and, hence, of $\left\{\varphi_{i_n}\right\}_{n\in\mathbb{N}}$) to φ, to (2.16), and to (2.5) we deduce that

$$\begin{aligned}&\varphi(v) + W^f(\xi, v) \leq \\ &\liminf_{n\to\infty} \varphi_{i_n}(\hat{x}_{i_n}) + \liminf_{n\to\infty} W^f(\xi_{i_n}, \hat{x}_{i_n}) \\ &\leq \liminf_{n\to\infty} \operatorname{Env}^f_{\varphi_{i_n}}(\xi_{i_n}) \leq \limsup_{n\to\infty} \operatorname{Env}^f_{\varphi_{i_n}}(\xi_{i_n}) \\ &\leq \limsup_{n\to\infty} \left\{\varphi_{i_n}(u_{i_n}) + W^f(\xi_{i_n}, u_{i_n})\right\} = \varphi(u) + W^f(\xi, u).\end{aligned} \tag{2.17}$$

Since u was arbitrarily chosen in $\operatorname{dom} f \cap \operatorname{dom} \varphi$, it follows that $v = \hat{x}$ and this proves Claim 3.

Now we are in position to show that (2.2) is also true. If we prove that, then the strong convergence of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ to $\hat{x}$ (i.e., (2.3)) when f is also totally convex results from [**15**, Theorem 1] and the proof of our theorem is complete. For proving (2.2) observe that, according to (1.10), the vector $\hat{x}$ belongs to $\operatorname{int}\operatorname{dom} f \cap \operatorname{dom} \varphi$ and, therefore, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that (2.15) holds for $u = \hat{x}$. Since the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ converges weakly to $v = \hat{x}$, the inequalities and equality in (2.17) remain true when v is replaced by $\hat{x}$ and i_n is replaced by n. Therefore, taking into account (1.12) and (2.4), we deduce

$$\begin{aligned}&\operatorname{Env}^f_{\varphi}(\xi) = \varphi(\hat{x}) + W^f(\xi, \hat{x}) \\ &\leq \liminf_{n\to\infty} \operatorname{Env}^f_{\varphi_n}(\xi_n) \leq \limsup_{n\to\infty} \operatorname{Env}^f_{\varphi_n}(\xi_n) \\ &= \varphi(\hat{x}) + W^f(\xi, \hat{x})\end{aligned}$$

and this implies (2.2). □

3. Consequences of the stability theorem

The following result shows that Theorem 1 applies to any constant sequence $\xi_n = \xi \in \operatorname{ran}\partial\varphi \cap \operatorname{int}\operatorname{dom} f^*$ since, for any such vector ξ, the sequence $\{\varphi_n^*(\xi)\}_{n\in\mathbb{N}}$ is bounded from above.

Corollary 1. *Suppose that the Legendre function f is coercive and $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ are functions contained in $\mathcal{F}_f$ such that M-$\lim_{n\to\infty} \varphi_n = \varphi$. If any of the conditions (A) or (B) of Theorem 1 is satisfied, and if $\xi \in \operatorname{ran}\partial\varphi \cap \operatorname{int}\operatorname{dom} f^*$, then*

$$\text{w-}\lim_{n\to\infty} \operatorname{Prox}^f_{\varphi_n}(\xi) = \operatorname{Prox}^f_{\varphi}(\xi) \tag{3.1}$$

and

$$\lim_{n\to\infty} \operatorname{Env}^f_{\varphi_n}(\xi) = \operatorname{Env}^f_{\varphi}(\xi). \tag{3.2}$$

Moreover, if the function f is also totally convex, then the convergence in (3.1) *is strong.*

Proof. According to Theorem 1, it is sufficient to show that if $\xi \in \operatorname{ran}\partial\varphi$, then the sequence $\{\varphi_n^*(\xi)\}_{n\in\mathbb{N}}$ is bounded from above. To this end, let $\bar{x} \in X$ be such that $\xi \in \partial\varphi(\bar{x})$. Then, by the convexity of φ, we have

$$\varphi(x) - \varphi(\bar{x}) \geq \langle \xi, x - \bar{x}\rangle\,,\ \forall x \in X,$$

showing that

$$\varphi(x) \geq \langle \xi, x\rangle - q,\ \forall x \in X, \tag{3.3}$$

where $q = \langle \xi, \bar{x}\rangle - \varphi(\bar{x})$ is a real number because $\bar{x} \in \operatorname{dom}\partial\varphi \subseteq \operatorname{dom}\varphi$. By the hypothesis that M-$\lim_{n\to\infty}\varphi_n = \varphi$ combined with (3.3) we deduce (see $(M1)$) that for any $x \in X$

$$q \geq \langle \xi, x\rangle - \varphi(x) \geq \langle \xi, x\rangle - \liminf_{n\to\infty}\varphi_n(x) = \limsup_{n\to\infty}\left[\langle \xi, x\rangle - \varphi_n(x)\right].$$

Hence,

$$q \geq \sup_{x\in X}\limsup_{n\to\infty}\left[\langle \xi, x\rangle - \varphi_n(x)\right] = \limsup_{n\to\infty}\sup_{x\in X}\left[\langle \xi, x\rangle - \varphi_n(x)\right] = \limsup_{n\to\infty}\varphi_n^*(\xi),$$

showing that the sequence $\{\varphi_n^*(\xi)\}_{n\in\mathbb{N}}$ is bounded from above. □

It is meaningful to note that, if the Banach space X has finite dimension, then the conditions (A) and (B) involved in Theorem 1 can be replaced by the requirement that

$$\operatorname{int}\operatorname{dom}\varphi \neq \varnothing. \tag{3.4}$$

Conditions (A) and (B) are only used in the proof of Theorem 1 in order to ensure (see Claim 1) that M-$\lim_{n\to\infty}(f+\varphi_n) = f+\varphi$ when M-$\lim_{n\to\infty}\varphi_n = \varphi$. According to [**19**, Theorem 5], if $\dim X < \infty$, then this happens whenever (3.4) holds because of (1.5) which implies that, in these circumstances, $0 \in \operatorname{int}(\operatorname{dom}\varphi - \operatorname{dom} f)$. So, we have the following result:

Corollary 2. *Suppose that* $\dim X < \infty$ *and that the Legendre function f is coercive. If $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ are functions contained in $\mathcal{F}_f$ such that M-$\lim_{n\to\infty}\varphi_n = \varphi$ and* (3.4) *holds, and if $\{\xi_n\}_{n\in\mathbb{N}}$ is a convergent sequence contained in* $\operatorname{int}\operatorname{dom} f^*$ *such that $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above and $\xi := \lim_{n\to\infty}\xi_n \in \operatorname{int}\operatorname{dom} f^*$, then* (2.1) *and* (2.2) *are true.*

Combining [**23**, Theorem 1 and Proposition 1] with (1.5), (1.9) and (1.10) one can see that the operator $\operatorname{Prox}^f_\varphi(\cdot)$ with $\varphi \in \mathcal{F}_f$ is maximal monotone and norm to weak continuous on $\operatorname{int}\operatorname{dom} f^*$. In other words, (2.1) holds for any constant sequence $\varphi_n = \varphi \in \mathcal{F}_f$ and for any sequence $\{\xi_n\}_{n\in\mathbb{N}}$ which is contained and converges in $\operatorname{int}\operatorname{dom} f^*$, even if f is not coercive or satisfies one of the conditions (A) or (B). A careful analysis of the proof of Theorem 1 shows that we have already proved that if (2.1) holds then (2.2) holds too. Also, carefully analyzing the proof of Theorem 1 one can observe that, if $\varphi_n = \varphi \in \mathcal{F}_f$ for all $n \in \mathbb{N}$, then the conditions (A) and (B) are superfluous (because in this case the conclusion of Claim 1 remains true even if these conditions do not hold). These remarks lead us to the following result:

Corollary 3. *If the Legendre function f is coercive and totally convex and if $\varphi \in \mathcal{F}_f$, then the following statements are true:*

(i) If B is a nonempty and bounded subset $\operatorname{dom} \partial\varphi$, *then* $\operatorname{Prox}_{\varphi}^{f}(\cdot)$ *is norm to norm continuous on* $\partial\varphi(B) \cap \operatorname{int}\operatorname{dom} f^*$*;*

(ii) If φ^* is bounded from above on bounded subsets of $\operatorname{int}\operatorname{dom} f^* \cap \operatorname{ran} \partial\varphi$, *then* $\operatorname{Prox}_{\varphi}^{f}(\cdot)$ *is norm to norm continuous on* $\operatorname{ran}\partial\varphi \cap \operatorname{int}\operatorname{dom} f^*$.

Proof. (i) Suppose that $\{\xi_n\}_{n\in\mathbb{N}}$ and ξ are contained in $\partial\varphi(B) \cap \operatorname{int}\operatorname{dom} f^*$ and satisfy $\lim_{n\to\infty} \xi_n = \xi$. Then, for each $n \in \mathbb{N}$, there exists a vector $\bar{x}_n \in B$ such that $\xi_n \in \partial\varphi(\bar{x}_n)$. By the convexity of φ we deduce that for any $x \in X$

$$\varphi(x) - \varphi(\bar{x}_n) \geq \langle \xi_n, x - \bar{x}_n \rangle, \ \forall n \in \mathbb{N}.$$

Hence, for any $x \in X$

$$\begin{aligned} \varphi(x) &\geq \langle \xi_n, x \rangle - \langle \xi_n, \bar{x}_n \rangle + \varphi(\bar{x}_n) \\ &\geq \langle \xi_n, x \rangle - \|\xi_n\|_* \|\bar{x}_n\| + \varphi(\bar{x}_n), \end{aligned} \tag{3.5}$$

where the sequence $\{\|\xi_n\|_* \|\bar{x}_n\|\}_{n\in\mathbb{N}}$ is bounded because both sequences $\{\|\xi_n\|_*\}_{n\in\mathbb{N}}$ and $\{\|\bar{x}_n\|\}_{n\in\mathbb{N}}$ are bounded, and the sequence $\{\varphi(\bar{x}_n)\}_{n\in\mathbb{N}}$ is bounded from below because $\{\bar{x}_n\}_{k\in\mathbb{N}}$ is contained in $\operatorname{dom}\varphi \cap \operatorname{int}\operatorname{dom} f$ and (1.6) holds. These facts, combined with (3.5) show that there exists a real number q such that for any $x \in X$

$$\varphi(x) \geq \langle \xi_n, x \rangle - q, \ \forall n \in \mathbb{N}.$$

In other words, the constant sequence $\varphi_n = \varphi$ satisfies (1.15) and, thus, the sequence $\{\varphi^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above. Applying Theorem 1 to the constant sequence $\varphi_n = \varphi$ and taking into account the remarks preceding this Corollary, we deduce that (2.3) holds in this case, i.e., $\operatorname{Prox}_{\varphi}^{f}(\cdot)$ is norm to norm continuous on $\partial\varphi(B) \cap \operatorname{int}\operatorname{dom} f^*$.

(ii) Suppose that $\{\xi_n\}_{n\in\mathbb{N}}$ and ξ are contained in $\operatorname{int}\operatorname{dom} f^* \cap \operatorname{ran}\partial\varphi$ and satisfy $\lim_{n\to\infty} \xi_n = \xi$. Then the sequence $\{\varphi^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above because $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded as being convergent. Application of Theorem 1 shows that (2.3) holds in this case too. □

References

[1] Alber, Ya. I., Generalized projection operators in Banach spaces: Properties and applications, in: M.E. Draklin and E. Litsyn (Eds.), "Functional Differential Equations", Vol. 1, *The research Institute of Judea and Samaria*, Kedumim-Ariel, 1993.

[2] Alber, Ya. I., Metric and generalized projection operators in Banach spaces: Properties and applications, in: G. Kartsatos (Ed.), "Theory and Applications of Nonlinear Operators of Accretive and Monotone Type", pp. 15-50, *Dekker*, New York, 1996.

[3] Alber Ya. I.: Stability of the proximal projection algorithm for nonsmooth convex optimization problems with perturbed constraint sets, *J. Nonlinear Convex Anal.* **4** (2003), 1–14.

[4] Alber, Ya. I.; Butnariu, D. Ryazantseva, I.: Regularization of monotone variational inequalities with Mosco approximations of the constraint sets. *Set-Valued Anal.* **13** (2005), 265–290.

[5] Attouch, H.: Variational Convergence for Functions and Operators, *Pitman*, Boston, 1984.

[6] Bauschke, H.H. and Borwein, J.M.: Legendre functions and the method of random Bregman projections, *J. Convex Anal.* **4** (1997), 27-67.

[7] Bauschke, H.H., Borwein, J.M. and Combettes, P.L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, *Commun. Contemp. Math.* **3** (2001), 615-647.

[8] Bauschke, H.H., Borwein, J.M. and Combettes, P.L.: Bregman monotone optimization algorithms, *SIAM J. Control Optim.* **42** (2003), 596-636.

[9] Bauschke, H.H., Combettes, P.L. and Noll, D.: Joint minimization with alternating Bregman proximity operators, preprint, 2005.
[10] Bonnans, J.F. and Shapiro, A.: Perturbation Analysis of Optimization Problems, *Springer*, New York, 2000.
[11] Butnariu, D. and Iusem, A.N.: Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, *Kluwer Academic Publishers*, Dordrecht, 2000.
[12] Butnariu, D., Iusem, A.N. and Zălinescu, C.: On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces, *J. Convex Anal.* **10** (2003) 35-61.
[13] Butnariu, D. and Kassay, G.: A proximal-projection method for finding zeros of set-valued operators, *preprint* 2007.
[14] Butnariu, D. and Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.* 2006, Art. ID 84919, 39 pp.
[15] Butnariu, D. and Resmerita, E.: Mosco stability of proximal mappings in reflexive Banach spaces, *J. Nonlinear Conv. Anal.* **8** (2007), 1-10.
[16] Dontchev, A.L. and Zolezzi, T.: Well-Posed Optimization Problems, *Springer Verlag*, Berlin, 1993.
[17] Ibaraki, T., Kimura, Y. and Takahashi, W.: Convergence theorems for generalized projections and maximal monotone operators in Banach spaces, *Abstr. Appl. Anal.* **10** (2003), 621-629.
[18] Kimura, Y.: Mosco convergence of closed subsets and resolvents of maximal monotone operators, *J. Nonlin. Convex Anal.* **4** (2003), 269-275.
[19] McLinden, I. and Bergstrom, C.: Preservation of convergence of convex sets and functions in finite dimensions, *Trans. Amer. Math. Soc.* **268** (1981), 127-142.
[20] Mosco, U.: Convergence of convex sets and of solutions of variational inequlities, *Adv. Math.* **4** (1969), 510-585.
[21] Penot, J.-P. and Zălinescu, C.: Continuity of usual operations and variational convergence, *Set-Valued Analysis* **11** (2003), 225-256.
[22] Resmerita, E.: On total convexity, Bregman projections and stability in Banach spaces, *J. Convex Anal.* **11** (2004), 1-16.
[23] Rockafellar, R.T.: On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.* **149** (1970), 75-88.
[24] Rockafellar, R.T. and Wets, R.J.-B.: Variational Analysis, *Springer Verlag*, Berlin, 1998.
[25] Zălinescu, C.: Convex Analysis in General Vector Spaces, *World Scientific*, 2002.

Dan Butnariu: Department of Mathematics, University of Haifa, 31905 Haifa, Israel
E-mail address: dbutnaru@math.haifa.ac.il

Elena Resmerita: Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, Linz 4040, Austria
E-mail address: elena.resmerita@ricam.oeaw.ac.at

Shoham Sabach: Department of Mathematics, University of Haifa, 31905 Haifa, Israel
E-mail address: ssabach@gmail.com