CONVERGENCE TO FIXED POINTS OF INEXACT ORBITS FOR BREGMAN-MONOTONE AND FOR NONEXPANSIVE OPERATORS IN BANACH SPACES

Dan Butnariu, Simeon Reich, and Alexander J. Zaslavski

ABSTRACT. Fixed points of Bregman-monotone operators are often solutions of problems occurring in applications. Finding fixed points of such operators is usually done by computing orbits which happen to converge to fixed points. Sometimes, the computation of the elements of the orbits of Bregman-monotone operators is itself the result of an approximation process which is affected by errors. We present sufficient conditions for inexact orbits of Bregman-monotone operators to be weakly convergent to fixed points.

1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$, let K be a nonempty, weakly closed subset of X and let $f : X \to (-\infty, +\infty]$ be a lower semicontinuous Gâteaux differentiable function which is uniformly convex on bounded subsets of X and satisfies $K \subseteq \operatorname{int} \operatorname{dom} f$. Recall (see [**11**, Proposition 4.2 and Corollary 4.3]) that lower semicontinuous functions which are uniformly convex on bounded sets and have domains with nonempty interiors exist only on reflexive Banach spaces. Therefore, by assuming that a function f as described above exists we implicitly assume that the space X is reflexive. We denote by X^* the dual space of X and by $\|\cdot\|_*$ the dual norm.

We follow [**4**] and call an operator $T : K \to K$ *Bregman-monotone* (*with respect to* f) if it satisfies the following condition:

$$z \in \operatorname{Fix} T \Rightarrow \forall x \in K : \ \langle f'(x) - f'(Tx), z - Tx \rangle \leq 0. \tag{1.1}$$

Bregman-monotone operators are the cornerstones of a large class of algorithms for solving variational inequalities, convex optimization problems, convex feasibility problems and equilibrium problems. Many algorithms in this class are built by taking a Bregman-monotone operator T (relative to a conveniently chosen function f) such that $\operatorname{Fix} T$ is contained in the solution set of the problem the algorithm is supposed to solve and, then, by using the *orbits* of T, i.e., the sequences defined by $x^k = T^k x$ for all $k \in \mathbb{N}$, in order to approximate fixed points of T (and, hence,

2000 *Mathematics Subject Classification.* Primary: 47H30, 47H09; Secondary: 47J25, 47H10.

Key words and phrases. Bregman-monotone operator, Bregman projection, Cimmino type operator, fixed point, inexact orbit, orbit, proximal point operator, resolvent.

solutions of the given problem). Among the best known Bregman-monotone operators involved in the build-up of such algorithms are the resolvents associated with maximal monotone operators, the proximal point operators analyzed in their full generality in [**4**] and the Cimmino-type operators presented in full generality in [**8**, Chapter 2]. A large body of literature is dedicated to the convergence analysis of the orbits of special classes Bregman-monotone operators (see [**13**], [**8**], [**4**] and the references therein). It is mostly aimed at ensuring that the orbits of the Bregman-monotone operators involved in computational procedures of interest in applications converge weakly, and sometimes strongly, to fixed points of those operators. It was repeatedly noted that, in applications, computing orbits of Bregman-monotone operators can not always be done with absolute precision and, instead, one has sometimes to content himself with *inexact orbits*, that is, with sequences $\{y^k\}_{k\in\mathbb{N}}$ such that y^k only approximates the value $T^k y$. This naturally leads to the question of whether, and in which conditions, inexact orbits of a Bregman-monotone operator T still converge to fixed points of T. In other words, the question is which errors can be tolerated in the computation of orbits for Bregman-monotone operators without altering their convergence behavior. That question was considered before. Conditions on the errors which are sufficient for ensuring weak convergence of inexact orbits for some Bregman-monotone operators are already known ([**25**], [**19**], [**15**], [**20**], [**26**], [**22**], [**23**]). The aim of this note is to prove that weak convergence to fixed points of inexact orbits for Bregman-monotone operators is a more general property. In Section 2 below we establish sufficient conditions for weak convergence to fixed points of inexact orbits for a large class of Bregman-monotone operators in reflexive Banach spaces. It should be noted that, even for (exact) orbits of Bregman-monotone operators, strong convergence to fixed points does not happen even in cases when weak convergence can be ensured (see [**17**] and [**6**] for a more details on this topic). In general, Bregman-monotone operators are not nonexpansive (see [**8**]), that is, they do not necessarily have the property that

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in K.$$

However, when X is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$ nonexpansivity is a property shared by many meaningful Bregman-monotone operators including all firmly nonexpansive operators and, among them, the resolvents of maximal monotone operators involved in proximal point optimization methods and the metric Cimmino-type operators involved in convex feasibility problem solving algorithms (cf. [**25**] and [**4**]). In Section 3 we show that for Bregman-monotone operators which are also nonexpansive weak convergence to fixed points of inexact orbits can be ensured under the quite usual summability of the errors requirement. This applies, in particular, to Cimmino-type operators commonly used in resolution of convex feasibility problems.

The results in Section 3 lead to the question whether inexact orbits with summable errors of nonexpansive Bregman-monotone operators $T : K \to K$ (with respect to $f = \frac{1}{2}\|\cdot\|^2$) still weakly converge to fixed points of T (provided that such fixed points exist) even when the underlying space X is no longer a Hilbert space. We do not know the answer to this question. However, in Section 4 we show that, for nonexpansive operators in any Banach space, the exact orbits and the inexact orbits with summable errors converge and diverge together. This fact may help us of find an answer to the question posed above.

2. Weak convergence of inexact orbits for Bregman-monotone operators in reflexive spaces

In this section the space X, the set K and the function f are assumed to satisfy all the requirements described at the beginning of Section 1. For any $x \in \operatorname{int}\operatorname{dom} f$ and $y \in \operatorname{dom} f$, we denote

$$D_f(y,x) = f(y) - f(x) - \langle f'(x), y - x\rangle . \tag{2.1}$$

With this notation, the Bregman-monotonicity condition (1.1) for the operator $T : K \to K$ can be equivalently re-written as

$$z \in \operatorname{Fix} T \Rightarrow \forall x \in K : D_f(z,Tx) + D_f(Tx,x) \leq D_f(z,x). \tag{2.2}$$

For any real number $t \geq 0$ and for each nonempty bounded subset E of $\operatorname{int}\operatorname{dom} f$ we denote

$$\nu_f(E;t) = \inf\left\{D_f(y,x) : y \in \operatorname{dom} f,\ x \in E,\ \ \|y-x\| = t\right\}. \tag{2.3}$$

The function $\nu_f(E;\cdot)$ is the so called *modulus of uniform total convexity of* f whose main properties were summarized in [**11**] and [**12**]. We only recall that $\nu_f(E;0) = 0$ and that f is uniformly convex on bounded sets if and only if $\nu_f(E;\cdot)$ is positive and strictly increasing on its domain.

In what follows in this section, we make the following assumption:

Assumption 1. *For each $x \in K$ and for any $\alpha > 0$, the set*

$$R_\alpha^f(x) = \{y \in K : D_f(x,y) \leq \alpha\}$$

is bounded.

This is an often assumed condition in the convergence analysis of algorithms based on Bregman-monotone operators. It is satisfied by many functions of practical interest. For instance, Assumption 1 holds in the case of the functions $\|\cdot\|^p$ with $p \in (1,\infty)$ in uniformly convex and smooth Banach spaces (see [**10**]) and the negentropy defined by $f(x) = \sum_{i=1}^n x_i \ln x_i$ if $x \in \mathbb{R}_+^n$ and $f(x) = +\infty$, otherwise, with the convention that $0\log 0 = 0$. More generally, it holds whenever f^*, the Fenchel conjugate of f, is Gâteaux differentiable and its derivative $f^{*\prime}$ is bounded on bounded sets.

A condition which is commonly involved in the convergence analysis of inexact orbits for Bregman-monotone operators is summability of the sequence $\left\{D_f(Ty^k, y^{k+1})\right\}_{k\in\mathbb{N}}$ which, when X is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$, amounts to

$$\sum_{k=0}^{\infty} \left\|Ty^k - y^{k+1}\right\|^2 < \infty. \tag{2.4}$$

However, it seems to us that summability of $\left\{D_f(Ty^k, y^{k+1})\right\}_{k\in\mathbb{N}}$ is not enough to ensure good convergence behavior of the inexact orbits. In what follows in this section we consider inexact orbits which simultaneously satisfy the following requirements:

$$\sum_{k=0}^{\infty} D_f(Ty^k, y^{k+1}) < \infty, \tag{2.5}$$

$$\sum_{k=0}^{\infty} \left\|f'(Ty^k) - f'(y^{k+1})\right\|_* < \infty, \tag{2.6}$$

$$\sum_{k=0}^{\infty} \left\| f'(Ty^k) - f'(y^{k+1}) \right\|_* \left\| Ty^k \right\| < \infty, \tag{2.7}$$

and

$$\sum_{k=0}^{\infty} \left\langle f'(y^{k+1}) - f'(Ty^k), y^{k+1} - Ty^k \right\rangle < \infty. \tag{2.8}$$

Note that, by the monotonicity of f', the terms of the series (2.8) are nonnegative.

It is obvious that exact orbits of T satisfy (2.6), (2.7) and (2.8). That exact orbits of Bregman-monotone operators satisfy (2.5) when $\operatorname{Fix} T \neq \varnothing$ immediately follows from (2.2).

The results we prove below show that these summability conditions are sufficient for ensuring weak convergence to fixed points for inexact orbits of many Bregman-monotone operators.

Theorem 1. *Suppose that $T : K \to K$ is a Bregman-monotone operator relative to the function f and that $\operatorname{Fix} T \neq \varnothing$. Let $\left\{y^k\right\}_{k\in\mathbb{N}}$ be an inexact orbit of T which satisfies the conditions (2.5), (2.6), (2.7) and (2.8). Then $\left\{y^k\right\}_{k\in\mathbb{N}}$ has the following properties:*

(i) The sequence $\left\{y^k\right\}_{k\in\mathbb{N}}$ is bounded, has weak accumulation points, and all its weak accumulation points are contained in K; if f' is bounded on bounded subsets of K, then we also have $\lim_{k\to\infty} \left\|y^k - y^{k+1}\right\| = 0$.

(ii) If, in addition, the function $x \to D_f(Tx, x)$ is sequentially weakly lower semicontinuous, then any weak accumulation point of $\left\{y^k\right\}_{k\in\mathbb{N}}$ is contained in $\operatorname{Fix} T$. In this case, if the function f also satisfies the condition that

$$\left\{ \begin{array}{c} \left\{x^k\right\}_{k\in\mathbb{N}}, \left\{u^k\right\}_{k\in\mathbb{N}} \subseteq K \\ x^k \rightharpoonup x, u^k \rightharpoonup u \\ x \neq u \end{array} \right\} \Rightarrow \liminf_{k\to\infty} \left|\left\langle f'(x^k) - f'(u^k), x - u \right\rangle\right| > 0, \tag{2.9}$$

then the sequence $\left\{y^k\right\}_{k\in\mathbb{N}}$ converges weakly to a fixed point of T.

Proof. Let $a, b, c \in \operatorname{int} \operatorname{dom} f$. Then, by the definition of D_f (see (2.1)) we deduce that

$$\begin{aligned} D_f(a,b) + D_f(b,c) &= D_f(a,c) + \langle f'(c) - f'(b), a - b \rangle \\ &= D_f(a,c) + \langle f'(c) - f'(b), c - b \rangle \\ &\quad + \langle f'(c) - f'(b), a - c \rangle . \end{aligned} \tag{2.10}$$

Suppose that $z \in \operatorname{Fix} T$ and that $\left\{y^k\right\}_{k\in\mathbb{N}}$ is a inexact orbit of T satisfying (2.5) - (2.8). Then, by letting in (2.10) $a = z \in \operatorname{int} \operatorname{dom} f$, $b = Ty^k$ and $c = y^{k+1}$, we obtain

$$\begin{aligned} & D_f(z, y^{k+1}) + \left\langle f'(y^{k+1}) - f'(Ty^k), y^{k+1} - Ty^k \right\rangle \\ = \; & D_f(z, Ty^k) + D_f(Ty^k, y^{k+1}) + \left\langle f'(y^{k+1}) - f'(Ty^k), y^{k+1} - z \right\rangle , \end{aligned}$$

for all $k \in \mathbb{N}$. Note that the second term on the left-hand side of this equality is nonnegative because f' is monotone. Hence, we have

$$D_f(z, y^{k+1}) \leq D_f(z, Ty^k) + D_f(Ty^k, y^{k+1}) + \left\langle f'(y^{k+1}) - f'(Ty^k), y^{k+1} - z \right\rangle , \tag{2.11}$$

for all $k \in \mathbb{N}$. If $z \in \operatorname{Fix} T$ then, by (2.2), one gets

$$D_f(z, Ty^k) + D_f(Ty^k, y^k) \leq D_f(z, y^k).$$

This and (2.11) imply that

$$D_f(z, y^{k+1}) + D_f(Ty^k, y^k) \leq D_f(z, y^k) + D_f(Ty^k, y^{k+1})$$
$$+ \left\langle f'(y^{k+1}) - f'(Ty^k), y^{k+1} - Ty^k \right\rangle + \left\langle f'(y^{k+1}) - f'(Ty^k), Ty^k - z \right\rangle$$

for all $k \in \mathbb{N}$ and for any $z \in \operatorname{Fix} T$. Hence, for all $k \in \mathbb{N}$ and for any $z \in \operatorname{Fix} T$, we have

$$\begin{aligned} & D_f(z, y^{k+1}) + D_f(Ty^k, y^k) \\ & \leq D_f(z, y^k) + \left[D_f(Ty^k, y^{k+1}) + \left\langle f'(y^{k+1}) - f'(Ty^k), y^{k+1} - Ty^k \right\rangle \right. \\ & \left. + \left\| f'(y^{k+1}) - f'(Ty^k) \right\|_* \left(\left\| Ty^k \right\| + \|z\| \right) \right]. \end{aligned} \tag{2.12}$$

The conditions (2.5) - (2.8) imply that the terms between the square brackets in this inequality are summable. Consequently, the sequence $\left\{ D_f(z, y^k) \right\}_{k \in \mathbb{N}}$ converges for any $z \in \operatorname{Fix} T$. Summing up the inequalities (2.12) for $k = 0, 1, ...n$ and letting $n \to \infty$ one deduces that the series $\sum_{k=0}^{\infty} D_f(Ty^k, y^k)$ converge and, hence, that

$$\lim_{k \to \infty} D_f(Ty^k, y^k) = 0. \tag{2.13}$$

Since $\left\{ D_f(z, y^k) \right\}_{k \in \mathbb{N}}$ is bounded, it follows that $\left\{ y^k \right\}_{k \in \mathbb{N}}$ is bounded too by Assumption 1. Let E be the bounded set of all terms of the sequence $\left\{ y^k \right\}_{k \in \mathbb{N}}$. By (2.3) we have that

$$0 \leq \nu_f(E, \left\| Ty^k - y^k \right\|) \leq D_f(Ty^k, y^k), \ \forall k \in \mathbb{N}.$$

This shows that $\lim_{k \to \infty} \nu_f(E, \left\| Ty^k - y^k \right\|) = 0$ and this can not happen unless

$$\lim_{k \to \infty} \left\| Ty^k - y^k \right\| = 0 \tag{2.14}$$

because, as noted above, for a lower semicontinuous function f which is uniformly convex on bounded sets the function $\nu_f(E, \cdot)$ vanishes at zero and is positive and strictly increasing on its domain. The space X being reflexive the sequence $\left\{ y^k \right\}_{k \in \mathbb{N}}$ has weakly convergent subsequences. Since K is weakly closed, all weak accumulation points of $\left\{ y^k \right\}_{k \in \mathbb{N}}$ are contained in K.

Now, suppose that f' is bounded on bounded subsets of K. By (2.10) we have that

$$\begin{aligned} & D_f(Ty^k, y^k) + D_f(y^k, y^{k+1}) \\ = \ & D_f(Ty^k, y^{k+1}) + \left\langle f'(y^{k+1}) - f'(Ty^k), y^k - Ty^k \right\rangle. \end{aligned}$$

This equality, together with (2.5), (2.13) and (2.14), implies that

$$\lim_{k \to \infty} D_f(y^k, y^{k+1}) = 0.$$

Therefore, by (2.3), we have

$$\lim_{k \to \infty} \nu_f(\left\{ y^{n+1} \right\}_{n \in \mathbb{N}}; \left\| y^k - y^{k+1} \right\|) = 0$$

which, in turn, implies that $\lim_{k \to \infty} \left\| y^k - y^{k+1} \right\| = 0$. The proof of (i) is complete.

In order to prove (ii) assume that the function $x \to D_f(Tx, x)$ is sequentially weakly lower semicontinuous. Let $\left\{ y^{i_k} \right\}_{k \in \mathbb{N}}$ be a weakly convergent subsequence of $\left\{ y^k \right\}_{k \in \mathbb{N}}$ and let y be the weak limit of this subsequence. Then

$$0 \leq D_f(Ty, y) \leq \liminf_{k \to \infty} D_f(Ty^{i_k}, y^{i_k}) = 0$$

and this implies that $Ty = y$ because f is strictly convex (since it is uniformly convex) and, thus, for $a, b \in \operatorname{int}\operatorname{dom} f$ we have that $D_f(a,b) = 0$ if and only if $a = b$. This shows that any weak accumulation point of $\{y^k\}_{k\in\mathbb{N}}$is contained in $\operatorname{Fix} T$. Suppose that, in addition to the function $x \to D_f(Tx, x)$ being sequentially weakly lower semicontinuous, the function f also satisfies (2.9). In this situation, assume by contradiction that the sequence $\{y^k\}_{k\in\mathbb{N}}$ has two different weak accumulation points y' and y''. Then $y', y'' \in \operatorname{Fix} T$. Let $\{y^{t_k}\}_{k\in\mathbb{N}}$ and $\{y^{s_k}\}_{k\in\mathbb{N}}$ be subsequences of $\{y^k\}_{k\in\mathbb{N}}$ such that $y^{s_k} \rightharpoonup y'$ and $y^{t_k} \rightharpoonup y''$. We have that

$$\left|\left\langle f'(y^{t_k}) - f'(y^{s_k}), y' - y''\right\rangle\right| =$$
$$\left|(D_f(y', y^{t_k}) - D_f(y', y^{s_k})) - (D_f(y'', y^{t_k}) - D_f(y'', y^{s_k}))\right| \leq$$
$$\left|D_f(y', y^{t_k}) - D_f(y', y^{s_k})\right| + \left|D_f(y'', y^{t_k}) - D_f(y'', y^{s_k})\right|,$$

for all $k \in \mathbb{N}$. Since the sequences $\{D_f(y', y^k)\}_{k\in\mathbb{N}}$ and $\{D_f(y'', y^k)\}_{k\in\mathbb{N}}$ converge (because $y', y'' \in \operatorname{Fix} T$), by letting $k \to \infty$ in this inequality we deduce that

$$\liminf_{k\to\infty} \left|\left\langle f'(y^{t_k}) - f'(y^{s_k}), y' - y''\right\rangle\right| = 0$$

and this is a contradiction. Hence, the sequence $\{y^k\}_{k\in\mathbb{N}}$ can not have two different weak accumulation points. So, the sequence $\{y^k\}_{k\in\mathbb{N}}$ converges weakly and its limit is a fixed point of T. □

Theorem 1 applies nicely when the space X has finite dimension and T is continuous. In this case the functions f and f' are continuous on $\operatorname{int}\operatorname{dom} f$ and, therefore, the function $x \to D_f(Tx, x)$ is continuous too (and, hence, lower semicontinuous). Also, continuity of f' implies that (2.9) is also satisfied. Thus, we obtain the following:

Corollary 1. *Suppose that X is a Banach space of finite dimension and $T : K \to K$ is a Bregman-monotone continuous operator such that* $\operatorname{Fix} T \neq \varnothing$. *If the sequence $\{y^k\}_{k\in\mathbb{N}} \subseteq K$ is an inexact orbit of T satisfying* (2.5) - (2.8), *then $\{y^k\}_{k\in\mathbb{N}}$ converges to a fixed point of T.*

Corollary 1 can be used as a tool for proving convergence of some algorithms for solving problems of practical interest. Here are examples suggesting how Corollary 1 can be used in order to recover and improve upon some known convergence criteria of procedures for solving optimization and feasibility problems.

Example 1. (*An inexact iterated resolvent method for finding zeros of maximal monotone operators*) Let X be a finite dimensional Banach space and assume that $\operatorname{dom} f = X$. Suppose that $A : X \to 2^{X^*}$ is a maximal monotone operator such that $A + f'$ is surjective (as happens when X is smooth, strictly convex and $f = \frac{1}{2}\|\cdot\|^2$). The (generalized) resolvent of A, as defined in [**4**], is the operator $R_A : X \to X$ given by

$$R_A = (A + f')^{-1} \circ f'$$

which is single valued on $\operatorname{dom} R_A = X$ and $\operatorname{Fix} R_A = A^{-1}0$ (cf. [**4**, Proposition 3.8]). In this case R_A is Bregman-monotone with respect to f (cf. [**4**, Corollary 3.14]) and continuous because $(A + f')^{-1}$ is maximal monotone. According to Corollary 1, the inexact orbits of R_A subjected to the summability conditions given above approximate fixed points of R_A, that is, solutions of the equation $0 \in Ax$, provided that such solutions exist.

Example 2. (*An inexact Cimmino method for solving feasibility problems*). Let X be finite dimensional and, for simplicity, suppose that $\operatorname{dom} f = X$ and that $D_f(\cdot, \cdot)$ is convex. Let $C_1, ..., C_m$ be a family of closed and convex subsets of X whose intersection is nonempty. The convex feasibility problem is that of finding a point in the intersection of the sets C_i. For each $i \in \{1, ...m\}$ let $P_i^f : X \to X$ be the Bregman projection on C_i defined by

$$P_i^f x = \arg\min \{D_f(y, x) : y \in C_i\}.$$

It is known that Bregman projection operators are continuous and Bregman-monotone (cf. [**12**]). Let $w_1, ..., w_m$ be positive real numbers with $\sum_{i=1}^m w_i = 1$. It follows easily from (2.2) that if the function $D_f(\cdot, \cdot)$ is jointly convex in both variables (as happens in some situations as those discussed in [**3**]), then the operator $T = \sum_{i=1}^m w_i P_i^f$ is also Bregman-monotone and continuous. Also, it can be shown that the fixed points of T are exactly the common points of the sets C_i (see [**9**]). It is well known (see [**13**]) that orbits of T converge to common points of the sets C_i whenever such common points exist. However, computation of the vectors $P_i^f x$ is rarely possible with absolute precision. Applying Corollary 1 one deduces that inexact orbits of T subjected to the summability conditions given above still converge to fixed points of T and, hence, to common points of the sets C_i whenever such points exist.

Theorem 1 also applies when X is infinite dimensional and f has weak to weak* continuous derivative f'. That is, for instance, the case when X is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$ as well as when $X = \ell_p$ with $p \in (1, +\infty)$ provided with $f = \frac{1}{p}\|\cdot\|^p$. Weak to weak* continuity of f implies that (2.9) holds. In such circumstances we have the following:

Corollary 2. *If f' is weak to weak* continuous, T is continuous with* $\operatorname{Fix} T \neq \varnothing$ *and $D_f(T(\cdot), \cdot)$ is convex, then any inexact orbit $\{y^k\}_{k\in\mathbb{N}}$ of T satisfying* (2.5) - (2.8) *converges weakly to a fixed point of T.*

Proof. As noted above (2.9) holds. The function f' being weak to weak* continuous is bounded on any closed ball contained in $\operatorname{int}\operatorname{dom} f$. Therefore, f is continuous on $\operatorname{int}\operatorname{dom} f$. Thus, by (2.1), the function $D_f(T(\cdot), \cdot)$ is convex and continuous and, hence, weakly lower semicontinuous. Therefore, application of Theorem 1 leads to the conclusion. □

Corollary 2 can be easily applied to extend the conclusion of Example 2 above to the infinite dimensional context of a Hilbert space provided with the function $f = \frac{1}{2}\|\cdot\|^2$. In the next section we will present this extension from a different perspective.

3. Weak convergence of inexact orbits for nonexpansive Bregman-monotone operators in Hilbert spaces

We have shown in Section 2 that, under certain conditions, inexact orbits of a Bregman-monotone operator $T : K \to K$ converge weakly to fixed points of T. In this section we assume that X is a Hilbert space and that $f = \frac{1}{2}\|\cdot\|^2$. Clearly, since in this case f' is the identity of the space X, condition (2.9) is satisfied no matter how the weakly closed set K is given in X. Also, condition (2.5) can be re-written here in the form (2.4) and it holds whenever (2.6) is satisfied. Therefore, in this current setting, Theorem 1 has a somewhat simplified form. However, even in this context, the remaining summability conditions (2.6), (2.7) and (2.8) as well as the requirement that the function $D_f(T(\cdot),\cdot)$ should be sequentially weakly lower semicontinuous (needed for ensuring that weak accumulation points of the inexact orbits are fixed points of the operator) are restrictive. We are going to show that inexact orbits of Bregman-monotone operators which are also nonexpansive converge to fixed points under much less demanding conditions. Note that in our particular context $D_f(y,x) = \frac{1}{2}\|y-x\|^2$ and that the Bregman-monotonicity requirement amounts to

$$z \in \operatorname{Fix} T \Rightarrow \forall x \in K : \|z - Tx\|^2 + \|Tx - x\|^2 \leq \|z - x\|^2 . \tag{3.1}$$

The following result describes the convergence behavior of inexact orbits with summable errors of nonexpansive Bregman-monotone operators. Observe that in our current setting firmly nonexpansive operators are necessarily nonexpansive Bregman-monotone operators and, therefore, the following result covers the firmly nonexpansive operators too.

Theorem 2. *If $T : K \to K$ is a nonexpansive Bregman-monotone operator and $\operatorname{Fix} T \neq \varnothing$, then each inexact orbit $\{y^k\}_{k\in\mathbb{N}}$ of T which satisfies*

$$\sum_{k=0}^{\infty} \left\|Ty^k - y^{k+1}\right\| < \infty \tag{3.2}$$

converges weakly to a fixed point of the operator T.

Proof. Let $z \in \operatorname{Fix} T$. Then we have

$$\begin{aligned} \left\|z - y^{k+1}\right\| &\leq \left\|z - Ty^k\right\| + \left\|y^{k+1} - Ty^k\right\| \\ &= \left\|Tz - Ty^k\right\| + \left\|y^{k+1} - Ty^k\right\| \\ &\leq \left\|z - y^k\right\| + \left\|y^{k+1} - Ty^k\right\|, \end{aligned} \tag{3.3}$$

for all $k \in \mathbb{N}$. By (3.2) and (3.3) it results that the sequence $\{\|z - y^k\|\}_{k\in\mathbb{N}}$ converges and, hence, that $\{y^k\}_{k\in\mathbb{N}}$ is bounded. Also by (3.2) we have that

$$\lim_{k\to\infty} \left\|y^{k+1} - Ty^k\right\| = 0. \tag{3.4}$$

Letting $k \to \infty$ in (3.3), we deduce that $\{\|z - Ty^k\|\}_{k\in\mathbb{N}}$ converges too and has the same limit as $\{\|z - y^k\|\}_{k\in\mathbb{N}}$. From (3.1) we deduce that

$$\left\|y^k - Ty^k\right\|^2 \leq \left\|z - y^k\right\|^2 - \left\|z - Ty^k\right\|^2, \ \forall k \in \mathbb{N}$$

Letting here $k \to \infty$ we get

$$\lim_{k\to\infty} \left\|y^k - Ty^k\right\| = 0. \tag{3.5}$$

Since

$$\left\|y^{k+1} - y^k\right\| \leq \left\|y^k - Ty^k\right\| + \left\|y^{k+1} - Ty^k\right\|$$

and the right hand side converges to zero (by (3.4) and (3.5)), it follows that

$$\lim_{k\to\infty} \left\|y^k - y^{k+1}\right\| = 0. \tag{3.6}$$

To any bounded sequence $\{u^k\}_{k\in\mathbb{N}}$ in X, we associate the convex function $F(\{u^k\}_{k\in\mathbb{N}};\cdot) : X \to \mathbb{R}_+$ defined by

$$F(\{u^k\}_{k\in\mathbb{N}}; x) = \limsup_{k\to\infty} \left\|u^k - x\right\| . \tag{3.7}$$

Recall that, according to [**16**], if $\{u^k\}_{k\in\mathbb{N}}$ is bounded, then the function $F(\{u^k\}_{k\in\mathbb{N}};\cdot)$ has a unique (global) minimizer which is the asymptotic center of $\{u^k\}_{k\in\mathbb{N}}$ (with respect to X). It coincides with the weak limit of $\{u^k\}_{k\in\mathbb{N}}$ whenever this limit exists.

Now, suppose that $\{y^{i_k}\}_{k\in\mathbb{N}}$ is a weakly convergent subsequence of the bounded sequence $\{y^k\}_{k\in\mathbb{N}}$ and $w-\lim_{k\to\infty} y^{i_k}=\bar{y}$. Then, $\bar{y}$ is the asymptotic center of $\{y^{i_k}\}_{k\in\mathbb{N}}$. By (3.6), $w-\lim_{k\to\infty} y^{i_k+1}=\bar{y}$ and, thus, $\bar{y}$ is the asymptotic center of $\{y^{i_k+1}\}_{k\in\mathbb{N}}$ too. We claim that $T\bar{y}$ is also an asymptotic center of $\{y^{i_k+1}\}_{k\in\mathbb{N}}$. If that is true, then $\bar{y}=T\bar{y}$ because the asymptotic center of $\{y^{i_k+1}\}_{k\in\mathbb{N}}$ is unique. In other words, if our claim is true, then any weak accumulation point of $\{y^k\}_{k\in\mathbb{N}}$ is a fixed point of T.

In order to prove our claim we first show that

$$F\left(\{y^{i_k+1}\}_{k\in\mathbb{N}};\cdot\right)=F\left(\{Ty^{i_k}\}_{k\in\mathbb{N}};\cdot\right). \tag{3.8}$$

To see that, note that

$$\left\|y^{i_k+1}-x\right\|\le\left\|y^{i_k+1}-Ty^{i_k}\right\|+\left\|Ty^{i_k}-x\right\|,\ \forall x\in X,\ \forall k\in\mathbb{N},$$

and that, by taking the upper limit as $k\to\infty$ on both sides of this inequality we get

$$F(\{y^{i_k+1}\}_{k\in\mathbb{N}};x)\le F(\{Ty^{i_k}\}_{k\in\mathbb{N}};x),\ \forall x\in X,$$

because of (3.4). By taking the upper limit as $k\to\infty$ on both sides of the inequality

$$\left\|Ty^{i_k}-x\right\|\le\left\|y^{i_k+1}-Ty^{i_k}\right\|+\left\|y^{i_k+1}-x\right\|,\ \forall x\in X,\ \forall k\in\mathbb{N},$$

one deduces (3.8). Now, note that, according to (3.8), we have

$$\begin{aligned}
F(\{y^{i_k+1}\}_{k\in\mathbb{N}};T\bar{y}) &= F(\{Ty^{i_k}\}_{k\in\mathbb{N}};T\bar{y})\\
&= \limsup_{k\to\infty}\left\|Ty^{i_k}-T\bar{y}\right\|\\
&\le \limsup_{k\to\infty}\left\|y^{i_k}-\bar{y}\right\|\\
&\le \limsup_{k\to\infty}\left[\left\|y^{i_k}-y^{i_k+1}\right\|+\left\|y^{i_k+1}-\bar{y}\right\|\right]\\
&= F(\{y^{i_k+1}\}_{k\in\mathbb{N}};\bar{y}),
\end{aligned}$$

where the first inequality follows from the nonexpansivity of T and the last equality follows from (3.6). Since $\bar{y}$ is a global minimizer of $F(\{y^{i_k+1}\}_{k\in\mathbb{N}};\cdot)$ it follows that $T\bar{y}$ is an asymptotic center of $\{y^{i_k+1}\}_{k\in\mathbb{N}}$.

The above considerations show that any weak accumulation point of $\{y^k\}_{k\in\mathbb{N}}$ is a fixed point of T. We will prove next that $\{y^k\}_{k\in\mathbb{N}}$ has an unique weak accumulation point. Suppose, by contradiction, that y' and y'' are two different weak accumulation points of $\{y^k\}_{k\in\mathbb{N}}$. As shown above, y' and y'' are contained in $\operatorname{Fix}T$. Therefore, (2.2) implies that the sequences $\{\|y'-y^k\|\}_{k\in\mathbb{N}}$ and $\{\|y''-y^k\|\}_{k\in\mathbb{N}}$ are convergent to some numbers a and b, respectively. We also have

$$\left\|y'-y^k\right\|^2=\left\|y''-y^k\right\|^2+\|y'-y''\|^2+2\left\langle y''-y^k,y'-y''\right\rangle\ \forall k\in\mathbb{N}.$$

If $\{y^{j_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{y^k\}_{k\in\mathbb{N}}$ such that $y' = w - \lim_{k\to\infty} y^{j_k}$ then, by writing the last equality with j_k instead of k and letting $k \to \infty$ we obtain

$$a^2 - b^2 = -\|y' - y''\|^2 .$$

A similar reasoning with y' and y'' interchanged implies that

$$b^2 - a^2 = -\|y' - y''\|^2 .$$

Summing up the last two equalities gives $\|y' - y''\| = 0$ and this is a contradiction. Hence, the sequence $\{y^k\}_{k\in\mathbb{N}}$ converges weakly and its weak limit belongs to $\operatorname{Fix} T$. □

Observe that if T is Bregman-monotone and nonexpansive, then so is $\lambda Id + (1-\lambda)T$ for each $\lambda \in (0,1)$. Clearly, $\operatorname{Fix} T = \operatorname{Fix} \lambda Id + (1-\lambda)T$. Therefore, applying Theorem 2 to $\lambda Id + (1-\lambda)T$ one deduces the following result which describes the convergence behavior of inexact orbits with summable errors of a class of procedures for computing fixed points of T by using "relaxed iterations".

Corollary 3. *If $T : K \to K$ is a nonexpansive and Bregman-monotone operator with* $\operatorname{Fix} T \neq \varnothing$ *and if $\lambda \in (0,1)$, then all inexact orbits $\{y^k\}_{k\in\mathbb{N}}$ of $\lambda Id + (1-\lambda)T$ satisfying* (3.2)*, converge weakly to fixed points of T.*

Another consequence of Theorem 2 is the following result concerning the behavior of inexact orbits of Cimmino type operators in Hilbert spaces. Since Cimmino's [**14**] use of (finite) averages of metric projections for the resolution of linear systems of equations, convergence of orbits for Cimmino type operators was continuously studied in many variants and from various points of view (see also [**18**], [**21**], [**7**], [**13**], [**9**], [**8**]and the references therein). For precising the terminology, let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\{C_\omega\}_{\omega\in\Omega}$ be a family of nonempty closed convex subsets of X such that the point-to-set function $\omega \to C_\omega$ is measurable. We say that the family $\{C_\omega\}_{\omega\in\Omega}$ is *square-integrable* (with respect to the probability space $(\Omega, \mathcal{A}, \mu)$) if it has a square-integrable selector, that is, if there exists a measurable function $\xi : \Omega \to X$ such that $\|\xi(\cdot)\|^2$ is integrable and $\xi(\omega) \in C_\omega$ for $\mu-$almost all $\omega \in \Omega$. In this case, for each $x \in X$, the function $\omega \to \|P_\omega x\|^2$ is integrable when P_ω denotes the metric projection onto the set C_ω (see [**8**, Chapter 2]). Consequently, the operator $P[\mu] : X \to X$ given by

$$P[\mu]x = \int_\Omega (P_\omega x) d\mu(\omega),$$

as well as the the function $g : X \to [0,\infty]$ given by

$$g(x) = \tfrac{1}{2} \textstyle\int_\Omega \|P_\omega x - x\|^2 \, d\mu(\omega), \tag{3.9}$$

are well defined and g is finite. Using the facts that the functions $x \to \|P_\omega x - x\|^2$ are convex and that each P_ω is a nonexpansive and Bregman-monotone operator relative to $f = \frac{1}{2}\|\cdot\|^2$, one can easily deduce that $P[\mu]$ is a nonexpansive Bregman-monotone operator (with respect to the function $f = \frac{1}{2}\|\cdot\|^2$).

Theorem 3. *Suppose that $\{C_\omega\}_{\omega\in\Omega}$ is a square-integrable family of nonempty, closed, convex subsets of X. If $\{y^k\}_{k\in\mathbb{N}}$ is an inexact orbit of the Cimmino type operator $P[\mu]$ associated with the family of sets $\{C_\omega\}_{\omega\in\Omega}$ such that*

$$\sum_{k=0}^{\infty} \left\|P[\mu]y^k - y^{k+1}\right\| < \infty, \tag{3.10}$$

then

(A) *The following statements are equivalent:*

(i) *The set* $\operatorname{Arg\,min} g$ *of (global) minimizers of the function* g *is nonempty;*

(ii) *The set* $\operatorname{Fix} P[\mu]$ *is nonempty;*

(iii) *The sequence* $\{y^k\}_{k\in\mathbb{N}}$ *converges weakly to a point in* $\operatorname{Arg\,min} g$*;*

(iv) *The sequence* $\{y^k\}_{k\in\mathbb{N}}$ *is bounded.*

(B) *If the set*

$$C := \{x \in X : x \in C_\omega,\ \mu\text{-a.e.}\}$$

of μ*-almost common points of the sets* C_ω *is nonempty, then the statements above are also equivalent to the following one:*

(v) *The sequence* $\{y^k\}_{k\in\mathbb{N}}$ *converges weakly to a point in* C*.*

(C) *If* $\{y^k\}_{k\in\mathbb{N}}$ *converges weakly, then*

(vi) *The sequence* $\{g(y^k)\}_{k\in\mathbb{N}}$ *converges to* $\min_{x\in X} g(x)$*;*

(vii) *If* $\min_{x\in X} g(x) = 0$*, then the sets* C_ω *have a* μ*-almost common point.*

Proof. Let P_ω be the metric projection onto the set C_ω. By [**1**, Proposition 1, p. 24], for each $\omega \in \Omega$, the function $g_\omega : x \to \frac{1}{2}\|x - P_\omega x\|^2$ is convex, continuously differentiable on X, and its gradient is exactly

$$g'_\omega(x) = x - P_\omega x.$$

Since the function g defined by (3.9) is exactly

$$g(x) = \tfrac{1}{2}\int_\Omega g_\omega(x) d\mu(\omega),$$

application of Lebesgue's bounded convergence theorem shows that g is convex, continuously differentiable on X and

$$g'(x) = x - P[\mu]x. \tag{3.11}$$

This implies that $\operatorname{Arg\,min} g = \operatorname{Fix} P[\mu]$. So, $\boxed{(i)\Leftrightarrow(ii)}$. The implication $\boxed{(ii)\Rightarrow(iii)}$ results from Theorem 2 above and the implication $\boxed{(iii)\Rightarrow(iv)}$ is obvious.

We prove next that $\boxed{(iv)\Rightarrow(ii)}$. To this end, we associate to any bounded sequence $\{u^k\}_{k\in\mathbb{N}}$ in X, the convex function $F(\{u^k\}_{k\in\mathbb{N}};\cdot)$ defined by (3.7). Suppose that the sequence $\{y^k\}_{k\in\mathbb{N}}$ is bounded. Then the sequence $\{P[\mu]y^k\}_{k\in\mathbb{N}}$ is bounded too because $P[\mu]$ is nonexpansive. From (3.10), we deduce that

$$\lim_{k\to\infty} \left\|P[\mu]y^k - y^{k+1}\right\| = 0.$$

Therefore, for any $x \in X$, we have

$$\begin{aligned} F(\{y^{k+1}\}_{k\in\mathbb{N}};x) &\le \limsup_{k\to\infty}\left[\left\|y^{k+1} - P[\mu]y^k\right\| + \left\|P[\mu]y^k - x\right\|\right] \\ &= \limsup_{k\to\infty}\left\|P[\mu]y^k - x\right\| = F(\{P[\mu]y^k\}_{k\in\mathbb{N}};x). \end{aligned}$$

We also have

$$\left\|P[\mu]y^k - x\right\| \le \left\|y^{k+1} - P[\mu]y^k\right\| + \left\|y^{k+1} - x\right\|,$$

so that

$$F(\{P[\mu]y^k\}_{k\in\mathbb{N}};x) \le F(\{y^{k+1}\}_{k\in\mathbb{N}};x).$$

Hence,

$$F(\{y^{k+1}\}_{k\in\mathbb{N}};x) = F(\{P[\mu]y^k\}_{k\in\mathbb{N}};x),\ \forall x \in X. \tag{3.12}$$

By the definition of $F(\{y^k\}_{k\in\mathbb{N}};\cdot)$, we have

$$F(\{y^{k+1}\}_{k\in\mathbb{N}};\cdot) = F(\{y^k\}_{k\in\mathbb{N}};\cdot).$$

This and (3.12) show that the sequences $\{y^k\}_{k\in\mathbb{N}}$ and $\{P[\mu]y^k\}_{k\in\mathbb{N}}$ have the same asymptotic center z. Hence, from (3.12) and since $P[\mu]$ nonexpansive , we deduce

$$\begin{aligned} F(\{y^k\}_{k\in\mathbb{N}};P[\mu]z) &= F(\{P[\mu]y^k\}_{k\in\mathbb{N}};P[\mu]z) \\ &= \limsup_{k\to\infty}\left\|P[\mu]y^k - P[\mu]z\right\| \\ &\leq \limsup_{k\to\infty}\left\|y^k - z\right\| = F(\{y^k\}_{k\in\mathbb{N}};z). \end{aligned}$$

Since, as noted above, z is the unique minimizer of $F(\{y^k\}_{k\in\mathbb{N}};\cdot)$, the last inequality implies that $P[\mu]z = z$, i.e., $\operatorname{Fix} P[\mu] \neq \varnothing$. This completes the proof of (A).

According to [**9**, Theorem 5.7(C)], if $C \neq \varnothing$, then $\operatorname{Arg\,min} g = C$. Thus, if $C \neq \varnothing$, then $\boxed{(v)\Leftrightarrow(i)}$ and this proves (B). In order to prove (C) note that, as shown at (A), whenever the sequence $\{y^k\}_{k\in\mathbb{N}}$ converges weakly, we have that $z := w - \lim_{k\to\infty} y^k \in \operatorname{Arg\,min} g$. According to Lemma 5.6 in [**9**] applied to the function $\frac{1}{2}\left\|\cdot\right\|^2$, if $u \in \operatorname{Fix}(P[\mu])$, then

$$\left\|u - P[\mu]x\right\|^2 + g(x) \leq \left\|u - x\right\|^2 + g(u),\ \forall x \in X.$$

Therefore, since $z \in \operatorname{Arg\,min} g = \operatorname{Fix}(P[\mu])$, we have that

$$\begin{aligned} 0 &\leq g(y^k) - g(z) \leq \left\|z - y^k\right\|^2 - \left\|z - P[\mu]y^k\right\|^2 \\ &= \left\|z - y^k\right\|^2 - \left\|(z - y^{k+1}) + (y^{k+1} - P[\mu]y^k)\right\|^2 \\ &= \left(\left\|z - y^k\right\|^2 - \left\|z - y^{k+1}\right\|^2\right) + \left\|y^{k+1} - P[\mu]y^k\right\|^2 \\ &\quad +2\left\langle z - y^{k+1}, y^{k+1} - P[\mu]y^k\right\rangle, \\ &\leq \left(\left\|z - y^k\right\|^2 - \left\|z - y^{k+1}\right\|^2\right) \\ &\quad + \left\|y^{k+1} - P[\mu]y^k\right\|\left(2\left\|z - y^{k+1}\right\| + \left\|y^{k+1} - P[\mu]y^k\right\|\right). \end{aligned} \tag{3.13}$$

Since (3.3) still holds with $P[\mu]$ instead of T, it results that the sequence $\{\left\|z - y^k\right\|\}_{k\in\mathbb{N}}$ converges. This implies that the right-hand side of (3.13) converges to zero as $k \to \infty$. Hence,

$$\lim_{k\to\infty} g(y^k) = g(z) = \min_{x\in X} g(x).$$

If $\min_{x\in X} g(x) = 0$, then $g(z) = 0$ and, by the definition of g, this implies that $\left\|P_\omega z - z\right\| = 0$ μ-a.e., that is, $z = P_\omega z \in C_\omega$, μ-a.e. $\square$

Among the potential applications of Theorem 3 we note that it can be an useful tool for analyzing and solving Fredholm equations of the form

$$\langle K(\omega), x\rangle = b(\omega),\ \mu\text{-a.e.,} \tag{3.14}$$

where $K : \Omega \to X$ and $b : \Omega \to \mathbb{R}$ are given square-integrable functions. Such equations are of interest in image processing (inverting the Radon transform) as well as in mathematical physics (heat equations can be often equivalently re-written in the form (3.14)). Taking

$$C_\omega := \{u \in X : \langle K(\omega), u\rangle = b(\omega)\},$$

we see that the sets C_ω are closed hyperplanes in X and that the collection of μ-almost common points of those sets is exactly the set of solutions of (3.14). Note that, in this particular case, if $K(\omega) \neq 0$, then

$$P_\omega x = x + \frac{b(\omega) - \langle K(\omega), x\rangle}{\|K(\omega)\|^2} K(\omega).$$

Clearly, if the sets C_ω have a square-integrable selector, then this formula holds for μ-almost all $\omega \in \Omega$. Therefore, in this case we have that

$$P[\mu]x = x + \int_\Omega \frac{b(\omega) - \langle K(\omega), x\rangle}{\|K(\omega)\|^2} K(\omega) d\mu(\omega). \tag{3.15}$$

and

$$g(x) = \tfrac{1}{2} \int_\Omega |b(\omega) - \langle K(\omega), x\rangle|^2 \|K(\omega)\|^{-2} d\mu(\omega). \tag{3.16}$$

Thus, by Theorem 3 we immediately obtain the following result giving a necessary and sufficient condition for the existence of solutions of (3.14):

Corollary 4. *Suppose that the functions $K : \Omega \to X$ and $b : \Omega \to \mathbb{R}$ are μ-square-integrable and that there exists a μ-square-integrable function $\xi : \Omega \to X$ such that*

$$\langle K(\omega), \xi(\omega)\rangle = b(\omega), \ \mu\text{-a.e.}$$

Then the following statements are true:

(*i*) *The equation* (3.14) *has solution if and only if there exists a bounded inexact orbit $\{y^k\}_{k\in\mathbb{N}}$ of the operator $P[\mu]$ defined at* (3.15) *such that* (3.2) *holds and*

$$\liminf_{k\to\infty} \int_\Omega \left|b(\omega) - \left\langle K(\omega), y^k\right\rangle\right|^2 \|K(\omega)\|^{-2} d\mu(\omega) = 0.$$

(*ii*) *If the equation* (3.14) *has solution, then any inexact orbit $\{y^k\}_{k\in\mathbb{N}}$ of $P[\mu]$ which satisfies* (3.2) *converges weakly to a solution of* (3.14).

4. Inexact orbits of nonexpansive operators

Theorem 2 shows that if X is a Hilbert space and if $T : K \to K$ is a nonexpansive and Bregman-monotone (with respect to $f = \frac{1}{2}\|\cdot\|^2$) operator which has fixed points, then *inexact orbits with summable errors* (i.e., inexact orbits satisfying (3.2)) of T converge weakly to fixed points of T. It is of interest to find out whether this property continues to hold outside Hilbert space. We do not know the answer to this question. However, we do have some results which may help us reach such an answer. We are going to show that, in any Banach space X, whenever all exact orbits of a nonexpansive operator $T : K \to K$ converge weakly (respectively strongly) to fixed points of T, the same is true for all inexact orbits with summable errors. That fact is significant because it reduces the convergence analysis of inexact orbits with summable errors to, the presumably easier, convergence analysis of the exact orbits.

The next result shows that, for nonexpansive operators in any Banach space, the weak convergence behavior of exact orbits and of inexact orbits with summable errors are equivalent.

Theorem 4. *Let $T : K \to K$ be a nonexpansive operator where K is a weakly closed subset of the Banach space X. Then the following two statements are equivalent:*

(i) All exact orbits of T converge weakly;
(ii) All inexact orbits with summable errors of T converge weakly.
Also, the following two statements are equivalent:
(iii) All exact orbits of T converge weakly to fixed points of T;
(iv) All inexact orbits with summable errors of T converge weakly to fixed points of T.

Proof. The implications $\boxed{(ii)\Rightarrow(i)}$ and $\boxed{(iv)\Rightarrow(iii)}$ are obvious. We first prove that $\boxed{(i)\Rightarrow(ii)}$. To this end, let $\{x^k\}_{k\in\mathbb{N}} \subset K$ be an inexact orbit with summable errors, that is, a sequence such that

$$\sum_{k=0}^{\infty} \left\| x^{n+1} - Tx^n \right\| < \infty. \tag{4.1}$$

Let $\{r_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers such that

$$\sum_{k=0}^{\infty} r_k < \infty \text{ and } \left\| x^{n+1} - Tx^n \right\| \leq r_n,\ \forall n \in \mathbb{N}. \tag{4.2}$$

According to (i), for any nonnegative integer k the sequence $\{T^n x^k\}_{n\in\mathbb{N}}$ converges weakly to some $y^k \in K$. By induction we will show that for each integer $i \geq 0$,

$$\left\| T^i x^k - x^{k+i} \right\| \leq \sum_{j=k-1}^{i+k-1} r_j - r_{k-1},\ \forall k \geq 1. \tag{4.3}$$

It is clear that the inequality (4.3) is true for $i = 0$. Assume that (4.3) holds for an integer $i \geq 0$. Since T is nonexpansive, it follows from (4.2) that, for each integer $k \geq 1$,

$$\begin{aligned} \left\| x^{k+i+1} - T^{i+1}x^k \right\| &\leq \left\| x^{k+i+1} - Tx^{k+i} \right\| + \left\| Tx^{k+i} - T(T^i x^k) \right\| \\ &\leq r_{k+i} + \left\| x^{k+i} - T^i x^k \right\| \\ &\leq r_{k+i} + \sum_{j=k-1}^{i+k-1} r_j - r_{k-1} = \sum_{j=k-1}^{i+k} r_j - r_{k-1}. \end{aligned}$$

Therefore (4.3) holds for all integers $i \geq 0$. Fix an integer $q \geq 1$. By (4.3) we have

$$\left\| T^q x^k - x^{k+q} \right\| \leq \sum_{j=k}^{\infty} r_j. \tag{4.4}$$

Hence, for each integer $i \geq 0$,

$$\left\| T^{q+i} x^k - T^i x^{k+q} \right\| \leq \left\| T^q x^k - x^{k+q} \right\| \leq \sum_{j=k}^{\infty} r_j.$$

Therefore

$$\left\| y^k - y^{q+k} \right\| \leq \sum_{j=k}^{\infty} r_j,\ \forall k \geq 1. \tag{4.5}$$

Since the above inequality holds for each pair of positive integers (q, k) and since $\sum_{j=0}^{\infty} r_j < \infty$, we conclude that $\{y^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence and, hence, it converges strongly to some $y^* \in K$. Letting $q \to \infty$ in (4.5) we get

$$\left\|y^k - y^*\right\| \le \sum_{j=k}^{\infty} r_j, \ \forall k \ge 1. \tag{4.6}$$

In order to complete the proof, it is sufficient to show that $\left\{x^k\right\}_{k\in\mathbb{N}}$ converges weakly to y^*. To this end, let ψ be a continuous linear functional on X such that $\|\psi\|_* \le 1$. Let ϵ be a positive real number. We show that $|\psi(y^* - x^i)| \le \epsilon$ for all large enough integers i. According to (4.2), there is an integer $k \ge 1$ such that

$$\sum_{j=k}^{\infty} r_j < \epsilon/4. \tag{4.7}$$

By (4.6) and (4.4), for each integer $i \ge 1$, we have

$$\begin{aligned} |\psi(y^* - x^{k+1})| &\le |\psi(y^* - y^k)| + |\psi(y^k - T^i x^k)| + |\psi(T^i x^k - x^{k+1})| \\ &\le \left\|y^* - y^k\right\| + |\psi(y^k - T^i x^k)| + \left\|T^i x^k - x^{k+1}\right\| \\ &\le \sum_{j=k}^{\infty} r_j + |\psi(y^k - T^i x^k)| + \sum_{j=k}^{\infty} r_j. \end{aligned} \tag{4.8}$$

Since y^k is the weak limit of $\left\{T^i x^k\right\}_{i\in\mathbb{N}}$, there exists a positive integer i_0 such that, for any integer $i \ge i_0$, we have

$$|\psi(y^k - T^i x^k)| \le \epsilon/4. \tag{4.9}$$

By (4.8) and (4.9), for each positive integer k for which (4.7) holds,

$$|\psi(y^* - x^{k+1})| \le \tfrac{3}{4}\varepsilon.$$

Hence, $\left\{x^k\right\}_{k\in\mathbb{N}}$ converges weakly to y^*. This completes the proof of $\boxed{(i)\Rightarrow(ii)}$.

Now we are going to prove $\boxed{(iii)\Rightarrow(iv)}$. To this end, let $\left\{x^k\right\}_{k\in\mathbb{N}}$ be a sequence in K satisfying (4.1), and let y^k and r_k be as above. Since (iii) holds, it results that (i) is also true and, therefore, the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ converges weakly to $y^* \in K$, where y^* is the limit (in norm) of the sequence $\left\{y^k\right\}_{k\in\mathbb{N}}$. We claim that y^* is a fixed point of $\left\{x^k\right\}_{k\in\mathbb{N}}$. Note that, according to (iii), for each $k \in \mathbb{N}$, $Ty^k = y^k$ because y^k is the weak limit of the exact orbit $\left\{T^n x^k\right\}_{n\in\mathbb{N}}$. Since T is continuous, we deduce that

$$y^* = \lim_{k\to\infty} y^k = \lim_{k\to\infty} Ty^k = Ty^*$$

and this completes the proof of Theorem 4. □

Theorem 4 deals with the weak convergence of orbits and inexact orbits with summable errors for nonexpansive operators in Banach spaces. Is strong convergence of all orbits of such an operator equivalent to strong convergence to fixed points of all its inexact orbits with summable errors? The next result shows that this is indeed the case and not only in Banach spaces. It improves upon a result in [**24**] which deals only with (strict) contractions.

Theorem 5. *Let (X, ρ) be a complete metric space, K a closed subset of X and $T : K \to K$ a nonexpansive operator with $Fix\, T \neq \emptyset$. Then the following two statements are equivalent:*

(i) All orbits of T converge in (X, ρ);

(ii) All inexact orbits with summable errors of T, that is, all sequences $\{x^k\}_{k\in\mathbb{N}} \subset K$ such that $\sum_{k=0}^{\infty} \rho(x^{k+1}, Tx^k) < \infty$, converge in (X, ρ) to fixed points of T.

Proof. We only have to prove the implication $\boxed{(ii) \Rightarrow (i)}$. Assume that for each $x \in X$ the sequence $\{T^k x\}_{k\in\mathbb{N}}$ converges in (X, ρ). Then the limit y of $\{T^k x\}_{k\in\mathbb{N}}$ is contained in $\operatorname{Fix} T$ because, for any positive integer k, we have

$$\begin{aligned} \rho(y, Ty) &\leq \rho(y, T^k x) + \rho(T^k x, Ty) \\ &\leq \rho(y, T^k x) + \rho(T^{k-1} x, y) \end{aligned}$$

and letting here $k \to \infty$ we get $y = Ty$. Let $\{x^k\}_{k\in\mathbb{N}}$ be an inexact orbit of T with summable errors and let $\{r_k\}_{k\in\mathbb{N}}$ be a summable sequence of real numbers such that

$$\rho(x^{k+1}, Tx^k) \leq r_k, \ \forall k \in \mathbb{N}. \tag{4.10}$$

Fix an arbitrary positive integer k and consider the sequence $\{T^n x^k\}_{n\in\mathbb{N}}$. By (i), this sequence converges to some y^k and, as noted above, $y^k \in \operatorname{Fix} T$. Reasoning by induction in a way similar to that which led to (4.3), we deduce that, for each $i \in \mathbb{N}$, we have

$$\rho(T^i x^k, x^{k+i}) \leq \sum_{j=k-1}^{i+k-1} r_j - r_{k-1}. \tag{4.11}$$

This implies that, for each integer $i \geq 1$, we also have

$$\begin{aligned} \rho(x^{k+i}, y^k) &\leq \rho(x^{k+i}, T^i x^k) + \rho(T^i x^k, y^k) \\ &\leq \sum_{j=k}^{\infty} r_j + \rho(T^i x^k, y^k). \end{aligned} \tag{4.12}$$

Since $\{T^i x^k\}_{i\in\mathbb{N}}$ converges to y^k in (X, ρ), there exists an integer $i_0 \geq 1$ such that for each integer $i \geq i_0$,

$$\rho(T^i x^k, y^k) \leq \tfrac{1}{4} \textstyle\sum_{j=k}^{\infty} r_j. \tag{4.13}$$

By (4.13) and (4.12), for each pair of integers $i_1, i_2 \geq i_0$, we have

$$\rho(x^{k+i_1}, x^{k+i_2}) \leq \rho(x^{k+i_1}, y^k) + \rho(y^k, x^{k+i_2}) \leq 3 \sum_{j=k}^{\infty} r_j.$$

Since k has been fixed arbitrarily in $\mathbb{N}$, it follows that for each $k \in \mathbb{N}$, there is an integer $i_0 \geq 1$ such that for each pair of integers $i_1, i_2 \geq i_0$,

$$\rho(x^{k+i_1}, x^{k+i_2}) \leq 3 \sum_{j=k}^{\infty} r_j.$$

Since $\sum_{j=0}^{\infty} r_j < \infty$, we conclude that $\{x^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence and, therefore, that there exists $\bar{x} = \lim_{n\to\infty} x^n$ in (X, ρ). Together with (4.12) this implies that

$$\rho(\bar{x}, y^k) \leq \sum_{j=k}^{\infty} r_j.$$

Letting here $k \to \infty$ and taking into account that the sequence $\{r_k\}_{k\in\mathbb{N}}$ is summable, we obtain that $\bar{x} = \lim_{k\to\infty} y^k$. Observing that T is continuous (because it is nonexpansive) and recalling that $Ty^k = y^k$ for all $k \in \mathbb{N}$,

$$\bar{x} = \lim_{k\to\infty} y^k = \lim_{k\to\infty} Ty^k = T\bar{x},$$

that is, the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges to a fixed point of T. $\square$

References

[1] Aubin, J.P. and Cellina, A.: Differential Inclusions, *Springer Verlag*, Heidelberg, 1984.

[2] Bauschke, H.H. and Combettes, P.L.: A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.,* **26** (2001), 248-264.

[3] Bauschke, H.H. and Borwein, J.M.: Joint and separate convexity of the Bregman distance. In: D. Butnariu, Y. Censor and S. Reich (eds.), Inherently parallel algorithms in feasibility and optimization and their applications (Haifa, 2000), 23–36, *Stud. Comput. Math.* **8**, North-Holland, Amsterdam, 2001.

[4] Bauschke, H.H., Borwein, J.M. and Combettes, P.L.: Bregman monotone optimization algorithms, *SIAM J. Control Optim.* **42** (2003), 596-636.

[5] Bauschke, H.H., Combettes, P.L. and Luke, R.: Finding best approximation pairs relative to two closed convex sets in Hilbert spaces, *J. Approx. Theory* **27** (2004), 178-192.

[6] Bauschke, H.H., Matoušková, E. and Reich, S.: Projection and proximal point methods: convergence results and counterexamples, *Nonlinear Anal.* **56** (2004), 715-738.

[7] Butnariu, D. and Censor, Y.: On the behavior of a block-iterative projection method for solving convex feasibility problems, *International Journal of Computer Mathematics* **34** (1990), 79-94.

[8] Butnariu, D. and Iusem, A.N.: Totally Convex Functions for Fixed Point Computation and Infinite Dimensional Optimization, *Kluwer Academic Publishers*, Dordrecht, 2000.

[9] Butnariu, D., Iusem, A., and Burachik, R.: Iterative Methods for Solving Stochastic Convex Feasibility Problems and Applications, *Comput. Optim. Appl.* **15** (2000), 269-307.

[10] Butnariu, D., Iusem, A., and Resmerita, E.: Total convexity for powers of the norm in uniformly convex Banach spaces, *J. Convex Anal.* **7** (2000), 319-334.

[11] Butnariu, D., Iusem, A.N. and Zălinescu, C., On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces, *J. Convex Anal.* **10** (2003), 35-61.

[12] Butnariu, D. and Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, to appear in *Abstract and Applied Analysis.*

[13] Censor, Y. and Zenios, S.: Parallel Optimization: Theory, Algorithms and Applications, *Oxford University Press*, New York 1997.

[14] Cimmino, G.: Calcolo approsimato per le soluzioni dei sistemi di ecuazioni lineari, *La Ricerca Scientifica, Roma*, **16** (1938), 326-333.

[15] Eckstein, J.: Nonlinear proximal point algorithms using Bregman functions, with applications to convex programming, *Math. Oper. Res.* **18** (1993), 202-226.

[16] Goebel, K. and Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, *Marcel Dekker*, New York and Basel, 1984.

[17] Güler, O., On the convergence of the proximal point algorithm for convex minimization, *SIAM J. Control Optim.* **29** (1991), 403-419.

[18] Kaczmarz, S.: Angenherte Auflösung von Systemen linearer Gleichungen, *Bulletin de l'Académie Polonaise des Sciences et Lettres*, **A-35**, (1937), 355-357.

[19] Kassay, G.: The proximal point algorithm for reflexive Banach spaces, *Studia Univ. Babes-Bolyai Math.* **30** (1985), 9-17.

[20] Kiwiel, K.: Proximal minimization methods with generalized Bregman functions, *SIAM J. Control Optimi.* **35** (1997), 1142-1168.

[21] Iusem, A.N. and De Pierro, A.R.: Convergence results for an accelerated Cimmino algorithm, *Numer. Math.* **49** (1986), 347-368.

[22] Iusem, A.N. and Otero, R.G.: Inexact versions of proximal point and augmented Lagrangian algorithms in Banach spaces. *Numer. Funct. Anal. Optim.* **22** (2001), no. 5-6, 609–640.

[23] Iusem, A.N. and Otero, R.G.: Proximal methods with penalization effects in Banach spaces, *Numer. Funct. Anal. Optim.* **25** (2004), 69-91.

[24] Ostrowski, A.M.: The round-off stability of iterations, *Z. Angew. Math. Mech.* **47** (1967), 77-81.

[25] Rockafellar, R.T.: Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.* **14** (1976), 877-898.

[26] Solodov, M.V. and Svaiter, B.F.: An inexact hybrid generalized proximal point algorithm and some new results in the theory of Bregman functions, *Math. Oper. Res.* **51** (2000), 479-494.

D. Butnariu: Department of Mathematics, University of Haifa, 31905 Haifa, Israel
E-mail address: `dbutnaru@math.haifa.ac.il`

S. Reich: Department of Mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: `sreich@techunix.technion.ac.il`

A. J. Zaslavski: Department of Mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: `ajzasl@techunix.technion.ac.il`