On The Behavior of Subgradient Projections Methods for Convex Feasibility Problems

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Abstract

We study some methods of subgradient projections for solving a convex feasibility problem with general (not necessarily hyperplanes or half-spaces) convex sets in the inconsistent case and propose a strategy that controls the relaxation parameters in a specific self-adapting manner. This strategy that controls the relaxation parameters in a specific manner leaves enough user-flexibility but gives a mathematical guarantee for the algorithm's behavior in the inconsistent case. We present numerical results of computational experiments that show the computational advantage of the method.

1 Introduction

In this paper we consider the method of simultaneous subgradient projections for solving a convex feasibility problem with general (not necessarily linear) convex sets in the inconsistent case. To cope with this situation we propose two algorithmic developments. One uses *steering parameters* instead of *relaxation parameters* in the simultaneous subgradient projection method, and the other is a strategy that controls the relaxation parameters in a specific self-adapting manner that leaves enough user-flexibility but gives some mathematical guarantees for the algorithm's behavior in the inconsistent case. For the algorithm that uses steering parameters there is currently no mathematical theory. We present numerical results of computational experiments that show the computational advantage of the mathematically-founded algorithm that implements our specific relaxation strategy. In the remainder of this section we elaborate upon the meaning of the above-made statements.

Given m closed convex subsets $Q_1, Q_2, \cdots, Q_m \subseteq R^n$ of the n-dimensional Euclidean space, expressed as

$$Q_i = \{x \in R^n \mid f_i(x) \leq 0\}, \tag{1.1}$$

where $f_i : R^n \to R$ is a convex function, the *convex feasibility problem* (CFP) is

$$\text{find a point } x^* \in Q := \cap_{i=1}^m Q_i. \tag{1.2}$$

Thus, it is required to solve the system of convex inequalities

$$f_i(x) \leq 0, \quad i = 1, 2, \ldots, m. \tag{1.3}$$

A fundamental question is how to approach the CFP in the inconsistent case, when $Q = \cap_{i=1}^m Q_i = \emptyset$. Logically, algorithms designed to solve the CFP by finding a point $x^* \in Q$ are bound to fail and should, therefore, not be employed. But this is not always the case. Projection methods that are commonly used for the CFP, particularly in some very large real-world applications (see details below) are applied to CFPs without prior knowledge whether or not the problem is consistent. In such circumstances it is imperative to know how would a method, that is originally known to converge for a consistent CFP, behave if consistency is not guaranteed.

We address this question for a particular type of projection methods. In general, sequential projection methods exhibit *cyclic convergence* in the inconsistent case. this means that the whole sequence of iterates does not converge, but it breaks up into m convergent subsequences (see Gubin, Polyak and Raik [32, Theorem 2]). In contrast, simultaneous projection methods generally converge, even in the inconsistent case, to a minimizer of a proximity function that "measures" the weighted sum of squared distances to all sets of the CFP, provided such a minimizer exists (see Iusem and De Pierro [35] for a local convergence proof and Combettes [25] for a global one).

Therefore, there is an advantage in using simultaneous projection methods from the point of view of convergence. Additional advantages are that (i) they are inherently parallel already at the mathematical formulation level due to the simultaneous nature, and (ii) they allow the user to assign weights (of importance) to the sets of the CFP. However, a severe limitation, common to sequential as well as simultaneous projection methods, is the need to solve an inner-loop distance-minimization step for the calculation of the orthogonal projection onto each individual set of the CFP. This need is alleviated only for convex sets that are simple to project onto, such as hyperplanes or half-spaces.

A useful path to circumvent this limitation is to use subgradient projections that rely on calculation of subgradients at the current (available) iteration points, see Censor and Lent [19] or [20, Section 5.3]. Iusem and Moledo [36] studied the simultaneous projection method with subgradient projections but only for consistent CFPs. To the best of our knowledge, there does not exist a study of the simultaneous projection method with subgradient projections for the inconsistent case. Our present results are a contribution towards this goal.

The CFP is a fundamental problem in many areas of mathematics and the physical sciences, see, e.g., Combettes [24, 26] and references therein. It has been used to model significant real-world problems in image reconstruction from projections, see, e.g., Herman [33], in radiation therapy treatment planning, see Censor, Altschuler and Powlis [16] and Censor [14], and in crystallography, see Marks, Sinkler and Landree [37], to name but a few, and has been used under additional names such as *set theoretic estimation* or the *feasible set approach.* A common approach to such problems is to use projection algorithms, see, e.g., Bauschke and Borwein [4], which employ *orthogonal projections* (i.e., nearest point mappings) onto the individual sets Q_i. The orthogonal projection $P_\Omega(z)$ of a point $z \in R^n$ onto a closed convex set $\Omega \subseteq R^n$ is defined by

$$P_\Omega(z) := \operatorname{argmin}\{\| z - x \| \mid x \in \Omega\}, \tag{1.4}$$

where, throughout this paper, $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and inner product, respectively, in R^n. Frequently a *relaxation parameter* is introduced so that

$$P_{\Omega,\lambda}(z) := (1 - \lambda)z + \lambda P_\Omega(z) \tag{1.5}$$

is the *relaxed projection* of z onto Ω with relaxation λ. Many iterative projection algorithms for the CFP were developed, see Subsection 1.1 below.

1.1 Projection methods: Advantages and earlier work

The reason why the CFP is looked at from the viewpoint of projection methods can be appreciated by the following brief comments, that we made in earlier publications, regarding projection methods in general. Projections onto sets are used in a variety of methods in optimization theory but not every method that uses projections really belongs to the class of projection methods. *Projection methods* are iterative algorithms that use projections onto sets while relying on the general principle that when a family of (usually closed and convex) sets is present then projections onto the given individual sets are easier to perform then projections onto other sets (intersections, image sets under some transformation, etc.) that are derived from the given individual sets.

A projection algorithm reaches its goal, related to the whole family of sets, by performing projections onto the individual sets. Projection algorithms employ projections onto convex sets in various ways. They may use different kinds of projections and, sometimes, even use different types of projections within the same algorithm. They serve to solve a variety of problems which are either of the feasibility or the optimization types. They have different algorithmic structures, of which some are particularly suitable for parallel computing, and they demonstrate nice convergence properties and/or good initial behavior patterns.

Apart from theoretical interest, the main advantage of projection methods, which makes them successful in real-world applications, is computational. They commonly have the ability to handle huge-size problems that are beyond the ability of more sophisticated, currently available, methods. This is so because the building blocks of a projection algorithm are the projections onto the given individual sets (assumed and actually easy to perform) and the algorithmic structure is either sequential or simultaneous (or in-between).

The field of projection methods is vast and we mention here only a few recent works that can give the reader some good starting points. Such a list includes, among many others, the works of Crombez [27, 28], the connection with variational inequalities, see, e.g., Aslam-Noor [38], Yamada's [42] which is motivated by real-world problems of signal processing, and the many contributions of Bauschke and Combettes, see, e.g., Bauschke, Combettes and Kruk [5] and references therein. Bauschke and Borwein [4] and Censor and Zenios [20, Chapter 5] provide reviews of the field.

Systems of linear equations, linear inequalities, or convex inequalities are all encompassed by the CFP which has broad applicability in many areas of mathematics and the physical and engineering sciences. These include, among others, optimization theory (see, e.g., Eremin [31], Censor and Lent [19] and Chinneck [21]), approximation theory (see, e.g., Deutsch [29] and references therein), image reconstruction from projections in computerized tomography (see, e.g., Herman [33, 34]) and control theory (see, e.g., Boyd et al. [7].)

2 Simultaneous subgradient projections with steering parameters

Subgradient projections have been incorporated in iterative algorithms for the solution of CFPs. The cyclic subgradient projections (CSP) method for the CFP was given by Censor and Lent [19] as follows.

Algorithm 2.1 *The method of cyclic subgradient projections (CSP).*
Initialization: $x^0 \in R^n$ *is arbitrary.*
Iterative step: *Given* x^k, *calculate the next iterate* x^{k+1} *by*

$$x^{k+1} = \begin{cases} x^k - \alpha_k \dfrac{f_{i(k)}(x^k)}{\| \ t^k \ \|^2} t^k, & \text{if } \ f_{i(k)}(x^k) > 0, \\ x^k, & \text{if } \ f_{i(k)}(x^k) \leq 0, \end{cases} \tag{2.1}$$

where $t^k \in \partial f_{i(k)}(x^k)$ *is a subgradient of* $f_{i(k)}$ *at the point* x^k, *and the relaxation parameters* $\{\alpha_k\}_{k=0}^{\infty}$ *are confined to an interval* $\epsilon_1 \leq \alpha_k \leq 2 - \epsilon_2$, *for all* $k \geq 0$, *with some, arbitrarily small,* $\epsilon_1, \epsilon_2 > 0$.

Control: *Denoting* $I := \{1, 2, \ldots, m\}$, *the sequence* $\{i(k)\}_{k=0}^{\infty}$ *is an almost cyclic control sequence on* I.

Observe that if $t^k = 0$, then $f_{i(k)}$ takes its minimal value at x^k, implying, by the nonemptiness of Q, that $f_{i(k)}(x^k) \leq 0$, so that $x^{k+1} = x^k$. Relations of the CSP method to other iterative methods for solving the convex feasibility problem and to the relaxation method for solving linear inequalities can be found, e.g., in [20, Chapter 5], see also, Bauschke and Borwein [4, Section 7]. Since sequential projection methods for CFPs commonly have fully-simultaneous counterparts, the simultaneous subgradient projections (SSP) method of Dos Santos [30] and Iusem and Moledo [36] is a natural algorithmic development.

Algorithm 2.2 *The method of simultaneous subgradient projections (SSP).*
Initialization: *$x^0 \in R^n$ is arbitrary.*
Iterative step: ***(i)*** *Given x^k, calculate, for all $i \in I = \{1, 2, \ldots, m\}$, intermediate iterates $y^{k+1,i}$ by*

$$y^{k+1,i} = \begin{cases} x^k - \alpha_k \dfrac{f_i(x^k)}{\| \ t^k \ \|^2} t^k, & \text{if } \ f_i(x^k) > 0, \\ x^k, & \text{if } \ f_i(x^k) \leq 0, \end{cases} \tag{2.2}$$

where $t^k \in \partial f_i(x^k)$ is a subgradient of f_i at the point x^k, and the relaxation parameters $\{\alpha_k\}_{k=0}^{\infty}$ are confined to an interval $\epsilon_1 \leq \alpha_k \leq 2 - \epsilon_2$, for all $k \geq 0$, with some, arbitrarily small, $\epsilon_1, \epsilon_2 > 0$.
(ii) *Calculate the next iterate x^{k+1} by*

$$x^{k+1} = \sum_{i=1}^{m} w_i y^{k+1,i} \tag{2.3}$$

where w_i are fixed, user-chosen, positive weights with $\sum_{i=1}^{m} w_i = 1$.

The convergence analysis for this algorithm is currently available only for consistent ($Q \neq \emptyset$) CFPs, see [30, 36]. In our experimental work, reported in the sequel, we applied Algorithm 2.2 to CFPs without knowing whether or not they are consistent. Convergence is diagnosed by performing plots of a *proximity function* that measures in some manner the infeasibility of the system. We used the weighted proximity function of the form

$$p(x) := (1/2) \sum_{i=1}^{m} w_i \parallel P_i(x) - x \parallel^2 \tag{2.4}$$

were $P_i(x)$ is the orthogonal projection of the point x onto Q_i. To combat instabilities in those plots that appeared occasionally in our experiments we used *steering parameters* σ_k instead of the relaxation parameters α_k in Algorithm 2.2. To this end we need the following definition.

Definition 2.3 *A sequence* $\{\sigma_k\}_{k=0}^{\infty}$ *of real numbers* $0 \leq \sigma_k < 1$ *is called a* `steering sequence` *if it satisfies the following conditions:*

$$\lim_{k\to\infty} \sigma_k = 0, \tag{2.5}$$

$$\sum_{k=0}^{\infty} \sigma_k = +\infty, \tag{2.6}$$

$$\sum_{k=0}^{\infty} \mid \sigma_k - \sigma_{k+m} \mid < +\infty. \tag{2.7}$$

A historical and technical discussion of these conditions can be found in [3]. The sequential and simultaneous Halpern-Lions-Wittmann-Bauschke (HLWB) algorithms discussed in Censor [15] employ the parameters of a steering sequence to "force" (steer) the iterates towards the solution of the best approximation problem (BAP). This steering feature of the steering parameters has a profound effect on the behavior of any sequence of iterates $\{x^k\}_{k=0}^{\infty}$. We return to this point in Section 6.

Algorithm 2.4 *The method of simultaneous subgradient projections (SSP) with steering.*

Initialization: $x^0 \in R^n$ *is arbitrary.*

Iterative step: ***(i)*** *Given* x^k, *calculate, for all* $i \in I = \{1, 2, \ldots, m\}$, *intermediate iterates* $y^{k+1,i}$ *by*

$$y^{k+1,i} = \begin{cases} x^k - \sigma_k \dfrac{f_i(x^k)}{\parallel t^k \parallel^2} t^k, & \text{if } f_i(x^k) > 0, \\ x^k, & \text{if } f_i(x^k) \leq 0, \end{cases} \tag{2.8}$$

where $t^k \in \partial f_i(x^k)$ *is a subgradient of* f_i *at the point* x^k, *and* $\{\sigma_k\}_{k=0}^{\infty}$ *is a sequence of steering parameters.*

(ii) *Calculate the next iterate* x^{k+1} *by*

$$x^{k+1} = \sum_{i=1}^{m} w_i y^{k+1,i} \tag{2.9}$$

where w_i *are fixed, user-chosen, positive weights with* $\sum_{i=1}^{m} w_i = 1$.

3 Subgradient projections with strategical relaxation: Preliminaries

Considering the CFP (1.2), the *envelope* of the family of functions $\{f_i\}_{i=1}^m$ is the function

$$f(x) := \max\{f_i(x) \mid i = 1, 2, \ldots, m\} \tag{3.1}$$

which is also convex. Clearly, the consistent CFP is equivalent to finding a point in

$$Q = \cap_{i=1}^m Q_i = \{x \in R^n \mid f(x) \leq 0\}. \tag{3.2}$$

The subgradient projections algorithmic scheme that we propose here employs a strategy for controlling the relaxation parameters in a specific manner, leaving enough user-flexibility while giving some mathematical guarantees for the algorithm's behavior in the inconsistent case. It is described as follows:

Algorithm 3.1

Initialization: *Let M be a positive real number and let $x^0 \in R^n$ be any initial point.*

Iterative step: *Given the current iterate x^k, set*

$$I(x^k) := \{i \mid 1 \leq i \leq m \ \ \text{and} \ \ f_i(x^k) = f(x^k)\} \tag{3.3}$$

and choose a nonnegative vector $w^k = (w_1^k, w_2^k, \ldots, w_m^k) \in R^m$ such that

$$\sum_{i=1}^m w_i^k = 1 \ \ \text{and} \ \ w_i^k = 0 \ \ \text{if} \ \ i \notin I(x^k). \tag{3.4}$$

Let λ_k be any real number such that

$$\max\big(0, \ f(x^k)\big) \leq \lambda_k M^2 \leq 2\max\big(0, \ f(x^k)\big) \tag{3.5}$$

and calculate

$$x^{k+1} = x^k - \lambda_k \sum_{i \in I(x^k)} w_i^k \xi_i^k, \tag{3.6}$$

where, for each $i \in I(x^k)$, we take a subgradient $\xi_i^k \in \partial f_i(x^k)$.

It is interesting to note that any sequence $\{x^k\}_{k=0}^{\infty}$ generated by this algorithm is well-defined, no matter how x^0 and M are chosen. Similarly to other algorithms described above, Algorithm 3.1 requires computing subgradients of convex functions. In case a function is differentiable this reduces to gradient calculations. Otherwise, one can use the subgradient computing procedure presented in Butnariu and Resmerita [11].

The procedure described above was previously studied in Butnariu and Mehrez [10]. The main result there shows that the procedure converges to a solution of the CFP under two conditions: (i) that the solution set Q has a nonempty interior and (ii) that the envelope f is uniformly Lipschitz on R^n, that is, there exists a positive real number L such that

$$|f(x) - f(y)| \leq L \|x - y\|, \quad \text{for all } \ x, y \in R^n. \tag{3.7}$$

Both conditions (i) and (ii) are restrictive and it is difficult to verify their validity in practical applications. In the following we show that this method converges to solutions of consistent CFPs under considerably less demanding conditions. In fact, we show that the requirement that f should be uniformly Lipschitz on R^n is superfluous. Also, we show that the method produces approximate solutions of consistent CFPs even if int $Q = \emptyset$, provided that f is strictly convex. Strict convexity of f may not be automatically satisfied by a CFP. However, any CFP can be rewritten in a practically-equivalent form so that the envelope of the functions involved in the rewritten problem is strictly convex. Indeed, for each $i = 1, 2, \dots, m+1$, let $\bar{f}_i : R^{n+1} \to R$ be the function

$$\bar{f}_i(x_1, x_2, \dots, x_{n+1}) := \begin{cases} f_i(x_1, x_2, \dots, x_n) + x_{n+1}^2, & \text{for } 1 \leq i \leq m, \\ \\ x_{n+1}^2, & \text{for } i = m+1. \end{cases} \tag{3.8}$$

Clearly, all functions $\bar{f}_i$ are strictly convex. Thus, the envelope

$$\bar{f}(x_1, x_2, \dots, x_{n+1}) = \max\{\bar{f}_i(x_1, x_2, \dots, x_{n+1}) \mid i = 1, 2, \dots, m+1\} \tag{3.9}$$

is strictly convex too. Consider the CFP

$$\bar{f}_i(x_1, x_2, \dots, x_{n+1}) \leq 0, \quad \text{for all } \ i = 1, 2, \dots, m+1. \tag{3.10}$$

The original CFP (1.2) and the CFP (3.10) are equivalent in the sense that if a vector $x = (x_1, x_2, \dots, x_n)$ is a solution of the original CFP (1.2) then the vector $\hat{x} = (x_1, x_2, \dots, x_n, 0)$ is a solution of (3.10) and vice versa. Hence, by solving the problem (3.10), which has a strictly convex envelope $\bar{f}$, one practically solves (1.2) for which the envelope f may not be strictly convex.

4 Subgradient projections with strategical relaxation: convergence analysis

In order to discuss the convergence behavior of the subgradient projections method with strategical relaxation, recall that convex functions defined on the whole space R^n are continuous and, consequently, are bounded on bounded sets in R^n. Therefore, application of Butnariu and Iusem [9, Proposition 1.1.11] to the convex function f shows that it is Lipschitz on bounded subsets of R^n, i.e., for any nonempty bounded subset $S \subseteq R^n$ there exists a positive real number $L(S)$, called *a Lipschitz constant of f over the set S*, such that

$$|f(x) - f(y)| \leq L(S) \|x - y\|, \text{ for all } x, y \in S. \tag{4.1}$$

We denote by $B(x^0, r)$ the closed ball with center at x^0 and radius r in R^n. Our goal is to prove the following result.

Theorem 4.1 *If a positive number M and an initial point x^0 in Algorithm 3.1 are chosen so that $M \geq L(B(x^0, r))$ for some positive real number r satisfying the condition that*

$$B(x^0, r/2) \cap Q \neq \emptyset, \tag{4.2}$$

and if at least one of the following conditions hold:
(i) $B(x^0, r/2) \cap \operatorname{int} Q \neq \emptyset$,
(ii) the function f is strictly convex,
then any sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm 3.1, converges to an element of Q.

We present the proof of Theorem 4.1 as a sequence of lemmas. To do so note that, for each integer $k \geq 0$, we have

$$x^{k+1} = x^k - \lambda_k \nu^k, \tag{4.3}$$

where

$$\nu^k := \sum_{i \in I(x^k)} w_i^k \xi_i^k \in \operatorname{conv} \cup_{i \in I(x^k)} \partial f_i(x^k). \tag{4.4}$$

Using (4.3), for any $z \in R^n$, we have

$$\left\|x^{k+1} - z\right\|^2 = \left\|x^k - z\right\|^2 + \lambda_k \left(\lambda_k \left\|\nu^k\right\|^2 - 2\left\langle \nu^k, x^k - z\right\rangle\right). \tag{4.5}$$

By Clarke [23, Proposition 2.3.12] we deduce that

$$\partial f(x^k) = \operatorname{conv} \cup_{i \in I(x^k)} \partial f_i(x^k) \tag{4.6}$$

and this implies that $\nu^k \in \partial f(x^k)$ because of (4.4). Therefore,

$$\left\langle \nu^k, z - x^k \right\rangle \leq f'_+(x^k; z - x^k) \tag{4.7}$$

where $f'_+(u; v)$ denotes the right-sided directional derivative at u in the direction v. Now suppose that M, r and x^0 are chosen according to the requirements of Theorem 4.1, that is,

$$r > 0, \ \ M \geq L(B(x^0, r)) \text{ and } B(x^0, r/2) \cap Q \neq \emptyset. \tag{4.8}$$

Next we prove the following basic fact.

Lemma 4.2 *If (4.8) is satisfied and if $z \in B(x^0, r/2) \cap Q$, then for all $k \geq 0$, we have for any sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by Algorithm 3.1,*

$$x^{k+1} \in B(x^0, r) \text{ and } \left\|x^{k+1} - z\right\| \leq \left\|x^k - z\right\| \leq r/2. \tag{4.9}$$

Proof. We first show that if, for some integer $k \geq 0$,

$$x^k \in B(x^0, r) \text{ and } \left\|x^k - z\right\| \leq r/2 \tag{4.10}$$

then (4.9) holds. If either $\lambda_k = 0$ or $\nu^k = 0$, then

$$\left\|x^{k+1} - z\right\| = \left\|x^k - z\right\| \leq r/2 \tag{4.11}$$

because of (4.5). In such a situation we have

$$\left\|x^{k+1} - x^0\right\| \leq \left\|x^{k+1} - z\right\| + \left\|x^0 - z\right\| \leq 2(r/2) = r, \tag{4.12}$$

showing that $x^{k+1} \in B(x^0, r)$, hence, (4.9) is true in this case. Assume now that $\lambda_k \neq 0$ and $\nu^k \neq 0$. Since, by (4.10), $x^k \in B(x^0, r)$, by (4.8) and by [23, Proposition 2.1.2(a)], we deduce that

$$M \geq L(B(x^0, r)) \geq \left\|\nu^k\right\|. \tag{4.13}$$

According to (3.5), we also have $f(x^k) > 0$ (otherwise $\lambda_k = 0$). Since $f(z) \leq 0$ we obtain from the subgradient inequality

$$\left\langle \nu^k, x^k - z \right\rangle \geq f(x^k) - f(z) \geq f(x^k) > 0. \tag{4.14}$$

This and (4.13) imply

$$2\left\langle \nu^k, x^k - z \right\rangle \geq 2f(x^k) \geq \lambda_k M^2 \geq \lambda_k \left\| \nu^k \right\|^2, \tag{4.15}$$

showing that the quantity inside the parentheses in (4.5) is nonpositive. Thus, we deduce that

$$\left\| x^{k+1} - z \right\| \leq \left\| x^k - z \right\| \leq r/2 \tag{4.16}$$

in this case too. This proves that if (4.10) is true for all $k \geq 0$, then so is (4.9). Now, we prove by induction that (4.10) is true for all $k \geq 0$. If $k = 0$ then (4.10) obviously holds. Suppose that (4.10) is satisfied for some $k = p$. As shown above, this implies that condition (4.9) is satisfied for $k = p$ and, thus, we have that

$$x^{p+1} \in B(x^0, r) \text{ and } \left\| x^{p+1} - z \right\| \leq r/2. \tag{4.17}$$

Hence, condition (4.10) also holds for $k = p + 1$. Consequently, condition (4.9) holds for $k = p + 1$ and this completes the proof. ■

Lemma 4.3 *If (4.8) is satisfied and* $\left\{x^k\right\}_{k=0}^{\infty}$ *is any sequence, generated by Algorithm 3.1, then there exists a sequence of natural numbers* $\{k_s\}_{s=0}^{\infty}$ *such that the following limits exist*

$$x^* = \lim_{s\to\infty} x^{k_s},\ \lambda_* = \lim_{s\to\infty} \lambda_{k_s}, \tag{4.18}$$

$$\xi_i^* = \lim_{s\to\infty} \xi_i^{k_s},\ w_i^* = \lim_{s\to\infty} w_i^{k_s},\ \text{for all } i = 1, 2, \ldots, m, \tag{4.19}$$

$$\nu^* = \lim_{s\to\infty} \nu^{k_s}, \tag{4.20}$$

and we have

$$w^* := (w_1^*, w_2^*, \ldots, w_m^*) \in R_+^m \text{ and } \sum_{i\in I(x^*)} w_i^* = 1 \tag{4.21}$$

and

$$\nu^* = \sum_{i\in I(x^*)} w_i^* \xi_i^* \in \partial f(x^*). \tag{4.22}$$

Moreover, if $\lambda_* = 0$*, then* x^* *is a solution of the CFP.*

Proof. From Lemma 4.2, the sequence $\{\|x^k - z\|\}_{k=0}^{\infty}$ is nonincreasing, since $z \in Q \cap B(x^0, r/2)$, and, therefore, convergent. Hence, the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded and, thus, has accumulation points. Let $\{x^{p_s}\}_{s=0}^{\infty}$ be a convergent subsequence of $\{x^k\}_{k=0}^{\infty}$ such that $x^* = \lim_{s\to\infty} x^{p_s}$. The function f is continuous (since it is real-valued and convex on R^n), hence, it is bounded on bounded subsets of R^n. Therefore, the sequence $\{f(x^{p_s})\}_{s=0}^{\infty}$ converges to $f(x^*)$ and the sequence $\{f(x^k)\}_{k=0}^{\infty}$ is bounded. By (3.5), boundedness of $\{f(x^k)\}_{k=0}^{\infty}$ implies that the sequence $\{\lambda_k\}_{k=0}^{\infty}$ is bounded. Since, for every $i = 1, 2, \ldots, m$, the operator $\partial f_i : R^n \to 2^{R^n}$ is monotone, it is locally bounded (cf. Pascali and Sburlan [39, Theorem on p. 104]).

Consequently, there exists a neighborhood U of x^* on which all ∂f_i, $i = 1, 2, \ldots, m$, are bounded. Clearly, since $x^* = \lim_{s\to\infty} x^{p_s}$, the neighborhood U contains all but finitely many terms of the sequence $\{x^{p_s}\}_{s=0}^{\infty}$. This implies that the sequences $\{\xi_i^{p_s}\}_{s=0}^{\infty}$ are uniformly bounded and, therefore, the sequence $\{\nu^{p_s}\}_{s=0}^{\infty}$ is bounded too.

Therefore, there exist a subsequence $\{k_s\}_{s=0}^{\infty}$ of $\{p_s\}_{s=0}^{\infty}$ such that the limits in (4.18)–(4.20) exist. Obviously, the vector $w^* = (w_1^*, w_2^*, \ldots, w_m^*) \in R_+^m$, and, according to [10, Lemma 1], we also have $\sum_{i\in I(x^*)} w_i^* = 1$. This and (4.4) imply that $\nu^* = \sum_{i\in I(x^*)} w_i^* \xi_i^*$.

Observe that, since $\nu^{k_s} \in \partial f(x^{k_s})$ for all $s \geq 0$, and since ∂f is a closed mapping (cf. Phelps [40, Proposition 2.5]), we have that $\nu^* \in \partial f(x^*)$. Now, if $\lambda_* = 0$ then, according to (3.5), and the continuity of f, we deduce

$$0 \leq \max\{0, f(x^*)\} = \lim_{s\to\infty} \max\{0, f(x^{k_s})\} \leq \lim_{s\to\infty} \lambda_{k_s} M^2 = \lambda_* M^2 = 0, \tag{4.23}$$

which implies that $f(x^*) \leq 0$, that is, $x^* \in Q$. ■

Lemma 4.4 *If (4.8) is satisfied and, if at least one of the conditions (i) or (ii) of Theorem 4.1 holds, then any accumulation point of any sequence* $\{x^k\}_{k=0}^{\infty}$*, generated by Algorithm 3.1, belongs to* Q*.*

Proof. As noted above, when (4.8) is satisfied then the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded and, hence, it has accumulation points. Let x^* be such an accumulation point and let $\{k_s\}_{s=0}^{\infty}$ be the sequence of natural numbers associated with x^* whose existence is guaranteed by Lemma 4.3. Since, for any $z \in C \cap B(x^0, r/2)$, the sequence $\{\|x^k - z\|\}_{k=0}^{\infty}$ is convergent (cf. Lemma

4.2) we deduce that

$$\|x^* - z\| = \lim_{s\to\infty} \left\|x^{k_s} - z\right\| = \lim_{k\to\infty} \left\|x^k - z\right\| = \lim_{s\to\infty} \left\|x^{k_s+1} - z\right\| \tag{4.24}$$

$$= \|x^* - \lambda_* \nu^* - z\| . \tag{4.25}$$

This implies

$$\|x^* - z\|^2 = \|x^* - z\|^2 + \lambda_* \left(\lambda_* \|\nu^*\|^2 - 2\langle \nu^*, x^* - z\rangle\right), \tag{4.26}$$

that is,

$$\lambda_* \|\nu^*\|^2 - 2\langle \nu^*, x^* - z\rangle = 0, \tag{4.27}$$

for all $z \in C \cap B(x^0, r/2)$. We distinguish now between two possible cases.

Case I: Assume that condition (i) of Theorem 4.1 is satisfied. According to (4.27), the set $Q \cap B(x^0, r/2)$ is contained in the hyperplane

$$H := \left\{x \in R^n \mid \langle \nu^*, x\rangle = (1/2)\left(2\langle \nu^*, x^*\rangle - \lambda_* \|\nu^*\|^2\right)\right\}. \tag{4.28}$$

By *condition (i) of Theorem 4.1*, it follows that int $(Q \cap B(x^0, r/2)) \neq \emptyset$ and this is an open set contained in int H. So, unless $\nu^* = 0$ (in which case $H = R^n$), we have reached a contradiction because int $H = \emptyset$. Therefore, we must have $\nu^* = 0$. According to Lemma 4.3, we have $0 = \nu^* \in \partial f(x^*)$ which implies that x^* is a global minimizer of f. Consequently, for any $z \in Q$ we have $f(x^*) \leq f(z) \leq 0$, that is, $x^* \in Q$.

Case II: Assume that condition (ii) of Theorem 4.1 is satisfied. According to (4.27), we have

$$\lambda_* \|\nu^*\|^2 = 2\langle \nu^*, x^* - z\rangle . \tag{4.29}$$

By (3.5) and the definition of M we deduce that

$$2f(x^{k_s}) \geq \lambda_{k_s} M^2 \geq \lambda_{k_s} \left\|\nu^{k_s}\right\|^2, \tag{4.30}$$

for all integers $s \geq 0$. Letting $s \to \infty$ we get

$$2f(x^*) \geq \lambda_* M^2 \geq \lambda_* \|\nu^*\|^2 = 2\langle \nu^*, x^* - z\rangle , \tag{4.31}$$

where the last equality follows from (4.29). Consequently, we have

$$f(x^*) \geq \langle \nu^*, x^* - z\rangle , \quad \text{for all} \quad z \in Q \cap B(x^0, r/2). \tag{4.32}$$

Convexity of f implies that, for all $z \in Q \cap B(x^0, r/2)$,

$$-f(x^*) \leq \langle \nu^*, z - x^* \rangle \leq f(z) - f(x^*) \leq -f(x^*). \tag{4.33}$$

Therefore, we have that

$$-f(x^*) = \langle \nu^*, z - x^* \rangle = f(z) - f(x^*), \text{ for all } \quad z \in Q \cap B(x^0, r/2). \tag{4.34}$$

This cannot hold unless $f(z) = 0$, for all $z \in Q \cap B(x^0, r/2)$. Hence, using again the convexity of f, we deduce that, for all $z \in Q \cap B(x^0, r/2)$,

$$f'_+(x^*; z - x^*) \leq f(z) - f(x^*) = -f(x^*) = \langle \nu^*, z - x^* \rangle \leq f'_+(x^*; z - x^*). \tag{4.35}$$

This implies

$$f'_+(x^*; z - x^*) = \langle \nu^*, z - x^* \rangle = f(z) - f(x^*), \text{ for all } \quad z \in Q \cap B(x^0, r/2). \tag{4.36}$$

Since, by *condition (ii) of Theorem 4.1*, f is strictly convex, we also have (see [9, Proposition 1.1.4]) that

$$f'_+(x^*; z - x^*) < f(z) - f(x^*), \text{ for all } \quad z \in \left(Q \cap B(x^0, r/2)\right) \setminus \{x^*\}. \tag{4.37}$$

Hence, the equalities in (4.36) cannot hold unless $Q \cap B(x^0, r/2) = \{x^*\}$ and, thus, $x^* \in Q$. ■

The previous lemmas show that if (4.8) holds and if one of the conditions *(i)* or *(ii)* of Theorem 4.1 is satisfied, then the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ is bounded and all its accumulation points are in Q. In fact, the results above say something more. Namely, in view of Lemma 4.2, they show that if (4.8) holds and if one of the conditions *(i)* or *(ii)* of Theorem 4.1 is satisfied, then all accumulation points x^* of $\left\{x^k\right\}_{k=0}^{\infty}$ are contained in $Q \cap B(x^0, r)$ because all x^k are in $B(x^0, r)$ by (4.9). In order to complete the proof of Theorem 4.1, it remains to show that the following result is true.

Lemma 4.5 *Under the conditions of Theorem 4.1 any sequence* $\left\{x^k\right\}_{k=0}^{\infty}$*, generated by Algorithm 3.1, has at most one accumulation point.*

Proof. Since, by Lemma 4.4, $x^* \in Q$ (4.14) and (4.15) hold with $z = x^*$. Thus, the sequence $\left\{\left\|x^k - x^*\right\|\right\}_{k=0}^{\infty}$ converges, see (4.5). Hence, if $\left\{x^{k_p}\right\}_{p=0}^{\infty}$ is a subsequence of $\left\{x^k\right\}_{k=0}^{\infty}$ converging to x^*, then

$$\lim_{k\to\infty} \left\|x^k - x^*\right\| = \lim_{p\to\infty} \left\|x^{k_p} - x^*\right\| = 0, \tag{4.38}$$

making the entire sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converge to x^*. ■

Application of Theorem 4.1 depends on our ability to choose numbers M and r and a vector x^0 such that condition (4.8) is satisfied. We show below that this can be done when the functions f_i of the CFP (1.2) are quadratic or affine and there is some a priori known ball which intersects Q. In actual applications it may be difficult to a priori decide whether the CFP (1.2) has or does not have solutions. However, as noted above, Algorithm 3.1 is well-defined and will generate sequences $\left\{x^k\right\}_{k=0}^{\infty}$ no matter how the initial data M, r and x^0 are chosen. This leads to the question whether it is possible to decide if Q is empty or not by simply analyzing the behavior of sequences $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 3.1. A partial answer to this question is contained in the following result.

Corollary 4.6 *Suppose that the CFP (1.2) has no solution and that the envelope f is strictly convex. Then, no matter how the initial vector x^0 and the positive number M are chosen, any sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by Algorithm 3.1, has the following properties:*

(i) If $\left\{x^k\right\}_{k=0}^{\infty}$ is bounded and

$$\lim_{k\to\infty} \left\|x^{k+1} - x^k\right\| = 0, \tag{4.39}$$

then f has a (necessarily unique) minimizer and $\left\{x^k\right\}_{k=0}^{\infty}$ converges to that minimizer while

$$\lim_{k\to\infty} f(x^k) = \inf\{f(x) \mid x \in R^n\}. \tag{4.40}$$

(ii) If f has no minimizer then either the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ is unbounded or the sequence $\left\{\left\|x^{k+1} - x^k\right\|\right\}_{k=0}^{\infty}$ does not converge to zero.

Proof. Clearly, (ii) is a consequence of (i). In order to prove (i) observe that, since the CFP (1.2) has no solution, all values of f are positive. Also, if f has a minimizer, then this minimizer is unique because f is strictly convex.

If $\left\{x^k\right\}_{k=0}^{\infty}$ is bounded then it has an accumulation point, say, x^*. By Lemma 4.3 there exists a sequence of positive integers $\{k_s\}_{s=0}^{\infty}$ such that (4.18) and (4.21)–(4.22) are satisfied. Using Lemma 4.3 again, we deduce that, if the limit λ_* in (4.18) is zero, then the vector $x^* = \lim_{s\to\infty} x^{k_s}$ is a solution of the CFP (1.2), i.e., $f(x^*) \leq 0$, contradicting the assumption that the CFP (1.2) has no solution. Hence, $\lambda_* > 0$. By (3.5), (4.39) and (4.18) we have that

$$0 = \lim_{s\to\infty} \lambda_{k_s} \nu^{k_s} = \lambda_* \nu^*. \tag{4.41}$$

Thus, we deduce that $\nu^* = 0$. From (4.21)–(4.22) and [23, Proposition 2.3.12] we obtain

$$0 = \nu^* = \sum_{i \in I(x^*)} w_i^* \xi_i^* \in \partial f(x^*), \tag{4.42}$$

showing that x^* is a minimizer of f. So, all accumulation points of $\{x^k\}_{k=0}^{\infty}$ coincide because f has no more than one minimizer. Consequently, the bounded sequence $\{x^k\}_{k=0}^{\infty}$ converges and its limit is the unique minimizer of f. ■

Remark 4.7 *An easy adaptation of the proof of Corollary 4.6 shows that if the sequence* $\{x^k\}_{k=0}^{\infty}$ *has a bounded subsequence* $\{x^{k_t}\}_{t=0}^{\infty}$ *such that the limit* $\lim_{t\to\infty}(x^{k_t+1} - x^{k_t}) = 0$, *then all accumulation points of* $\{x^{k_t}\}_{t=0}^{\infty}$ *are minimizers of* f *(even if* f *happens to be not strictly convex).*

Remark 4.8 *The fact that for some choice of* x^0 *and* M *a sequence* $\{x^k\}_{k=0}^{\infty}$, *generated by Algorithm 3.1, has the property that* $\lim_{k\to\infty} f(x^k) = 0$, *does not imply that the CFP (1.2) has a solution. For example, take in (1.2)* $m = n = 1$ *and* $f_1(x) = e^{-x}$. *Clearly, in this case (1.2) has no solution and* $f = f_1$. *However, for* $x^0 = 0$, $M = 1$ *and* $\lambda_k = (3/2)f(x^k)$, *we have* $\lim_{k\to\infty} f(x^k) = 0$.

A meaningful implication of Corollary 4.6 is the following result.

Corollary 4.9 *Suppose that the CFP (2) has no solution and that* f *is strictly convex. Then, no matter how the initial vector* x^0 *and the positive number* M *are chosen in Algorithm 3.1, the following holds: If the series* $\sum_{k=0}^{\infty} \|x^k - x^{k+1}\|$ *converges, then the function* f *has a unique global minimizer and the sequence* $\{x^k\}_{k=0}^{\infty}$, *generated by Algorithm 3.1, converges to that minimizer while the sequence* $\{f(x^k)\}_{k=0}^{\infty}$ *converges to* $\inf\{f(x) \mid x \in R^n\}$.

Proof. When $\sum_{k=0}^{\infty} \|x^k - x^{k+1}\|$ converges to some number S we have

$$\|x^0 - x^{k+1}\| \le \sum_{\ell=0}^{k} \|x^\ell - x^{\ell+1}\| \le S, \tag{4.43}$$

for all integers $k \ge 0$. This implies that the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded and $\lim_{k\to\infty} \|x^k - x^{k+1}\| = 0$. Hence, by applying Corollary 4.6, we complete the proof. ■

Remark 4.10 *Finding an initial vector x^0, the radius r and a positive number M satisfying condition (4.2) when there is no a priori knowledge about the existence of a solution of the CFP can be quite easily done when at least one of the sets Q_i, say Q_{i_0}, is bounded and the functions f_i are differentiable. In this case it is sufficient to determine a vector x^0 and a positive number r large enough so that the ball $B(x^0, r)$ contains Q_{i_0}. Clearly, for such a ball condition (4.2) holds. Once the ball $B(x^0, r)$ is determined, finding a number $M \geq L(B(x^0, r))$ can be done by taking into account that the gradients of the differentiable convex functions $f_i : R^n \to R$ are necessarily continuous and, therefore, the numbers*

$$L_i = \sup\{\|\nabla f_i(x)\| \mid x \in B(x^0, r)\} \tag{4.44}$$

are necessarily finite. Since $L := \max\{L_i \mid 1 \leq i \leq m\}$ is necessarily a Lipschitz constant of f over $B(x^0, r)$, one can take $M = L$.

Remark 4.11 *The method of choosing x^0, r and M presented in Remark 4.10 does not require a priori knowledge of the existence of a solution of the CFP and can be applied even when Q is empty. In such a case one should compute, along the iterative procedure of Algorithm 3.1, the sums $S_k = \sum_{\ell=0}^{k} \left\|x^\ell - x^{\ell+1}\right\|$. Theorem 4.1 and Corollary 4.9 then provide the following insights and tools for solving the CFP:*

- *If along the computational process the sequence S_k remains bounded from above by some number S^* while the sequence $\{f(x^k)\}_{k=0}^{\infty}$ stabilizes itself asymptotically at some* ***positive*** *value, then the given CFP has no solution, but the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ still approximates a global minimum of f which may be taken as a surrogate solution of the given CFP.*

- *If along the computational process the sequence S_k remains bounded from above by some number S^* while the sequence $\{f(x^k)\}_{k=0}^{\infty}$ stabilizes itself asymptotically at some* ***nonpositive*** *value, then the given CFP has a solution, and the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ approximates such a solution.*

- *If along the computational process of the iterates, generated by Algorithm 3.1, either the sequence S_k grows towards ∞ or the sequence $\{f(x^k)\}_{k=0}^{\infty}$ does not stabilize itself, then one can conclude that the CFP has no solution and that f has no minimizer.*

5 Implementation of Algorithm 3.1 when f_i are linear or quadratic functions

In order to implement Algorithm 3.1 we have to determine numbers r and M required by the algorithm. We deal first with the problem of determining a number M such that

$$M \geq L(B(x^0, r)), \tag{5.1}$$

provided that an $r > 0$ is given. Recall that if $g : R^n \to R$ is a continuously differentiable function then, by Taylor's formula, we have that, whenever $x, y \in B(x^0, r)$, there exists a $u \in [x, y]$ such that

$$\begin{aligned} &| g(y) - g(x) | = | \langle \nabla g(u), y - x \rangle | \leq \|\nabla g(u)\| \, \|y - x\| \\ &\leq \|y - x\| \max\{\|\nabla g(u)\| \mid u \in B(x^0, r)\}. \end{aligned} \tag{5.2}$$

This shows that

$$\max\{\|\nabla g(u)\| \mid u \in B(x^0, r)\} \tag{5.3}$$

is a Lipschitz constant for g on $B(x^0, r)$. Suppose now that each function f_i is either linear or quadratic. Denote $I_1 = \{i \mid 1 \leq i \leq m, \ \ f_i \text{ is linear}\}$ and $I_2 = \{i \mid 1 \leq i \leq m, \ \ f_i \text{ is quadratic}\}$. Namely,

$$f_i(x) = \left\langle a^i, x \right\rangle + b_i, \quad \text{for all} \ \ i \in I_1, \tag{5.4}$$

with $a^i \in R^n \backslash \{0\}$ and $b_i \in R$, and

$$f_i(x) = \langle x, U_i x \rangle + \left\langle a^i, x \right\rangle + b_i, \ \text{for all} \ \ i \in I_2, \tag{5.5}$$

where $U_i = (u^i_{\ell,k})$ is a symmetric positive semidefinite $n \times n$ matrix, $a^i \in R^n \backslash \{0\}$ and $b_i \in R$. We have, of course,

$$\nabla f_i(x) = \begin{cases} a^i, & \text{if} \ \ i \in I_1, \\ 2U_i x + a^i, & \text{if} \ \ i \in I_2, \end{cases} \tag{5.6}$$

so that (5.3) can give us Lipschitz constants for each f_i over $B(x^0, r)$. Denote

$$L_i := \begin{cases} \left\|a^i\right\|, & \text{if } i \in I_1, \\ 2 \left\|U_i\right\|_\infty (\|x^0\| + r) + \left\|a^i\right\|, & \text{if } i \in I_2, \end{cases} \tag{5.7}$$

where $\left\|U_i\right\|_\infty = \max\left\{\left|u^i_{\ell,k}\right| \mid 1 \leq \ell, k \leq n\right\}$. Due to (4.6), this implies that $\cup_{x \in B(x^0,r)} \partial f(x) \subseteq B(0, L)$ where

$$L := \max\{L_i \mid 1 \leq i \leq m\}. \tag{5.8}$$

Taking $\xi \in \partial f(x)$ and $\zeta \in \partial f(y)$, for some $x, y \in B(x^0, r)$, we have

$$L \|x - y\| \geq \|\zeta\| \, \|x - y\| \geq \langle \zeta, y - x \rangle \geq f(y) - f(x) \\ \geq \langle \xi, y - x \rangle \geq - \|\xi\| \, \|x - y\| \geq -L \|x - y\| , \qquad (5.9)$$

which implies

$$|f(y) - f(x)| \leq L \|x - y\| , \text{ for all } \; x, y \in B(x^0, r). \qquad (5.10)$$

In other words, L is a Lipschitz constant of f over $B(x^0, r)$. Thus, given an $r > 0$, we can take M to be any number such that $M \geq L$. Note that choosing x^0 such that the corresponding r is small may speed up the computational process by reducing the number of iterations needed to reach a reasonably good approximate solution of the CFP. In general, determining a number r is straightforward when one has some information about the range of variation of the coordinates of some solutions to the CFP.

For instance, if one knows a priori that the solutions of the CFP are vectors $x = (x_j)_{j=1}^n$ such that

$$\ell_j \leq x_j \leq u_j, \; 1 \leq j \leq n, \qquad (5.11)$$

where, $\ell_j, u_j \in R$, for all j, then the set Q is contained in the hypercube of edge length $\delta = u_{\max} - \ell_{\min}$, whose faces are parallel to the axes of the coordinates, and centered at the point x^0 whose coordinates are $x_j^0 = \frac{1}{2}(\ell_{\min} + u_{\max})$, where

$$\ell_{\min} := \min\{\ell_j \mid 1 \leq j \leq n\} \text{ and } u_{\max} := \max\{u_j \mid 1 \leq j \leq n\}. \qquad (5.12)$$

Therefore, by choosing this x^0 as the initial point for Algorithm 3.1 and choosing $r = \sqrt{2}\delta$, condition (4.2) holds.

6 Computational results

In this section, we compare the performance of Algorithms 2.2, 2.4 and 3.1 by examining a few test problems. There are a number of degrees-of-freedom used to evaluate and compare the performance of the algorithms. These are the maximum number of iterations, the number of constraints, the lower and upper bounds of the box constraints, the values of the relaxation parameters, the initial values of the steering parameters and the steering sequence. In all our experiments, the steering sequence of Algorithm 2.4 assumed the form

$$\sigma_k = \frac{\sigma}{k+1} \tag{6.1}$$

with a fixed user-chosen constant σ. The main performance measure is the value of $f(x^k)$, plotted as a function of the iteration index k.

6.1 Test problem description

There are three types of constraints in our test problems: Box constraints, linear constraints and quadratic constraints. Some of the numerical values used to generate the constraints are uniformly distributed random numbers, lying in the interval $\tau = [\tau_1, \tau_2]$, where τ_1 and τ_2 are user-chosen pre-determined values.

The n box constraints are defined by

$$\ell_j \leq x_j \leq u_j, \quad j = 1, 2, \ldots, n \tag{6.2}$$

where $\ell_j, u_j \in \tau$ are the lower and upper bounds, respectively. Each of the N_q quadratic constrains is generated according to

$$G_i(x) = \langle x, U_i x \rangle + \langle v^i, x \rangle + \beta_i, \quad i = 1, 2, \ldots, N_q. \tag{6.3}$$

Here U_i is are $n \times n$ matrices defined by

$$U_i = W_i \Lambda_i W_i^T, \tag{6.4}$$

the $n \times n$ matrices Λ_i are diagonal, positive definite, given by

$$\Lambda_i = \operatorname{diag}\left(\delta_1^i,\, \delta_2^i, \ldots, \delta_n^i\right) \tag{6.5}$$

where $0 < \delta_1^i \leq \delta_2^i \leq \ldots \leq \delta_n^i \in \tau$ are generated randomly. The matrices W_i are generated by orthonormalizing an $n \times n$ random matrix, whose entries lie in the interval τ. Finally, the vector $v^i \in R^n$ is constructed so that all its components lie in the interval τ and similarly the scalar $\beta_i \in \tau$. The N_ℓ linear constraints are constructed in a similar manner according to

$$L_i(x) = \langle y^i, x \rangle + \gamma_i, \quad i = 1, 2, \ldots, N_\ell. \tag{6.6}$$

Thus, the total number of constraints is $n + N_q + N_\ell$.

Table 6.1 summarizes the test cases used to evaluate and compare the performance of Algorithms 2.2, 2.4 and 3.1. In these eight experiments, we modified the value of the constant σ in (6.1), the interval τ, the number of constraints, the number of iterations, and the relative tolerance ε, used as a termination criterion between subsequent iterations.

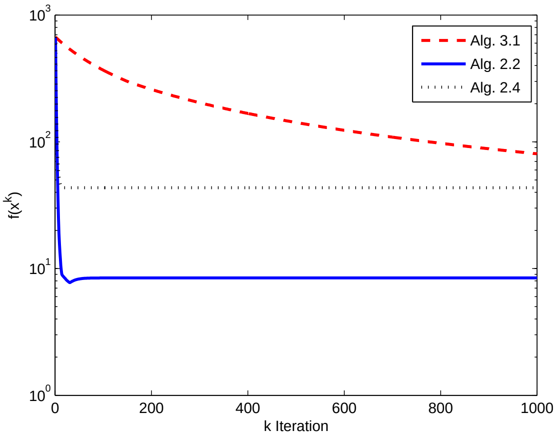
Case	$\alpha/\sigma/\lambda$	τ	n	N_q	N_ℓ	Iterations	ε
1	1.1		3	5	5	1,000	
2	1.1		3	5	5	1,000	
3	1.98		3	5	5	1,000	
4	1.98	$[-0.1, 0.1]$	30	50	50	1,000	0.1
5	1.98	$[-10, 10]$	30	50	50	100,000	0.1
6	2	$[-0.1, 0.1]$	30	50	50	1,000	0.1
7	3	$[-10, 10]$	3	5	5	1,000	0.1
8	5	$[-0.1, 0.1]$	3	5	5	1,000	0.1

Table 1: Test cases for performance evaluation

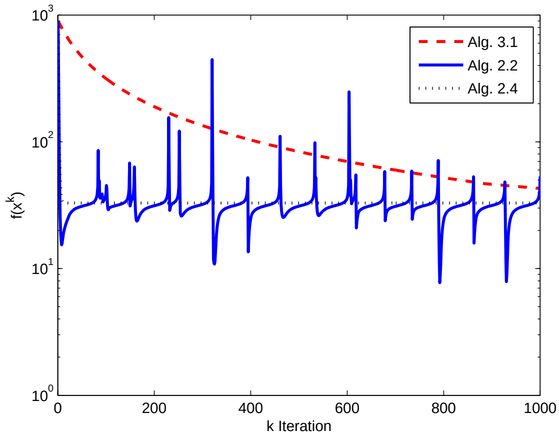
In Table 6.1, Cases 1 and 2 represent small-scale problems, with a total of 13 constraints, whereas Cases 4–6 represent mid-scale problems, with a total of 130 constraints. Cases 6–8 examine the case of over relaxation, wherein the initial steering (relaxation) parameter is at least 2.

6.2 Results

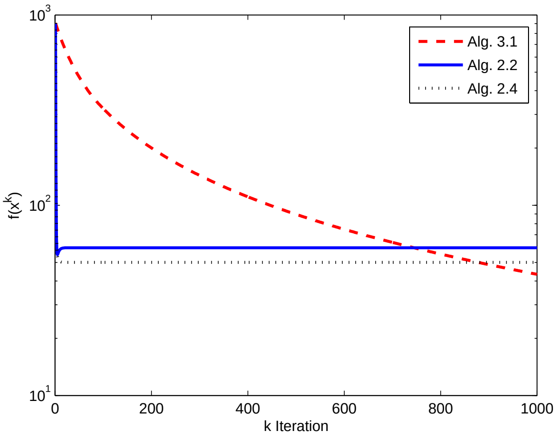
The results of our experiments are depicted in Figures 1–3. The results of Cases 1–3 are shown in Figures 1(a)–1(c), respectively. It is seen that in Case 1 Algorithm 2.2 has better initial convergence than Algorithms 2.4 and 3.1. However, in Case 2, Algorithm 2.4 yields fast and smooth initial behavior, while Algorithm 2.2 oscillates chaotically. Algorithm 3.1 exhibits slow initial convergence, similarly to Case 1. In Case 3, Algorithm 3.1 supersedes the performance of the other two algorithm, since it continues to converge toward zero. However, none of the algorithms detects a feasible solution, since none converged to the tolerance threshold after the maximum number of iterations.



(a) Case 1

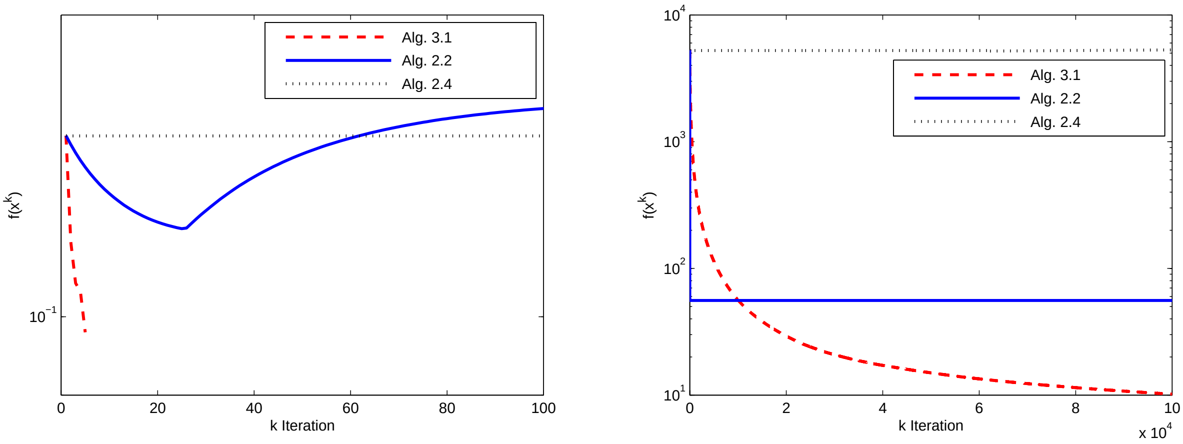


(b) Case 2



(c) Case 3

Figure 1: Simulation results for a small-scale problem, comparing Algorithms 2.2, 2.4 and 3.1.



(a) Case 4 (b) Case 5

Figure 2: Simulation results for a mid-scale problem, comparing Algorithms 2.2, 2.4 and 3.1.

The mid-sized problems of Cases 4 and 5 are depicted by Figures 2(a) and 2(b). Figure 2(a) shows that Algorithm 3.1 detects a feasible solution, while both Algorithms 2.2 and 2.4 fail to detect such a solution. Figure 2(b) shows a phenomenon similar to the one observed in the small-scale problem: Algorithm 3.1 continues to seek for a feasible solution, while Algorithms 2.2 and 2.4 converge to a steady-state, indicating failure to detect a feasible solution.

In the experiments, Cases 6–8, Algorithm 3.1 outperforms the other algorithms, arriving very close to finding feasible solutions. It should be observed that the behavior of Algorithm 3.1 observed above is the result of the way in which the relaxation parameters λ_k are self-regulating their sizes. In Algorithm 3.1 the relaxation parameter λ_k can be chosen (see Equation (3.6)) to be any number of the form

$$\lambda_k = \beta_k \frac{\max(0, f(x^k))}{M^2} + 2(1-\beta_k)\frac{\max(0, f(x^k))}{M^2} = (2-\beta_k)\frac{\max(0, f(x^k))}{M^2}, \tag{6.7}$$

where β_k runs over the interval $[0,1]$. Consequently, the size of λ_k can be very close to zero when x^k is close to a feasible solution (no matter how β_k is chosen in $[0,1]$). Also, λ_k may happen to be much larger then 2 when x^k is far from a feasible solution and the number $f(x^k)$ is large enough (note that $2-\beta_k$

stays between 1 and 2). So, Algorithm 3.1 is naturally under- or over- relaxing the computational process according to the relative position of the current iterate x^k to the feasibility set of the problem. As our experiments show, in some circumstances, this makes Algorithm 3.1 behave better then the other procedures we compare it with. At the same time, the self-regulation of the relaxation parameters, which is essential in Algorithm 3.1, may happen to reduce the initial speed of convergence of this procedure, that is, Algorithm 3.1 may require more computational steps in order to reach a point x^k which is close enough to the feasibility set such that its self-regulatory features to be really advantageous for providing a very precise solution of the given problem (which the other procedures may fail to do since they may became stationary in the vicinity of the feasibility set). Another interesting feature of Algorithm 3.1, which differentiates it from the other algorithms we compare it with, is its essentially non-simultaneous character: Algorithm 3.1 does not necessarily ask for $w_i^k > 0$ for all $i \in \{1, ..., m\}$. The set of positive weights w_i^k which condition the progress of the algorithm at step k essentially depends on the current iterate x^k (see (3.4)) and allows reducing the number of subgradients needed to be computed at each iterative step (in fact, one can content himself with only one $w_i^k > 0$ and, thus, with a single subgradient ξ_i^k). This may be advantageous in cases when computing subgradients is difficult and, therefore, time consuming.

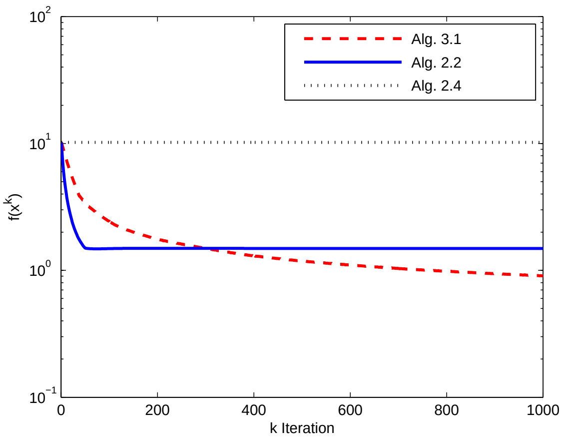
The main observations can be summarized as follows:

1. Algorithm 3.1 exhibits faster initial convergence than the other algorithms in the vicinity of points with very small $f(x^k)$. When the algorithms reach points with small $f(x^k)$ values, then Algorithm 3.1 tends to further reduce the value of $f(x^k)$, while the other algorithms tend to converge onto a constant steady-state value.

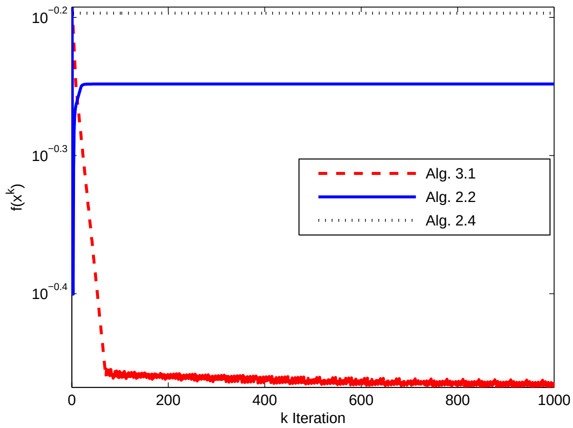
2. The problem dimensions in our experiments have little impact on the behavior of the algorithms.

3. All the examined small-scale problems have no feasible solutions. This can be seen from the fact that all three algorithms stabilize around $f(x^k) = 50$.

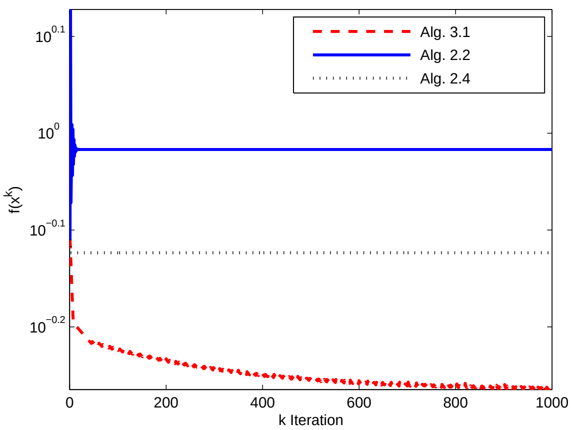
4. The chaotic oscillations of Algorithm 2.2 in the underrelaxed case is due to the fact that this algorithm has no internal mechanism to self-adapt its progress to the distance between the current iterates and



(a) Case 6



(b) Case 7



(c) Case 8

Figure 3: Simulation results for small- and mid-scale problems with overrelaxation, comparing Algorithms 2.2, 2.4 and 3.1.

the sets whose intersections are to be found. This phenomenon can hardly happen in Algorithm 3.1 because its relaxation parameters are self-adapting to the size of the current difference between successive iterations. This is an important feature of this algorithm. However, this feature also renders it somewhat slower than the other algorithms.

5. In some cases, Algorithms 2.2 and 2.4 indicate that the problem has no solution. In contrast, Algorithm 3.1 continues to make progress and seems to indicate that the problem has a feasible solution. This phenomenon is again due to the self-adaptation mechanism, and can be interpreted in one of the following ways: (a) The problem indeed has a solution but Algorithms 2.2 and 2.4 are unable to detect it (because they stabilize too fast). Algorithm 3.1 detects a solution provided that it is given enough running time; (b) The problem has no solution and then Algorithm 3.1 will stabilize close to zero, indicating that the problem has no solution, but this may be due to computing (round-off) errors. Thus, a very small perturbation of the functions involved in the problem may render the problem feasible.

7 Conclusions

We have studied here mathematically and experimentally subgradient projections methods for the convex feasibility problem. The behavior of the fully simultaneous subgradient projections method in the inconsistent case is not known. Therefore, we studied and tested two options. One is the use of steering parameters instead of relaxation parameters and the other is a variable relaxation strategy which is self-adapting. Our small-scale and mid-scale experiments are not decisive in all aspects and call for further research. But one general feature of the algorithm with the self-adapting strategical relaxation is its stability (non-oscillatory) behavior and its relentless improvement of the iterations towards a solution in all cases.

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