# ASYMPTOTIC BEHAVIOR OF QUASI-NONEXPANSIVE MAPPINGS

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Let $K$ be a closed convex subset of a Banach space $X$ and let $F$ be a nonempty closed convex subset of $K$. We consider complete metric spaces of self-mappings of $K$ which fix all the points of $F$ and are quasi-nonexpansive with respect to a given convex function $f$ on $X$. We prove (under certain assumptions on $f$) that the iterates of a generic mapping in these spaces converge strongly to a retraction on $F$.

## 1. INTRODUCTION

Assume that $(X, ||\cdot||)$ is a Banach space, $f: X \to R^1 \cup \{\infty\}$ is convex, the algebraic interior $\mathcal{D}^0$ of

$$\mathcal{D} = \mathrm{dom}(f) = \{x \in X : f(x) < \infty\}$$

is nonempty, and $K$ is a nonempty closed convex subset of $\mathcal{D}^0$.

For each $x \in \mathcal{D}^0$ and $y \in X$ set

$$D_f(y,x) = f(y) - f(x) + f^0(x, x-y),$$

where

$$f^0(x,v) = \lim_{t \to 0^+} t^{-1}(f(x+tv) - f(x)).$$

The basic properties of the Bregman distance $D_f$ are presented in detail in [9,14]. It is not a distance in the usual sense of the term because, although nonnegative, it is not, in general, symmetric nor does it satisfy the triangle inequality. Denote by $\mathcal{M}$ the set of all

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mappings $T: K \to K$ which are bounded on bounded subsets of $K$. We endow the set $\mathcal{M}$ with the uniformity determined by the following base:

$$E(N,\epsilon) = \{(T_1,T_2) \in \mathcal{M} \times \mathcal{M} : \ ||T_1x - T_2x|| \le \epsilon \\ \text{for all } x \in K \text{ satisfying } ||x|| \le N\},$$

where $N, \epsilon > 0$. Clearly this uniform space is metrizable and complete [21]. We also endow the space $\mathcal{M}$ with the topology induced by this uniformity. Denote by $\mathcal{M}_c$ the set of all continuous $T \in \mathcal{M}$. Clearly $\mathcal{M}_c$ is a closed subset of $\mathcal{M}$. We consider the topological subspace $\mathcal{M}_c \subset \mathcal{M}$ with the relative topology.

Denote by $\mathcal{M}_u$ the set of all those mappings $T: K \to K$ which are uniformly continuous on bounded subsets of $K$. Clearly any $T \in \mathcal{M}_u$ is bounded on bounded subsets of $K$ and $\mathcal{M}_u$ is a closed subset of the complete uniform space $\mathcal{M}$. We consider the topological subspace $\mathcal{M}_u \subset \mathcal{M}$ with the relative topology.

Let $F \subset K$ be a nonempty closed convex subset of $K$. Denote by $\mathcal{M}^{(F)}$ the set of all $T \in \mathcal{M}$ such that

$$Tx = x \text{ for all } x \in F \tag{1}$$

and

$$D_f(z,Tx) \le D_f(z,x) \text{ for all } z \in F \text{ and all } x \in K. \tag{2}$$

Set

$$\mathcal{M}_c^{(F)} = \mathcal{M}^{(F)} \cap \mathcal{M}_c \text{ and } \mathcal{M}_u^{(F)} = \mathcal{M}^{(F)} \cap \mathcal{M}_u.$$

We assume that

$$D_f(z,\cdot): \ K \to R^1 \text{ is convex and continuous for all } z \in F. \tag{3}$$

This assumption implies that $\mathcal{M}^{(F)}$, $\mathcal{M}_c^{(F)}$ and $\mathcal{M}_u^{(F)}$ are closed subsets of $\mathcal{M}$. We consider the topological subspaces $\mathcal{M}^{(F)}$, $\mathcal{M}_c^{(F)}, \mathcal{M}_u^{(F)} \subset \mathcal{M}$ with the relative topologies.

We also assume that for each $x \in K$, each $y, z \in F$ and each $\gamma \in (0,1)$,

$$D_f((1-\gamma)z + \gamma y, (1-\gamma)x + \gamma y) \le (1-\gamma) D_f(z,x). \tag{4}$$

Clearly (3) and (4) are true if the function $D_f(\cdot,\cdot): F \times K \to R^1$ is convex as a function of two variables.

Although these two assumptions are rather restrictive we note that there are quite a few examples of convex functions $f$ for which the corresponding Bregman distance $D_f$ is convex as a function of two variables. These examples include the square of the norm in Hilbert space, the negentropy in $R^n$ [7] and the examples discussed in [3].

For each $x \in K$ and each nonempty $G \subset K$ set

$$\rho_f(x,G) = \inf\{D_f(z,x): \ z \in G\}. \tag{5}$$

Finally, we assume the following uniform continuity condition:

For any $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in K, \ z \in F$ and

$$D_f(z,x) \le \delta, \text{ then } ||z - x|| \le \epsilon. \tag{6}$$

If $K$ is bounded, then condition (6) is a weakened version of the so-called sequential compatibility condition studied in [9].

The study of the asymptotic behavior of various classes of operators of contractive type is a central topic in applied nonlinear analysis and optimization. In recent years Bregman functions and distances [5,12] have been playing an ever-increasing role in the literature devoted to this topic. See, for instance, [6–10,13,14,24] and the references mentioned therein.

In this paper we study the asymptotic behavior of quasi-nonexpansive (see (2)) self-mappings of $K$. For appropriate complete metric spaces of such mappings we establish the existence of an everywhere dense $G_\delta$ subset such that the iterates of any of its elements converge strongly to a retraction onto $F$. Results of this kind for powers of a nonexpansive operator were already established by De Blasi and Myjak [15]. This approach, when a certain property is investigated not for a single point of a complete metric space, but for the whole space, has also been successfully applied in the theory of dynamical systems [10,16,23,25], approximation theory [17], optimization [18–20], variational analysis [1,4, 22], the calculus of variations [2,26], as well as in optimal control [27,28].

Our paper is organized as follows. In Section 2 we state a theorem on the convergence of powers for the space $\mathcal{M}_u^{(F)}$. This theorem is proved in Section 3. In Section 4 we state two convergence theorems for the space $\mathcal{M}^{(F)}$ when the Bregman distance $D_f$ is uniformly continuous. These two theorems are established in Section 5. In Section 6 we consider the asymptotic behavior of continuous mappings without assuming uniform continuity of either the mappings or the Bregman distance. We obtain three theorems which are proved in Section 7.

## 2. CONVERGENCE OF POWERS FOR A CLASS OF UNIFORMLY CONTINUOUS MAPPINGS

In this section we make the following assumption:

$$\begin{aligned}&\text{For each bounded set } K_0 \subset K \text{ and each } \epsilon > 0 \text{ there is } \delta > 0\\ &\text{such that if } x \in K_0,\ z \in F \text{ and } \|z - x\| \le \delta, \text{ then } D_f(z,x) \le \epsilon. \end{aligned} \tag{7}$$

*Remark.* Note that (7) is true if the function $f$ is Lipschitzian on each bounded subset of $K$.

We also assume that for each bounded set $K_0 \subset K$,

$$\sup\{\rho_f(x,F) :\ x \in K_0\} < \infty. \tag{8}$$

**Theorem 2.1** *Assume that $P \in \mathcal{M}_u^{(F)}$ and $P(K) = F$. Then there exists a set $\mathcal{F} \subset \mathcal{M}_u^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_u^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

*(i) There exists $P_B \in \mathcal{M}_u^{(F)}$ such that $P_B(K) = F$ and $B^n x \to P_B x$ as $n \to \infty$, uniformly on bounded subsets of $K$;*

*(ii) for each $\epsilon > 0$ and each bounded set $C \subset K$ there exist a neighborhood $U$ of $B$ in $\mathcal{M}_u^{(F)}$ and an integer $N \ge 1$ such that for each $S \in U$, each $x \in C$ and each integer $n \ge N$, we have $\|S^n x - P_B x\| \le \epsilon$*

In this theorem, as well as in our subsequent theorems, we assume that certain retractions belong to $\mathcal{M}^{(F)}$. Sufficient conditions for the existence of such retractions can be found in [11].

## 3. PROOF OF THEOREM 2.1

We first recall the following result established in [10].

**Proposition 3.1** *Let $K_0$ be a bounded subset of $K$, $T \in \mathcal{M}_u$, $\epsilon > 0$, and let $n \geq 1$ be an integer. Then there exists a neighborhood $U$ of $T$ in $\mathcal{M}_u$ such that for each $S \in U$ and each $x \in K_0$, the inequality $||T^n x - S^n x|| \leq \epsilon$ holds.*

Now let $\gamma \in (0,1)$ and $T \in \mathcal{M}_u^{(F)}$. Define a mapping $T_\gamma : \ K \to K$ by

$$T_\gamma x = \gamma P x + (1-\gamma)Tx, \ x \in K. \tag{9}$$

Clearly, $T_\gamma \in \mathcal{M}_u$ and $T_\gamma x = x$ for all $x \in F$. By (9), (3) and (2), for each $z \in F$ and each $x \in K$,

$$D_f(z, T_\gamma x) = D_f(z, \gamma P x + (1-\gamma)Tx) \leq \gamma D_f(z, Px) +$$

$$(1-\gamma) D_f(z, Tx) \leq D_f(z, x). \tag{10}$$

Clearly for each $T \in \mathcal{M}_u^{(F)}$ we have

$$T_\gamma \to T \text{ as } \gamma \to 0 \text{ in } \mathcal{M}. \tag{11}$$

**Lemma 3.1** *Let $T \in \mathcal{M}_u^{(F)}$ and $\gamma \in (0,1)$. Then for each $x \in K$,*

$$\rho_f(T_\gamma x, F) \leq (1-\gamma)\rho_f(x, F). \tag{12}$$

**Proof.** Let $x \in K$ and $\epsilon > 0$. There exists $z \in F$ such that (see (5))

$$D_f(z, x) \leq \rho_f(x, F) + \epsilon. \tag{13}$$

Then $\gamma P x + (1-\gamma) z \in F$ and by (9), (5), (4), (2) and (13),

$$\rho_f(T_\gamma x, F) \leq D_f(\gamma P x + (1-\gamma) z, (1-\gamma) Tx + \gamma P x) \leq$$

$$D_f(z, Tx)(1-\gamma) \leq (1-\gamma) D_f(z, x) \leq (1-\gamma)\rho_f(x, F) + \epsilon(1-\gamma).$$

Since $\epsilon$ is an arbitrary positive number we conclude that (12) is true. Lemma 3.1 is proved.

**Lemma 3.2** *Let $T \in \mathcal{M}_u^{(F)}$, $\gamma \in (0,1)$ and let $K_0$ be a nonempty bounded subset of $K$. Then the set $\{T_\gamma^i x : \ x \in K_0, \ i \geq 1\}$ is bounded.*

**Proof.** By (6) there exists $\delta \in (0,1)$ such that

$$\text{if } x \in K,\ z \in F \text{ and } D_f(z,x) \le 2\delta, \text{ then } ||z-x|| \le 2^{-1}. \tag{14}$$

By (8) there is a number $c_0 > 0$ such that

$$\rho_f(x,F) \le c_0 \text{ for all } x \in K_0. \tag{15}$$

Choose a natural number $N$ such that

$$(1-\gamma)^N(c_0+2) \le 4^{-1}\delta. \tag{16}$$

It follows from Lemma 3.1, (15) and (16) that for each $x \in K_0$,

$$\rho_f(T_\gamma^N x, F) = (1-\gamma)^N \rho_f(x,F) \le (1-\gamma)^N c_0 \le 4^{-1}\delta.$$

Thus (see (5)) for each $x \in K_0$ there is $Qx \in F$ such that

$$D_f(Qx, T_\gamma^N x) \le 2^{-1}\delta.$$

Combined with (10) this inequality implies that

$$D_f(Qx, T_\gamma^n x) \le 2^{-1}\delta$$

for each $x \in K_0$ and each integer $n \ge N$. It follows from the latter inequality and (14) that

$$||Qx - T_\gamma^n x|| \le 2^{-1} \text{ for each } x \in K_0 \text{ and each integer } n \ge N. \tag{17}$$

Since the operator $T_\gamma$ is bounded on bounded subsets of $K$ it follows from (17) that the sets $\{T_\gamma^i x :\ x \in K_0\}$, $i = 1, 2, \ldots, N$, $\{Qx :\ x \in K_0\}$ and $\{T_\gamma^n x :\ x \in K_0,\ n \ge N\}$ are all bounded. Lemma 3.2 is proved.

**Lemma 3.3** *Let $T \in \mathcal{M}_u^{(F)}$, $\gamma \in (0,1)$, $\epsilon > 0$, and let $K_0$ be a nonempty bounded subset of $K$. Then there exist a neighborhood $U$ of $T_\gamma$ in $\mathcal{M}_u^{(F)}$ and a natural number $N$ such that for each $x \in K_0$ there exists $Qx \in F$ such that for each integer $n \ge N$ and each $S \in U$, we have*

$$||S^n x - Qx|| \le \epsilon. \tag{18}$$

**Proof.** Set

$$K_1 = K_0 \cup \{T_\gamma^i x :\ x \in K_0,\ i \ge 1\}$$

and

$$K_2 = \{y \in K :\ \inf\{||y-x|| :\ x \in K_1\} \le 2\}. \tag{19}$$

By Lemma 3.2 the sets $K_1$ and $K_2$ are bounded. By (8) there is $c_0 > 0$ such that

$$\rho_f(x,F) \le c_0 \text{ for all } x \in K_2. \tag{20}$$

By (6) there exists $\delta \in (0,\epsilon)$ such that

$$\text{if } x \in K,\ z \in F \text{ and } D_f(z,x) \le 2\delta, \text{ then } ||z-x|| \le 2^{-1}\epsilon. \tag{21}$$

By (7) there is $\epsilon_0 \in (0, 2^{-1}\delta)$ such that

$$\text{if } x \in K_2, \ z \in F \text{ and } ||x - z|| \le 2\epsilon_0, \text{ then } D_f(z, x) \le 2^{-1}\delta. \tag{22}$$

By (6) there is $\delta_0 \in (0, 2^{-1}\epsilon)$ such that

$$\text{if } x \in K, \ z \in F, \ D_f(z, x) \le 2\delta_0, \text{ then } ||x - z|| \le 2^{-1}\epsilon_0. \tag{23}$$

Choose a natural number $N$ such that

$$(1 - \gamma)^N (c_0 + 2) \le 4^{-1}\delta_0.$$

It follows from this inequality, Lemma 3.1, (20) and (19) that for each $x \in K_0$,

$$\rho_f(T_\gamma^N x, F) \le (1 - \gamma)^N \rho_f(x, F) \le (1 - \gamma)^N c_0 \le 4^{-1}\delta_0.$$

Thus for each $x \in K_0$ there is $Qx \in F$ such that $D_f(Qx, T_\gamma^N x) \le 2^{-1}\delta_0$. Combined with (23) this implies that

$$||T_\gamma^N x - Qx|| \le 2^{-1}\epsilon_0 \text{ for all } x \in K_0. \tag{24}$$

By Proposition 3.1 there exists a neighborhood $U$ of $T_\gamma$ in $\mathcal{M}_u^{(F)}$ such that for each $x \in K_0$ and each $S \in U$,

$$||S^N x - T_\gamma^N x|| \le 4^{-1}\epsilon_0. \tag{25}$$

Assume that $x \in K_0$ and $S \in U$. Then (25) is true and by (19) we have

$$S^N x \in K_2. \tag{26}$$

By (25) and (24), $||S^N x - Qx|| \le 3 \cdot 4^{-1}\epsilon_0$. It follows from this inequality, (26) and (22) that $D_f(Qx, S^N x) \le 2^{-1}\delta$. Since $S \in \mathcal{M}_u^{(F)}$ it follows from the latter inequality that $D_f(Qx, S^n x) \le 2^{-1}\delta$ for all integers $n \ge N$. Combined with (21) this implies that $||Qx - S^n x|| \le \epsilon$ for all integers $n \ge N$. Lemma 3.3 is proved.

**Completion of the proof of Theorem 2.1.** By (11), the set $\{T_\gamma : \ T \in \mathcal{M}_u^{(F)}, \ \gamma \in (0, 1)\}$ is an everywhere dense subset of $\mathcal{M}_u^{(F)}$. Fix $\theta \in K$. For each natural number $i$ set

$$K_i = \{x \in K : \ ||x - \theta|| \le i\}. \tag{27}$$

By Lemma 3.3, for each $T \in \mathcal{M}_u^{(F)}$, each $\gamma \in (0, 1)$ and each integer $i \ge 1$, there exist an open neighborhood $\mathcal{U}(T, \gamma, i)$ of $T_\gamma$ in $\mathcal{M}_u^{(F)}$ and a natural number $N(T, \gamma, i)$ such that the following property holds:

P(i) For each $x \in K_{2^i}$ there is $Qx \in F$ such that

$$||S^n x - Qx|| \le 2^{-i} \text{ for all integers } n \ge N(T, \gamma, i) \text{ and all } S \in \mathcal{U}(T, \gamma, i).$$

Define

$$\mathcal{F} = \cap_{q=1}^{\infty} \cup \{\mathcal{U}(T, \gamma, i) : \ T \in \mathcal{M}_u^{(F)}, \ \gamma \in (0, 1), \ i \ge q\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\mathcal{M}_u^{(F)}$.

Let $B \in \mathcal{F}$, $\epsilon > 0$ and let $C$ be a bounded subset of $K$. There exists an integer $q \geq 1$ such that

$$C \subset K_{2^q} \text{ and } 2^{-q} < 4^{-1}\epsilon. \tag{28}$$

There exist $T \in \mathcal{M}_u^{(F)}$, $\gamma \in (0,1)$ and an integer $i \geq q$ such that

$$B \in \mathcal{U}(T, \gamma, i). \tag{29}$$

It follows from Property P(i), (28) and (29) that the following property also holds:

P(ii) For each $x \in C$ there is $Qx \in F$ such that

$$||S^n x - Qx|| \leq 4^{-1}\epsilon$$

for each integer $n \geq N(T, \gamma, i)$ and each $S \in \mathcal{U}(T, \gamma, i)$.

Property P(ii) and (29) now imply that for each $x \in C$ and each integer $n \geq N(T, \gamma, i)$,

$$||B^n x - Qx|| \leq 4^{-1}\epsilon. \tag{30}$$

Since $\epsilon$ is an arbitrary positive number and $C$ is an arbitrary bounded subset of $K$ we conclude that for each $x \in K$, $\{B^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore for each $x \in K$ there exists

$$P_B x = \lim_{n \to \infty} B^n x. \tag{31}$$

By (30) and (31), for each $x \in C$ we have

$$||P_B x - Qx|| \leq 4^{-1}\epsilon. \tag{32}$$

Once again, since $\epsilon$ is an arbitrary positive number and $C$ is an arbitrary bounded subset of $K$, we conclude that

$$P_B(K) = F. \tag{33}$$

It follows from Property (Pii) and (32) that for each $x \in C$, each $S \in \mathcal{U}(T, \gamma, i)$ and each integer $n \geq N(T, \gamma, i)$,

$$||S^n x - P_B x|| \leq 2^{-1}\epsilon.$$

This completes the proof of Theorem 2.1.

## 4. CONVERGENCE OF POWERS FOR A CLASS OF MAPPINGS WITH A UNIFORMLY CONTINUOUS BREGMAN DISTANCE

In this section we continue to assume that for each bounded set $K_0 \subset K$,

$$\sup\{\rho_f(x, F) : \ x \in K_0\} < \infty. \tag{34}$$

We also assume that there exists $P \in \mathcal{M}^{(F)}$ for which $P(K) = F$ and that the function $D_f(\cdot, \cdot) : F \times K \to R^1$ is uniformly continuous on bounded subsets of $F \times K$. In the next section we will establish the following two theorems.

**Theorem 4.1** *There exists a set $\mathcal{F} \subset \mathcal{M}^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

*1. There exists $P_B \in \mathcal{M}^{(F)}$ such that $P_B(K) = F$ and $B^n x \to P_B x$ as $n \to \infty$, uniformy on bounded subses of $K$; if $B \in \mathcal{M}_c^{(F)}$, then $P_B \in \mathcal{M}_c^{(F)}$.*

*2. For each $\epsilon > 0$ and each nonempty bounded set $C \subset K$, there exists a neighborhood $U$ of $B$ in $\mathcal{M}^{(F)}$ and a natural number $N$ such that for each $S \in U$ and each $x \in C$, there is $z(S,x) \in F$ such that $||S^n x - z(S,x)|| \le \epsilon$ for all integers $n \ge N$.*

*Moreover, if $P \in \mathcal{M}_c^{(F)}$, then there exists a set $\mathcal{F}_c \subset \mathcal{F} \cap \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$.*

**Theorem 4.2** *Let the set $\mathcal{F} \subset \mathcal{M}^{(F)}$ be as guaranteed by Theorem 4.1, $B \in \mathcal{F} \cap \mathcal{M}_c^{(F)}$, $P_B z = \lim_{n\to\infty} B^n z$, $z \in K$, and let $x \in K$, $\epsilon > 0$. Then there exist a neighborhood $U$ of $B$ in $\mathcal{M}^{(F)}$, a number $\delta > 0$ and a natural number $N$ such that for each $y \in K$ satisfying $||x - y|| \le \delta$, each $S \in U$ and each integer $n \ge N$, we have $||S^n y - P_B x|| \le \epsilon$.*

## 5. PROOFS OF THEOREMS 4.1 AND 4.2

We begin with several lemmas.

**Lemma 5.1** *Let $F_0$ be a nonempty bounded subset of $F$ and let $\beta$ be a positive number. Then the set $\{(z,y) \in F_0 \times K : \ D_f(z,y) \le \beta\}$ is bounded.*

**Proof.** Assume the contrary. Then there exists a sequence $\{(z_i, x_i)\}_{i=1}^{\infty} \subset F_0 \times K$ such that

$$D_f(z_i, x_i) \le \beta, \ i = 1, 2, \ldots, \text{ and } ||x_i|| \to \infty \text{ as } i \to \infty. \tag{35}$$

We may assume that $||z_i - x_i|| \ge 16$, $i = 1, 2, \ldots$. For each integer $i \ge 1$ there exists $\alpha_i > 0$ such that

$$||[(1 - \alpha_i) z_i + \alpha_i x_i] - z_i|| = 1. \tag{36}$$

Clearly $\alpha_i \to 0$ as $i \to \infty$. By (3) and (35), for each integer $i \ge 1$ we have

$$D_f(z_i, (1 - \alpha_i) z_i + \alpha_i x_i) \le \alpha_i D_f(z_i, x_i) \le \alpha_i \beta \to 0 \text{ as } i \to \infty.$$

Combined with (6) this implies that

$$||z_i - [(1 - \alpha_i) z_i + \alpha_i x_i]|| \to 0 \text{ as } i \to \infty,$$

which contradicts (36). This proves Lemma 5.1.

In our next lemma we use the convention that the empty set is bounded.

**Lemma 5.2** *Let $K_0$ be a nonempty bounded subset of $K$ and let $\beta$ be a positive number. Then the set $\{(z,y) \in F \times K_0 : \ D_f(z,y) \le \beta\}$ is bounded.*

**Proof.** Let us suppose the contrary. Then there exists a sequence $\{(z_i, x_i)\}_{i=1}^{\infty} \subset F \times K_0$ such that

$$D_f(z_i, x_i) \le \beta, \ i = 1, 2, \ldots, \text{ and } ||z_i|| \to \infty \text{ as } i \to \infty. \tag{37}$$

We may assume that $||z_i - x_i|| \geq 16$, $i = 1, 2, \ldots$. For each integer $i \geq 1$ there exists $\alpha_i > 0$ such that

$$||[(1-\alpha_i)z_i + \alpha_i x_i] - z_i|| = 1. \tag{38}$$

Clearly $\alpha_i \to 0$ as $i \to \infty$. By (3), for each integer $i \geq 1$,

$$D_f(z_i, (1-\alpha_i)z_i + \alpha_i x_i) \leq \alpha_i D_f(z_i, x_i) \leq \alpha_i \beta \to 0 \text{ as } i \to \infty.$$

Combined with (6) this implies that $||z_i - [(1-\alpha_i)z_i + \alpha_i x_i]|| \to 0$ as $i \to \infty$. This contradicts (38). Thus Lemma 5.2 is proved.

Let $T \in \mathcal{M}^{(F)}$ and $\gamma \in (0,1)$. Define a mapping $T_\gamma : K \to K$ by

$$T_\gamma x = (1-\gamma)Tx + \gamma Px, \; x \in K. \tag{39}$$

Clearly $T_\gamma \in \mathcal{M}$, $T_\gamma x = x$, $x \in F$, and if $T, P \in \mathcal{M}_c$, then $T_\gamma \in \mathcal{M}_c$. By (39) and (3), for each $z \in F$ and each $x \in K$,

$$D_f(z, T_\gamma x) = D_f(z, (1-\gamma)Tx + \gamma Px) \leq (1-\gamma)D_f(z, Tx) + \gamma D_f(z, Px) \leq D_f(z, x).$$

Therefore,

$$T_\gamma \in \mathcal{M}^{(F)} \text{ and if } T, P \in \mathcal{M}_c, \text{ then } T_\gamma \in \mathcal{M}_c^{(F)}. \tag{40}$$

Clearly, for each $T \in \mathcal{M}^{(F)}$ we have

$$T_\gamma \to T \text{ as } \gamma \to 0 \text{ in } \mathcal{M}^{(F)}. \tag{41}$$

**Lemma 5.3** *Let $T \in \mathcal{M}^{(F)}$, $\gamma, \epsilon \in (0,1)$ and let $K_0$ be a nonempty bounded subset of $K$. Then there exists a neighborhood $U$ of $T_\gamma$ in $\mathcal{M}^{(F)}$ such that the set $\{Sx : \; S \in U, \; x \in K_0\}$ is bounded and for each $S \in U$ and each $x \in K_0$ satisfying $\rho_f(x, F) > \epsilon$, the following inequality holds:*

$$\rho_f(Sx, F) \leq \rho_f(x, F) - \epsilon\gamma/4. \tag{42}$$

**Proof.** Set

$$U_0 = \{S \in \mathcal{M}^{(F)} : \; ||Sx - T_\gamma x|| \leq 1 \text{ for all } x \in K_0\}. \tag{43}$$

Clearly $U_0$ is a neighborhood of $T_\gamma$ in $\mathcal{M}^{(F)}$ and

$$\sup\{||Sx|| : \; S \in U_0, \; x \in K_0\} < \infty. \tag{44}$$

By (34) there is $c_0 > 0$ such that

$$4 + \sup\{\rho_f(x, F) : \; x \in K_0\} < c_0. \tag{45}$$

By Lemma 5.2 there exists a number $c_1 > 0$ such that

$$\text{if } (z, x) \in F \times K_0 \text{ and } D_f(z, x) \leq c_0 + 2, \text{ then } ||z|| \leq c_1. \tag{46}$$

We may assume without loss of generality that

$$c_1 > \sup\{||Px|| : \; x \in K_0\}. \tag{47}$$

Since $D_f(\cdot,\cdot)$ is assumed to be uniformly continuous on bounded subsets of $F \times K$, there exists a number $\delta \in (0, 2^{-1})$ such that for each

$$(z, x_1),\ (z, x_2) \in \{\xi \in F :\ ||\xi|| \le c_1\} \times \cup\{S(K_0) :\ S \in U_0\}$$

satisfying $||x_1 - x_2|| \le \delta$, the following inequality holds:

$$|D_f(z, x_1) - D_f(z, x_2)| \le 4^{-1}\epsilon\gamma. \tag{48}$$

Set

$$U = \{S \in \mathcal{M}^{(F)} :\ ||Sx - T_\gamma x|| \le \delta \text{ for all } x \in K_0\}. \tag{49}$$

Clearly $U$ is a neighborhood of $T_\gamma$ in $\mathcal{M}^{(F)}$ and the set $\{Sx :\ S \in U,\ x \in K_0\}$ is bounded.

Assume that

$$S \in U,\ x \in K_0 \text{ and } \rho_f(x, F) > \epsilon. \tag{50}$$

We will show that (42) is valid. Let

$$\Delta \in (0, 4^{-1}\gamma\epsilon). \tag{51}$$

There is

$$z \in F \text{ such that } D_f(z, x) \le \rho_f(x, F) + \Delta. \tag{52}$$

By (45), (46), (50) and (52),

$$D_f(z, x) \le c_0 \text{ and } ||z|| \le c_1. \tag{53}$$

By (49) and (4),

$$\begin{aligned} D_f((1-\gamma)z + \gamma Px, T_\gamma x) &= D_f((1-\gamma)z + \gamma Px, (1-\gamma)Tx + \gamma Px) \\ &\le (1-\gamma)D_f(z, Tx) = (1-\gamma)D_f(z, x). \end{aligned} \tag{54}$$

By (50) and (49) we have

$$||T_\gamma x - Sx|| \le \delta. \tag{55}$$

By (53) and (47),

$$\begin{aligned} &((1-\gamma)z + \gamma Px, T_\gamma x), ((1-\gamma)z + \gamma Px, Sx) \\ &\in \{\xi \in F :\ ||\xi|| \le c_1\} \times \cup\{H(K_0) :\ H \in U_0\}. \end{aligned} \tag{56}$$

By (55), (56) and the definition of $\delta$ (see (48)),

$$|D_f((1-\gamma)z + \gamma Px, T_\gamma x) - D_f((1-\gamma)z + \gamma Px, Sx)| \le 4^{-1}\epsilon\gamma.$$

Combined with (54), (52) and (51) this inequality implies that

$$\begin{aligned} \rho_f(Sx, F) \le D_f((1-\gamma)z + \gamma Px, Sx) \le 4^{-1}\epsilon\gamma + D_f((1-\gamma)z + \gamma Px, T_\gamma x) \le \\ 4^{-1}\epsilon\gamma + (1-\gamma)D_f(z, x) \le 4^{-1}\epsilon\gamma + (1-\gamma)\rho_f(x, F) + \\ (1-\gamma)\Delta \le (1-\gamma)\rho_f(x, F) + 2^{-1}\epsilon\gamma. \end{aligned}$$

Since (42) follows from the latter inequality and (50), Lemma 5.3 is proved.

We use the convention that $S^0 = I$, the identity operator, for each $S \in \mathcal{M}^{(F)}$.

**Lemma 5.4** *Let $T \in \mathcal{M}^{(F)}$, $\gamma, \epsilon \in (0,1)$ and let $K_0$ be a nonempty bounded subset of $K$. Then there exists a neighborhood $U$ of $T_\gamma$ in $\mathcal{M}^{(F)}$ and a natural number $N$ such that for each $S \in U$ and each $x \in K_0$,*

$$\rho_f(S^N x, F) \le \epsilon. \tag{57}$$

**Proof.** By (34) there is

$$c_0 > \sup\{\rho_f(x, F) : \ x \in K_0\}. \tag{58}$$

Choose a natural number $N \ge 2$ for which

$$8^{-1}\epsilon\gamma N > c_0 + 1. \tag{59}$$

Using Lemma 5.3 and induction we construct a sequence of nonempty bounded sets $K_i \subset K$, $i = 0, \dots, N$, and a sequence of neighborhoods $U_i$, $i = 0, \dots N-1$, of $T_\gamma$ in $\mathcal{M}^{(F)}$ such that for all $i = 0, \dots, N-1$ the following properties hold:

(Pi) $K_{i+1} = \{Sx : \ S \in U_i, \ x \in K_i\}$;

(Pii) for each $S \in U_i$ and each $x \in K_i$ satisfying $\rho_f(x, F) > \epsilon$ the following inequality holds:

$$\rho_f(Sx, F) \le \rho_f(x, F) - \epsilon\gamma/4.$$

Set $U = \cap_{i=0}^{N-1} U_i$.

Assume that $S \in U$ and $x \in K_0$. We will show that (57) is valid. Assume the contrary. Then $\rho(S^i x, F) > \epsilon, \ i = 0, \dots, N$. Combined with Property (Pii) this implies that for all $i = 0, \dots, N-1$ we have

$$\rho_f(S^{i+1}x, F) \le \rho_f(S^i x, F) - \epsilon\gamma/4.$$

Therefore

$$\rho_f(S^N x, F) \le \rho_f(x, F) - \epsilon\gamma N/8.$$

Using this inequality, (58) and (59) we obtain

$$0 \le \rho_f(S^n x, F) \le c_0 - 8^{-1}\epsilon\gamma N \le -1,$$

a contradiction. Thus (57) is indeed true and Lemma 5.4 is proved.

**Lemma 5.5** *Let $T \in \mathcal{M}^{(F)}$, $\gamma, \epsilon \in (0,1)$ and let $K_0$ be a nonempty bounded subset of $K$. Then there exists a neighborhood $U$ of $T_\gamma$ in $\mathcal{M}^{(F)}$ and a natural number $N$ such that for each $S \in U$ and each $x \in K_0$ there is $z(S, x) \in F$ for which*

$$||S^i x - z(S, x)|| \le \epsilon \text{ for all integers } i \ge N. \tag{60}$$

**Proof.** By (41) there is $\delta \in (0,1)$ such that

$$\text{if } x \in K, \ z \in F \text{ and } D_f(z, x) \le \delta, \ \text{then } ||x - z|| \le 2^{-1}\epsilon. \tag{61}$$

By Lemma 5.4 there exist a neighborhood $U$ of $T_\gamma$ in $\mathcal{M}^{(F)}$ and a natural number $N$ such that

$$\rho_f(S^N x, F) \leq \delta/2 \text{ for each } S \in U \text{ and } x \in K_0.$$

This implies that for each $x \in K_0$ and each $S \in U$, there exists $z(S,x) \in F$ for which $D_f(z(S,x), S^N x) < \delta$. When combined with (51) this inequality implies that for each $x \in K_0$, each $S \in U$, and each integer $i \geq N$ we have

$$D_f(z(S,x), S^i x) < \delta \text{ and } ||S^i x - z(S,x)|| \leq 2^{-1}\epsilon.$$

Lemma 5.5 is proved.

**Completion of the proof of Theorem 4.1.** By (41), the set $\{T_\gamma : T \in \mathcal{M}^{(F)}, \gamma \in (0,1)\}$ is an everywhere dense subset of $\mathcal{M}^{(F)}$ and if $P \in \mathcal{M}_c^{(F)}$, then $\{T_\gamma : T \in \mathcal{M}_c^{(F)}, \gamma \in (0,1)\}$ is an everywhere dense subset of $\mathcal{M}_c^{(F)}$. Fix $\theta \in K$. For each natural number $i$ set

$$K_i = \{x \in K : ||x - \theta|| \leq i\}. \tag{62}$$

By Lemma 5.5, for each $T \in \mathcal{M}^{(F)}$, each $\gamma \in (0,1)$ and each integer $i \geq 1$ there exist an open neighborhood $\mathcal{U}(T,\gamma,i)$ of $T_\gamma$ in $\mathcal{M}^{(F)}$ and a natural number $N(T,\gamma,i)$ such that the following property holds:

P(iii) For each $x \in K_{2^i}$ and each $S \in \mathcal{U}(T,\gamma,i)$, there is $z(S,x) \in F$ for which

$$||S^n x - z(S,x)|| \leq 2^{-i} \text{ for all integers } n \geq N(T,\gamma,i).$$

Define

$$\mathcal{F} = \cap_{q=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,i) : T \in \mathcal{M}^{(F)}, \gamma \in (0,1), i \geq q\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}^{(F)}$. If $P \in \mathcal{M}_c^{(F)}$, then define

$$\mathcal{F}_c = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,i) : T \in \mathcal{M}_c^{(F)}, \gamma \in (0,1), i \geq q\}] \cap \mathcal{M}_c^{(F)}.$$

In this case $\mathcal{F}_c \subset \mathcal{F}$ and $\mathcal{F}_c$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$.

Let $B \in \mathcal{F}$, $\epsilon > 0$ and let $C$ be a bounded subset of $K$. There exists an integer $q \geq 1$ such that

$$C \subset K_{2^q} \text{ and } 2^{-q} < 4^{-1}\epsilon. \tag{63}$$

There also exist $T \in \mathcal{M}^{(F)}$, $\gamma \in (0,1)$ and an integer $i \geq q$ such that

$$B \in \mathcal{U}(T,\gamma,i). \tag{64}$$

Note that if $P \in \mathcal{M}_c^{(F)}$ and $B \in \mathcal{F}_c$, then $T \in \mathcal{M}_c^{(F)}$.

It follows from Property P(iii), (63) and (64) that the following property holds:

(Piv) for each $S \in \mathcal{U}(T,\gamma,i)$ and each $x \in C$, there is $z(S,x) \in F$ such that $||S^n x - z(S,x)|| \leq 4^{-1}\epsilon$ for each integer $n \geq N(T,\gamma,i)$.

Property P(iv) and (64) imply that for each $x \in C$ and each integer $n \geq N(T,\gamma,i)$,

$$||B^n x - z(B,x)|| \leq 4^{-1}\epsilon. \tag{65}$$

Since $\epsilon$ is an arbitrary positive number and $C$ is an arbitrary bounded subset of $K$, we conclude that for each $x \in K$, $\{B^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore for each $x \in K$ there exists the limit

$$P_B x = \lim_{n\to\infty} B^n x.$$

In view of (65) we see that, for each $x \in C$,

$$||P_B x - z(B,x)|| \le 4^{-1}\epsilon. \tag{66}$$

Once again, since $\epsilon$ is an arbitrary positive number and $C$ is an arbitrary bounded subset of $K$, we conclude that $P_B(K) = F$. It follows from (65) and (66) that for each $x \in C$ and each integer $n \ge N(T,\gamma,i)$,

$$||B^n x - P_B x|| \le 2^{-1}\epsilon.$$

This implies that $P_B \in \mathcal{M}^{(F)}$ and if $B \in \mathcal{M}_c^{(F)}$, then $P_B \in \mathcal{M}_c^{(F)}$. This completes the proof of Theorem 4.1.

In the proof of Theorem 4.2 we will use the following lemma which can be obtained by induction on $n$.

**Lemma 5.6** *Let $B \in \mathcal{M}_c^{(F)}$, $x \in K$, $\epsilon \in (0,1)$ and let $n \ge 1$ be an integer. Then there exists a neighborhood $U$ of $B$ in $\mathcal{M}^{(F)}$ and a number $\delta > 0$ such that for each $S \in U$ and each $y \in K$ satisfying $||y - x|| \le \delta$, the following inequality holds:*

$$||S^n y - B^n x|| \le \epsilon.$$

**Proof of Theorem 4.2.** By Theorem 4.1 there exists a natural number $N$ and a neighborhood $U_0$ of $B$ in $\mathcal{M}^{(F)}$ such that

$$||P_B y - B^n y|| \le 8^{-1}\epsilon \text{ for each } y \in K \text{ satisfying } ||y - x|| \le 1; \tag{67}$$

for each $S \in U_0$ and each $y \in K$ satisfying $||y - x|| \le 1$, there is $z(S,y) \in F$ such that

$$||S^n y - z(S,y)|| \le 8^{-1}\epsilon \text{ for all integers } n \ge N. \tag{68}$$

By Lemma 5.6 there exist a number $\delta \in (0,1)$ and a neighborhood $U$ of $B$ in $\mathcal{M}^{(F)}$ such that $U \subset U_0$ and

$$||S^N y - B^N x|| \le 8^{-1}\epsilon \text{ for each } S \in U \text{ and each } y \in K \text{ for which } ||y - x|| \le \delta. \tag{69}$$

Assume that

$$y \in K,\ ||x - y|| \le \delta \text{ and } S \in U. \tag{70}$$

By (70), (69) and (67),

$$||S^N y - B^N x|| \le 8^{-1}\epsilon,\ ||S^N y - z(S,y)|| \le 8^{-1}\epsilon \text{ and } ||P_B x - B^N x|| \le 8^{-1}\epsilon.$$

These three inequalities imply that

$$||z(S,y) - P_B x|| \le 3 \cdot 8^{-1}\epsilon.$$

Combined with (68) the latter inequality implies that

$$||S^n y - P_B x|| \le 2^{-1}\epsilon \text{ for all integers } n \ge N.$$

Theorem 4.2 is proved.

## 6. CONVERGENCE OF POWERS FOR CONTINUOUS MAPPINGS

In this section we assume that there is $P \in \mathcal{M}_c^{(F)}$ such that

$$P(K) = F. \tag{71}$$

The following three theorems will be established in Section 7.

**Theorem 6.1** *Let $x \in K$. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

*1. There exists $\lim_{n\to\infty} B^n x \in F$.*

*2. For each $\epsilon > 0$ there exist a neighborhood $U$ of $B$ in $\mathcal{M}_c^{(F)}$, a natural number $N$ and a number $\delta > 0$ such that for each $S \in U$, each $y \in K$ satisfying $||y - x|| \le \delta$ and each integer $n \ge N$, $||S^n y - \lim_{i\to\infty} B^i x|| \le \epsilon$.*

In the next two theorems we consider the space $K \times \mathcal{M}_c^{(F)}$ with the product topology.

**Theorem 6.2** *There exists a set $\mathcal{F} \subset K \times \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $K \times \mathcal{M}_c^{(F)}$ such that for each $(z, B) \in \mathcal{F}$ the following two assertions hold:*

*1. There exists $\lim_{n\to\infty} B^n z \in F$.*

*2. For each $\epsilon > 0$ there exist a neighborhood $U$ of $(z, B)$ in $K \times \mathcal{M}_c^{(F)}$ and a natural number $N$ such that for each $(y, S) \in U$ and each integer $n \ge N$,*

$$||S^n y - \lim_{i\to\infty} B^i z|| \le \epsilon.$$

**Theorem 6.3** *Assume that the set $K_0$ is a nonempty separable closed subset of $K$. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$ such that for each $T \in \mathcal{F}$ there exists a set $\mathcal{K}_T \subset K_0$ which is a countable intersection of open everywhere dense subsets of $K_0$ with the relative topology such that the following two assertions hold:*

*1. For each $x \in \mathcal{K}_T$ there exists $\lim_{n\to\infty} T^n x \in F$.*

*2. For each $x \in \mathcal{K}_T$ and each $\epsilon > 0$ there exist an integer $N \ge 1$ and a neighborhood $U$ of $(x, T)$ in $\mathcal{K} \times \mathcal{M}_c^{(F)}$ such that for each $(y, S) \in U$ and each integer $i \ge N$, $||S^i y - \lim_{n\to\infty} T^n x|| \le \epsilon$.*

## 7. PROOFS OF THEOREMS 6.1-6.3

Let $T \in \mathcal{M}_c^{(F)}$ and $\gamma \in (0, 1)$. Define a mapping $T_\gamma : K \to K$ by

$$T_\gamma x = (1 - \gamma)Tx + \gamma Px, \; x \in K. \tag{72}$$

Clearly, $T_\gamma \in \mathcal{M}_c$, $T_\gamma x = x$, $x \in F$, and for each $z \in F$ and each $x \in K$,

$$D_f(z, T_\gamma x) = D_f(z, (1 - \gamma)Tx + \gamma Px) \le$$
$$(1 - \gamma)D_f(z, Tx) + \gamma D_f(z, Px) \le D_f(z, x).$$

Therefore $T_\gamma \in \mathcal{M}_c^{(F)}$. Clearly, for each $T \in \mathcal{M}_c^{(F)}$ we have

$$T_\gamma \to T \text{ as } \gamma \to 0 \text{ in } \mathcal{M}_c^{(F)}. \tag{73}$$

In the proofs of Theorems 6.1-6.3 we will use the followings two lemmas. The proof of the first one is identical with that of Lemma 3.1 and therefore we omit it.

**Lemma 7.1** *Let $T \in \mathcal{M}_c^{(F)}$ and $\gamma \in (0,1)$. Then for each $x \in K$,*

$$\rho_f(T_\gamma x, F) \leq (1-\gamma)\rho_f(x, F).$$

**Lemma 7.2** *Let $T \in \mathcal{M}_c^{(F)}$, $\gamma, \epsilon \in (0,1)$ and let $x \in K$. Then there exist a neighborhood $U$ of $T_\gamma$ in $\mathcal{M}_c^{(F)}$, a natural number $N$, $\widehat{z} \in F$ and a number $\delta > 0$ such that for each $S \in U$, each $y \in K$ satisfying $||y - x|| \leq \delta$ and each integer $n \geq N$,*

$$||S^n y - \widehat{z}|| \leq \epsilon. \tag{74}$$

**Proof.** By (6) there is $\epsilon_0 \in (0, \epsilon/2)$ such that

$$\text{if } z \in F,\ y \in K \text{ and } D_f(z,y) \leq 2\epsilon_0, \text{ then } ||z-y|| \leq \epsilon/2. \tag{75}$$

Choose a natural number $N$ for which

$$(1-\gamma)^N(\rho_f(x,F)+1) < \epsilon_0/8. \tag{76}$$

When combined with Lemma 7.1 this inequality implies that

$$\rho_f(T_\gamma^N x, F) < \epsilon_0/8.$$

Hence there exists $\widehat{z} \in F$ for which

$$D_f(\widehat{z}, T_\gamma^N x) < \epsilon_0/8. \tag{77}$$

Since the function $D_f(\widehat{z}, \cdot) : K \to R^1$ is continuous (see (3)), there exists $\epsilon_1 \in (0, \epsilon_0/2)$ such that

$$D_f(\widehat{z}, \xi) < \epsilon_0/8 \text{ for all } \xi \in K \text{ satisfying } ||\xi - T_\gamma^N x|| \leq \epsilon_1. \tag{78}$$

It follows from the continuity of $T_\gamma$ that there exist a neighborhood $U$ of $T_\gamma$ in $\mathcal{M}_c^{(F)}$ and a number $\delta > 0$ such that for each $S \in U$ and each $y \in K$ satisfying $||y-x|| \leq \delta$ we have

$$||S^N y - T_\gamma^N x|| \leq \epsilon_1. \tag{79}$$

Assume that

$$S \in U,\ y \in K \text{ and } ||y - x|| \leq \delta.$$

By the definition of $U$ and $\delta$, the inequality (79) is valid. By (79) and (78), $D_f(\widehat{z}, S^N y) < \epsilon_0/8$. This implies that $D_f(\widehat{z}, S^n y) < \epsilon_0/8$ for all integers $n \geq N$. Combined with (75) the latter inequality implies that $||\widehat{z} - S^n y|| \leq \epsilon$ for all integers $n \geq N$. This completes the proof of Lemma 7.2.

**Proof of Theorem 6.1.** Let $x \in K$. By Lemma 7.2, for each $T \in \mathcal{M}_c^{(F)}$, each $\gamma \in (0,1)$ and each integer $i \geq 1$, there exist an open neighborhood $\mathcal{U}(T,\gamma,i)$ of $T_\gamma$ in $\mathcal{M}_c^{(F)}$, a natural number $N(T,\gamma,i)$, a point $z(T,\gamma,i) \in F$ and a number $\delta(T,\gamma,i) > 0$ such that the following property holds:

(Pi) For each $S \in \mathcal{U}(T,\gamma,i)$, each $y \in K$ satisfying $||x-y|| \leq \delta(T,\gamma,i)$ and each integer $n \geq N(T,\gamma,i)$,

$$||S^n y - z(T,\gamma,i)|| \leq 2^{-i}.$$

Define

$$\mathcal{F} = \cap_{q=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,i) :\ T \in \mathcal{M}_c^{(F)},\ \gamma \in (0,1),\ i \geq q\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$.

Let $B \in \mathcal{F}$ and $\epsilon > 0$. There exists an integer $q \geq 1$ such that

$$2^{-q} < 4^{-1}\epsilon. \tag{80}$$

There also exist $T \in \mathcal{M}_c^{(F)}$, $\gamma \in (0,1)$ and an integer $i \geq q$ such that

$$B \in \mathcal{U}(T,\gamma,i). \tag{81}$$

It follows from Property (Pi) and (80) that the following property also holds:

(Pii) For each $S \in \mathcal{U}(T,\gamma,i)$, each $y \in K$ satisfying $||y-x|| \leq \delta(T,\gamma,i)$ and each integer $n \geq N(T,\gamma,i)$,

$$||S^n y - z(T,\gamma,i)|| \leq 4^{-1}\epsilon. \tag{82}$$

Since $\epsilon$ is an arbitrary positive number we conclude that $\{B^n x\}_{n=1}^{\infty}$ is a Cauchy sequence and there exists $\lim_{n\to\infty} B^n x$. The inequality (82) implies that

$$|| \lim_{n\to\infty} B^n x - z(T,\gamma,i)|| \leq 4^{-1}\epsilon.$$

Since $\epsilon$ is an arbitrary positive number we conclude that $\lim_{n\to\infty} B^n x \in F$. It follows from this relation and Property (Pi) that for each $S \in \mathcal{U}(T,\gamma,i)$, each $y \in K$ satisfying $||y-x|| \leq \delta(T,\gamma,i)$ and each integer $n \geq N(T,\gamma,i)$,

$$||S^n y - \lim_{i\to\infty} B^i x|| \leq 2^{-1}\epsilon.$$

Theorem 6.1 is established.

**Proof of Theorem 6.2.** By Lemma 7.2, for each $(x,T) \in K \times \mathcal{M}_c^{(F)}$, each $\gamma \in (0,1)$ and each integer $i \geq 1$, there exist an open neighborhood $\mathcal{U}(x,T,\gamma,i)$ of $(x,T_\gamma)$ in $K \times \mathcal{M}_c^{(F)}$, a natural number $N(x,T,\gamma,i)$ and a point $z(x,T,\gamma,i) \in F$ such that the following property holds:

(Piii) For each $(y,S) \in \mathcal{U}(x,T,\gamma,i)$ and each integer $n \geq N(x,T,\gamma,i)$,

$$||S^n y - z(x,T,\gamma,i)|| \leq 2^{-i}.$$

Define

$$\mathcal{F} = \cap_{q=1}^{\infty} \cup \{\mathcal{U}(x,T,\gamma,i) :\ (x,T) \in K \times \mathcal{M}_c^{(F)},\ \gamma \in (0,1),\ i \geq q\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $K \times \mathcal{M}_c^{(F)}$.

Let $(z, B) \in \mathcal{F}$ and $\epsilon > 0$. There exists an integer $q \geq 1$ such that

$$2^{-q} < 4^{-1}\epsilon. \tag{83}$$

There also exist $x \in K$, $T \in \mathcal{M}_c^{(F)}$, $\gamma \in (0,1)$ and an integer $i \geq q$ such that

$$(z, B) \in \mathcal{U}(x, T, \gamma, i). \tag{84}$$

By (83) and property (Piii) the following property holds:

(Piv) For each $(y, S) \in \mathcal{U}(x, T, \gamma, i)$ and each integer $n \geq N(x, T, \gamma, i)$,

$$||S^n y - z(x, T, \gamma, i)|| \leq 4^{-1}\epsilon. \tag{85}$$

Since $\epsilon$ is an arbitrary positive number we conclude that $\{B^n z\}_{n=1}^{\infty}$ is a Cauchy sequence and there exists $\lim_{n\to\infty} B^n z$. Property (Piv) and (84) imply that

$$||\lim_{n\to\infty} B^n z - z(x, T, \gamma, i)|| \leq 4^{-1}\epsilon. \tag{86}$$

Since $\epsilon$ is an arbitrary positive number we conclude that $\lim_{n\to\infty} B^n z \in F$. It follows from (86) and Property (Piv) that for each $(y, S) \in \mathcal{U}(x, T, \gamma, i)$ and each integer $n \geq N(x, T, \gamma, i)$,

$$||S^n y - \lim_{i\to\infty} B^i z|| \leq 2^{-1}\epsilon.$$

Theorem 6.2 is proved.

**Proof of Theorem 6.3.** Assume that $K_0$ is a nonempty closed separable subset of $K$. Let the sequence $\{x_j\}_{j=1}^{\infty}$ be dense in $K_0$. By Theorem 6.1, for each integer $p \geq 1$ there exists a set $\mathcal{F}_p \subset \mathcal{M}_c^{(F)}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$ such that for each $T \in \mathcal{F}_p$ the following two properties hold:

C(i) There exists $\lim_{n\to\infty} T^n x_p \in F$.

C(ii) For each $\epsilon > 0$, there exist a neighborhood $U$ of $T$ in $\mathcal{M}_c^{(F)}$, a number $\delta > 0$ and a natural number $N$ such that for each $S \in U$, each $y \in K$ satisfying $||y - x_p|| \leq \delta$ and each integer $m \geq N$,

$$||S^m y - \lim_{n\to\infty} T^n x_p|| \leq \epsilon.$$

Set

$$\mathcal{F} = \cap_{p=1}^{\infty} \mathcal{F}_p. \tag{87}$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c^{(F)}$.

Assume that $T \in \mathcal{F}$. Then for each $p \geq 1$ there exists $\lim_{n\to\infty} T^n x_p \in F$.

Now we will construct the set $\mathcal{K}_T \subset K_0$. By property C(ii), for each pair of natural numbers $q, i$ there exist a neighborhood $\mathcal{U}(q, i)$ of $T$ in $\mathcal{M}_c^{(F)}$, a number $\delta(q, i) > 0$ and a natural number $N(q, i)$ such that the following property holds:

C(iii) For each $S \in \mathcal{U}(q, i)$, each $y \in K$ satisfying $||y - x_q|| \leq \delta(q, i)$ and each integer $m \geq N(q, i)$,

$$||S^m y - \lim_{n\to\infty} T^n x_q|| \leq 2^{-i}.$$

Define

$$\mathcal{K}_T = \cap_{n=1}^{\infty} \cup \{\{y \in K_0 :\ ||y - x_q|| < \delta(q,i)\} :\ q \geq 1,\ i \geq n\}. \tag{88}$$

Clearly $\mathcal{K}_T$ is a countable intersection of open everywhere dense subsets of $K_0$.

Assume that $x \in \mathcal{K}_T$ and $\epsilon > 0$. There exists an integer $n \geq 1$ such that

$$2^{-n} < 4^{-1}\epsilon. \tag{89}$$

By (88) there exist a natural number $q$ and an integer $i \geq n$ such that

$$||x - x_q|| < \delta(q,i). \tag{90}$$

It follows from (89) and C(iii) that the following property also holds:

C(iv) For each $S \in \mathcal{U}(q,i)$, each $y \in K$ satisfying $||y - x_q|| \leq \delta(q,i)$ and each integer $m \geq N(q,i)$,

$$||S^m y - \lim_{j\to\infty} T^j x_q|| \leq 4^{-1}\epsilon.$$

By property C(iv) and (90) we have

$$||T^m x - \lim_{j\to\infty} T^j x_q|| \leq 4^{-1}\epsilon$$

for all integers $m \geq N(q,i)$. Since $\epsilon$ is an arbitrary positive number we conclude that $\{T^m x\}_{m=1}^{\infty}$ is a Cauchy sequence, there exists $\lim_{m\to\infty} T^m x$, and

$$|| \lim_{m\to\infty} T^m x - \lim_{m\to\infty} T^m x_q|| \leq 4^{-1}\epsilon.$$

Since $\epsilon$ is an arbitrary positive number we conclude that $\lim_{m\to\infty} T^m x \in F$. By this relation and property (Civ), for each $S \in \mathcal{U}(q,i)$, each $y \in K$ satisfying $||y - x|| < \delta(q,i) - ||x - x_q||$, and each integer $m \geq N(q,i)$ we have

$$||S^m y - \lim_{j\to\infty} T^j x|| \leq 2^{-1}\epsilon.$$

Theorem 6.3 is established.

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