A RELAXED CIMMINO TYPE METHOD FOR COMPUTING ALMOST COMMON FIXED POINTS OF TOTALLY NONEXPANSIVE FAMILIES OF OPERATORS

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Abstract. We study the convergence behavior of a relaxed Cimmino type method of finding almost common fixed points for totally nonexpansive families of operators in Banach spaces. We show that this method preserves the convergence properties it is already known to have when applied to firmly nonexpansive operators in Hilbert spaces and to families of Bregman projections in reflexive Banach spaces.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability space, B a separable reflexive Banach space and C a closed, convex, nonempty subset of B. Suppose that $T_\omega : C \to C$, $\omega \in \Omega$, is measurable family of operators, that is, for each $x \in B$, the function $T_\bullet(x) : \Omega \to B$, given by $T_\bullet(x)(\omega) = T_\omega(x)$ is measurable.

Our aim is to prove well-definedness and convergence of an algorithm for computing almost common fixed points of measurable families of operators. Recall that a point $x^* \in C$ such that

$$\mu(\{\omega \in \Omega : \ T_\omega(x^*) = x^*\}) = 1,$$

is called an *almost common fixed point* of the measurable family of operators T_ω, $\omega \in \Omega$. The collection of almost common fixed points of T_ω, $\omega \in \Omega$, is denoted $Afix(T_\bullet)$.

The algorithm we have in mind is the following iterative procedure of generating points x^k in B starting from an initial point x^0 chosen arbitrarily in C :

$$x^{k+1} = (1 - \lambda_k)x + \lambda_k \int_\Omega T_\omega(x) d\mu_k(\omega), \tag{1.1}$$

where $\{\lambda_k\}_{k \in \mathbb{N}}$ is a sequence in $(0, 1]$ and μ_k is a sequence of complete probability measures on $(\Omega, \mathcal{A})$. It is usually called the *relaxed Cimmino type method.*

The basic idea of algorithm (1.1) originates in Cimmino's classical method of solving systems of linear equalities (see [13]). Kammerer and Nashed [14] have expanded this idea to solving linear operator equations in Hilbert spaces. The Cimmino type algorithm they consider is exactly algorithm (1.1) with T_ω being metric projections on some closed convex sets, all $\lambda_k = 1$ and all $\mu_k = \mu$ (when B is finite dimensional and Ω is finite this is precisely the algorithm proposed by Cimmino). Convergence of the same unrelaxed Cimmino type procedure was analyzed in Banach spaces by [1]-[4]. In a finite dimensional setting and for a finite set Ω the relaxed Cimmino type algorithm was extensively studied during the

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last decade. The main results in this respect are summarized in [12] (see also the references therein). In Hilbert spaces and for firmly nonexpansive operators the relaxed Cimmino type algorithm was discussed in [11]. In the paper [8] is proved weak convergence of it in Banach spaces for the case where T_ω are Bregman projections on closed convex nonempty sets with nonempty intersection. This note is aimed at showing that the relaxed Cimmino type algorithm still produces weak approximations of almost common fixed points of the operators T_ω when placed in (not necessarily Hilbertian) Banach spaces provided that the family of operators is totally nonexpansive. The practical meaning of this result is that when applied to (eventually infinite) totally nonexpansive families of operators, even out of the Hilbertian context, the Cimmino type algorithms preserves its weak convergence properties even if some errors occur in the process of computing the iterates. This is relevant because computing iterates in Cimmino type procedures involves determining integrals of often inexactly computable functions. Little is known today about the strong convergence of the Cimmino type method. From [10] we know that the set of totally nonexpansive operators, whose orbits do not converge strongly, is "rare". This means that unrelaxed Cimmino procedures will more often than not converge strongly. For a discussion of some cases in which strong convergence occurs see [2], [4] and [14].

2. Preliminaries

In this section we present several notions, notations and results, which will be used in our convergence analysis of the relaxed Cimmino type method. We start by recalling that the measurable family of operators $T_\omega : C \to C$, $\omega \in \Omega$, is called *integrable* if, for each $x \in C$, the Bochner integral $\int_\Omega T_\omega(x)d\mu(\omega)$ exists. A lower semicontinuous convex function $f : B \to (-\infty, +\infty]$ is called a *Bregman function on the set* $C \subseteq Int(Dom(f))$ (cf. [12]) if, for each $x \in C$, the following conditions are satisfied:

(i) f is Gâteaux differentiable and totally convex at x (see [7]);

(ii) For any $\alpha \geq 0$, the set

$$R_\alpha^f(x; C) = \{y \in C; D_f(x, y) \leq \alpha\}$$

is bounded.

The most typical example of Bregman functions are the functions $\|\cdot\|^r$, with $r \in (1, +\infty)$, when B is a uniformly convex and smooth Banach space (see [6]). The measurable family of operators T_ω, $\omega \in \Omega$, is called *totally nonexpansive* with respect to the Bregman function $f : B \to (-\infty, +\infty]$ on the set C, if there exists a point $z \in C$ such that, for each $x \in C$,

$$D_f(z, T_\omega(x)) + D_f(T_\omega(x), x) \leq D_f(z, x),\ \mu - a.e.(\Omega) \tag{2.1}$$

A point $z \in C$ such that (2.1) holds for each $x \in C$ is called a *nonexpansivity pole* with respect to f of T_ω, $\omega \in \Omega$. The set of all nonexpansivity poles with respect to f of the family T_ω, $\omega \in \Omega$, is denoted $Nex_f(T_\bullet)$. Clearly, $Nex_f(T_\bullet) \subseteq Afix(T_\bullet)$. Recall (cf. [7, Corollary 2.2.5]) that totally nonexpansive families of operators are always integrable with respect to any complete probability measure on Ω.

Our purpose is to show that, if there exists a continuously differentiable Bregman function $f : B \to (-\infty, +\infty]$ on the closed, convex, nonempty subset $C \subseteq Int(Dom(f))$ with respect to which the measurable family of operators T_ω, $\omega \in \Omega$, is totally nonexpansive and that, for some $z \in Nex_f(T_\bullet)$, the function $D_f(z, \cdot)$ is convex then, for any sequence of complete probability measures $\{\mu_k\}_{k\in\mathbb{N}}$ on $(\Omega, \mathcal{A})$, and for each sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subseteq (0,1]$ bounded away from zero, the operators $\mathbf{T}_k : C \to C$ given by

$$\mathbf{T}_k(x) = (1-\lambda_k)x + \lambda_k \int_\Omega T_\omega(x) d\mu_k(\omega), \tag{2.2}$$

are well defined for $k \in \mathbb{N}$ and, under some additional conditions, their orbits converge weakly to almost common fixed points of the family of operators T_ω, $\omega \in \Omega$. The following result shows that, under some conditions, orbits of the sequence of operators $\mathbf{T}_k$ approximate weakly almost common fixed points of the totally nonexpansive family of operators T_ω. To be precise, an *orbit* of the sequence of operators $\{\mathbf{T}_k\}$, $k \in \mathbb{N}$, is a sequence $\{x^k\}_{k\in\mathbb{N}}$ in B such that $x^{k+1} = \mathbf{T}_k(x^k)$, $k \in \mathbb{N}$ that is, a relaxed Cimmino type method generated sequence. Recall that the measure μ_k on $(\Omega, \mathcal{A})$ is called *absolutely continuous* with respect to the measure μ, and we write $\mu_k \ll \mu$, if $\mu_k(E) = 0$ for any measurable set E for which $\mu(E) = 0$. The *Radon-Nikodym derivative* (see, for instance, [15]) of the μ - absolutely continuous measure μ_k with respect to μ, is the μ - a.e. unique measurable function $\frac{d\mu_k}{d\mu} : \Omega \to [-\infty, +\infty]$ such that

$$\mu_k(A) = \int_A \frac{d\mu_k}{d\mu} d\mu(\omega), \quad A \in \mathcal{A}.$$

The function f is said to satisfy the *separability requirement* on C, if for any two sequences $\{y^k\}_{k\in\mathbb{N}}$ and $\{z^k\}_{k\in\mathbb{N}}$ in C which converge weakly to y and z, respectively, $y \neq z$ implies that

$$\liminf_{k\to\infty} |\langle f'(y^k) - f'(z^k), y - z\rangle| > 0.$$

Our convergence argument involves the following property of the operators $\mathbf{T}_k$, $k \in \mathbb{N}$, given by (2.2).

Lemma 1 *If there exists a differentiable Bregman function $f : B \to (-\infty, +\infty]$ on C, with respect to which the measurable family of operators T_ω, $\omega \in \Omega$, is totally nonexpansive and for some $z \in Nex_f(T_\bullet)$, the function $D_f(z, \cdot)$ is convex on C then, for any sequence of probability measures $\{\mu_k\}_{k\in\mathbb{N}}$ on $(\Omega, \mathcal{A})$, and for any sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subseteq (0,1]$, the operators T_k, $k \in \mathbb{N}$, given by (2.2) are totally nonexpansive with respect to f.*

Proof: Let $z \in Nex_f(T_\bullet)$ be such that $D_f(z, \cdot)$ is convex then, for any $x \in C$, we have

$$D_f(z, \mathbf{T}_k(x)) + D_f(\mathbf{T}_k(x), x) \leq$$

$$(1-\lambda_k)D_f(z,x) + \lambda_k \left[D_f(z, \int_\Omega T_\omega(x)d\mu_k(\omega) + D_f(\int_\Omega T_\omega(x)d\mu_k(\omega), x) \right] \leq$$

$$(1-\lambda_k)D_f(z,x) + \lambda_k \int_\Omega \left[D_f(z, T_\omega(x)) + D_f(T_\omega(x), x) \right] d\mu_k(\omega) \leq$$

$$(1-\lambda_k)D_f(z,x) + \lambda_k \int_\Omega D_f(z,x) d\mu_k(\omega) = D_f(z,x),$$

where the second inequality follows from Jensen's inequality. Hence, the operators $\mathbf{T}_k$, are totally nonexpansive with respect to the Bregman function f.

3. The main result

We are now in position to prove a convergence result for the relaxed Cimmino type algorithm.

Theorem 1 *Suppose that the measurable family of operators $T_\omega, \omega \in \Omega$, is totally nonexpansive with respect to the continuously differentiable Bregman function f on C and that, for some $z \in Nex_f(T_\bullet)$, the function $D_f(z, \cdot)$ is convex. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence of numbers such that for some positive number λ we have $\lambda_k \in [\lambda, 1]$, for all $k \in \mathbb{N}$. If for each $k \in \mathbb{N}$ the complete probability measure μ_k is absolutely continuous with respect to μ and if, for $\mu-$almost all $\omega \in \Omega$,*

$$\liminf_{k\to\infty} \frac{d\mu_k}{d\mu}(\omega) > 0, \tag{3.1}$$

then, any orbit $\{x^k\}_{k\in\mathbb{N}}$ of $\{\mathbf{T}_k\}_{k\in\mathbb{N}}$, has the following properties:

(i) The sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded, has weak accumulation points and, for $\mu-$almost all $\omega \in \Omega$, we have

$$\liminf_{k\to\infty} D_f(T_\omega(x^k), x^k) = 0; \tag{3.2}$$

(ii) If, in addition, for $\mu-$almost all $\omega \in \Omega$, the function $x \to D_f(T_\omega(x), x)$ is sequentially weakly lower semicontinuous, then

(a) Any weak accumlation point of $\{x^k\}_{k\in\mathbb{N}}$ is contained in $Afix(T_\bullet)$;

(b) The sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to a point in $Afix(T_\bullet)$ whenever either $Afix(T_\bullet)$ is a singleton or $Afix(T_\bullet) = Nex_f(T_\bullet)$ and f satisfies the separability requirement.

Proof: Define the function $\Upsilon_k^f : C \to [0, +\infty]$ by

$$\Upsilon_k^f(x) = \int_\Omega D_f(T_\omega(x), x)d\mu_k(\omega). \tag{3.3}$$

Note that this function is well defined and everywhere finite on C since it was proven in [7, Corollary 2.2.5] that for any $x \in C$, the function $\omega \to D_f(T_\omega(x), x)$ is integrable (this function is measurable because $D_f(z, \cdot)$ is continuous since f is continuously differentiable). Let $z \in Nex_f(T_\bullet)$ be such that $D_f(z, \cdot)$ is convex. For any $x \in C$, we have

$$D_f(z, T_\omega(x)) + D_f(T_\omega(x), x) \le D_f(z, x), \quad \mu - a.e. \tag{3.4}$$

Integrating (3.4) we get

$$D_f(z, \int_\Omega T_\omega(x)d\mu_k(\omega)) \le \int_\Omega D_f(z, T_\omega(x))d\mu_k(\omega) \le D_f(z, x) - \Upsilon_k^f(x),$$

where the first inequality follows from Jensen's inequality since $D_f(z, \cdot)$ is convex and continuous. Using the last inequality, we get

$$\begin{aligned} D_f(z, \mathbf{T}_k(x)) &\le (1 - \lambda_k)D_f(z, x) + \lambda_k D_f(z, \int_\Omega T_\omega(x)d\mu_k(\omega)) \\ &\le (1 - \lambda_k)D_f(z, x) + \lambda_k(D_f(z, x) - \Upsilon_k^f(x)) \\ &= D_f(z, x) - \lambda_k \Upsilon_k^f(x) \end{aligned} \tag{3.5}$$

Writing this for $x = x^k$ we deduce that the sequence $\{D_f(z, x^k)\}_{k\in\mathbb{N}}$ is nonincreasing and, hence, bounded by $\alpha = D_f(z, x^0)$. Therefore, the sequence $\{x^k\}_{k\in\mathbb{N}}$ is contained in the bounded set $R_\alpha^f(z, C)$. By consequence, the sequence $\{x^k\}_{k\in\mathbb{N}}$

has weakly convergent subsequences and, thus, it has weak accumulation points. Observe that, according to (3.5), for any nonnegative integer k, we have

$$0 \leq \Upsilon_k^f(x^k) \leq \frac{1}{\lambda_k}(D_f(z,x^k) - D_f(z,x^{k+1})) \leq \frac{1}{\lambda}(D_f(z,x^k) - D_f(z,x^{k+1})).$$

Since the sequence $\{D_f(z,x^k)\}_{k\in\mathbb{N}}$ is nonincreasing and bounded, it is convergent. By consequence, the last inequality implies that $\lim_{k\to\infty}\Upsilon_k^f(x^k) = 0$.

According to the Radon-Nikodym Theorem (see, e.g., [15, Theorem 4.6]), we have

$$\Upsilon_k^f(x) = \int_\Omega D_f(T_\omega(x),x)\frac{d\mu_k}{d\mu}(\omega)d\mu(\omega). \tag{3.6}$$

Combining the last equation with Fatou's lemma, we get

$$\begin{aligned} 0 &\leq \int_\Omega \liminf_{k\to\infty}\left[D_f(T_\omega(x^k),x^k)\frac{d\mu_k}{d\mu}(\omega)\right]d\mu(\omega) \\ &\leq \liminf_{k\to\infty}\int_\Omega \left[D_f(T_\omega(x^k),x^k)\frac{d\mu_k}{d\mu}(\omega)\right]d\mu(\omega) \\ &= \lim_{k\to\infty}\Upsilon_k^f(x^k) = 0, \end{aligned}$$

and we deduce that

$$\liminf_{k\to\infty} D_f(T_\omega(x^k),x^k)\frac{d\mu_k}{d\mu}(\omega) = 0, \;\; \mu - a.e. \tag{3.7}$$

Therefore, for μ-almost all $\omega\in\Omega$

$$\begin{aligned} 0 &= \liminf_{k\to\infty} D_f(T_\omega(x^k),x^k)\frac{d\mu_k}{d\mu}(\omega) \\ &\geq \left[\liminf_{k\to\infty} D_f(T_\omega(x^k),x^k)\right]\cdot\left[\liminf_{k\to\infty}\frac{d\mu_k}{d\mu}(\omega)\right] \geq 0, \end{aligned}$$

which, according to (3.1), proves (3.2). This completes the proof of (i).

For proving (ii), let $\{x^{j_k}\}_{k\in\mathbb{N}}$ be any weakly convergent subsequence of $\{x^k\}_{k\in\mathbb{N}}$ and denote by x^* its weak limit. Let $\omega\in\Omega$ be such that (3.2) holds and such that the function $x\to D_f(T_\omega(x),x)$ is sequentially weakly lower semicontinuous. According to the definition of the modulus of total convexity (see [7]) we have

$$\begin{aligned} \nu_f(x^*,\|T_\omega(x^*)-x^*\|) &\leq D_f(T_\omega(x^*),x^*) \\ &\leq \liminf_{k\to\infty} D_f(T_\omega(x^{j_k}),x^{j_k}) = 0, \;\; \mu - a.e., \end{aligned}$$

where the second inequality results from the sequentially weak lower semicontinuity of $x\to D_f(T_\omega(x),x)$. Consequently, we have

$$\nu_f(x^*,\|T_\omega(x^*)-x^*\|) = 0, \;\; a.e.,$$

and this cannot hold unless $T_\omega(x^*) = x^*$, a.e., because f is totally convex. Since $\omega\in\Omega$ is any of the elements for which (3.2) is satisfied, it follows that $x^*\in Afix(T_\bullet)$. This proves (a).

For proving (b) suppose, by contradiction, that the sequence $\{x^k\}_{k\in\mathbb{N}}$ does not converge weakly. Then, $\{x^k\}_{k\in\mathbb{N}}$ has two weakly convergent subsequences $\{x^{t_k}\}_{k\in\mathbb{N}}$

and $\{x^{s_k}\}_{k\in\mathbb{N}}$ whose weak limits, x' and x'', respectively, are different. Note that

$$
\begin{aligned}
(3.8) \qquad & |\langle f'(x^{t_k}) - f'(x^{s_k}), x' - x''\rangle| \\
= \; & |(D_f(x', x^{t_k}) - D_f(x', x^{s_k})) - (D_f(x'', x^{t_k}) - D_f(x'', x^{s_k}))| \\
\leq \; & |D_f(x', x^{t_k}) - D_f(x', x^{s_k})| + |D_f(x'', x^{t_k}) - D_f(x'', x^{s_k})|.
\end{aligned}
$$

According to (a), $x', x'' \in Afix(T_\bullet) = Nex_f(T_\bullet)$. Therefore, for $x = x^k$, Lemma 1 implies that the sequences $\{D_f(x', x^k)\}_{k\in\mathbb{N}}$ and $\{D_f(x'', x^k)\}_{k\in\mathbb{N}}$ converge nonincreasingly. By taking in inequality (3.8) the limit for $k \to \infty$, one obtains

$$\liminf_{k\to\infty} |\langle f'(x^{t_k}) - f'(x^{s_k}), x' - x''\rangle| = 0$$

and this contradicts the separability requirement.

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