Regularization and Resolution of Monotone Variational Inequalities with Operators Given by Hypomonotone Approximations

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Abstract

We study the stability in reflexive, smooth and strictly convex Banach spaces of the classical Tikhonov-Browder operator regularization method for monotone variational inequalities with data perturbations. We prove that this regularization method is stable even if the perturbed data contain operators which fail to be monotone, but are strictly hypomonotone. We use this stability result in order to prove convergence in smooth uniformly convex spaces of an iterative algorithm for approximating solutions of monotone variational inequalities. The algorithm we analyze involves in computations the perturbed data only and it converges even if the perturbed operators are not necessarily monotone, but strictly hypomonotone.

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1 Introduction

Let X be a reflexive, smooth, strictly convex Banach space and let $A : X \to 2^{X^*}$ be a monotone operator which is demiclosed and convex valued on the interior of its domain. Let Ω be a nonempty, closed, convex subset of $\mathrm{Int}\,(\mathrm{Dom}\, A)$ and let $f \in X^*$. We consider the variational inequality

$$\langle Ax - f, y - x \rangle \geq 0, \ \forall y \in \Omega. \tag{1}$$

By a *solution* of the variational inequality (1) we mean a vector $x^* \in \Omega$ such that, for some $\xi^* \in Ax^*$, we have

$$\langle \xi^* - f, y - x^* \rangle \geq 0, \ \forall y \in \Omega. \tag{2}$$

We denote by $\mathcal{S}(A, f, \Omega)$ the set of solutions of (1). We assume that the problem data A, f and Ω are given by sequences of approximations $\{A_k\}_{k\in\mathbb{N}}$, $\{f^k\}_{k\in\mathbb{N}}$ and $\{\Omega_k\}_{k\in\mathbb{N}}$, respectively.

The problem of finding a solution of (1) may be ill-posed in the sense that either $\mathcal{S}(A, f, \Omega)$ is not a singleton and/or small perturbations of the problem data lead to relatively large perturbations of the solution set. In such situations, solving variational inequalities

$$\left\langle A_k x - f^k, y - x \right\rangle \geq 0, \ \forall y \in \Omega_k \tag{3}$$

instead of (1) makes little sense because the problem of finding solutions of (3) can be as ill-posed as the original problem and its solutions may be far from the set $\mathcal{S}(A, f, \Omega)$. The aim of this paper is two folds. First we study the stability of a regularization method for situations in which the variational inequality (1) is ill-posed. Second, we use the stability properties of that regularization method in order to establish convergence of an iterative algorithm for finding solutions of the variational inequality (1) by using in computations the approximative data only.

The regularization method we consider in this paper goes back to Tikhonov [50] and Browder [19]. Its underlying idea is to associate to the variational inequality (1) the perturbed regularized variational inequalities

$$\left\langle (A_k + \alpha_k J)\, x - f^k, y - x \right\rangle \geq 0, \forall y \in \Omega_k, \tag{4}$$

where J is the normalized duality mapping and $\{\alpha_k\}_{k\in\mathbb{N}}$ is a sequence of positive real numbers (regularization parameters) such that $\lim_{k\to\infty} \alpha_k = 0$. It is well known that, under various conditions concerning the original and the approximative data, each variational inequality (4) has a unique solution x^k and that the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ converges weakly, and sometimes strongly, to a solution of (1) – see [50], [19], [51], [37], [38], [29], [30], [2], [31], [32], [46], [33], [34], [4], [35], [36], [16], [13], [28], [8], [9]. In all these works it is presumed that the approximating operators A_k preserve the basic continuity and monotonicity features of the operator A involved in the original variational inequality (1). In this paper we consider the situation in which, *by contrast to* A, *the operators* A_k *may fail to be monotone, although they are required to satisfy a less demanding property which we call "strong hypomonotonicity" in that follows.* Precisely, we study the above described regularization method under the following assumptions on the approximative data:

(**A1**) For each $k\in\mathbb{N}$, we have that $\Omega_k\subseteq \operatorname{Int}(\operatorname{Dom} A_k)$, the operator A_k is demiclosed on Ω_k and, for any $x\in\Omega_k$, the set $A_k x$ is convex.

(**A2**) For each $k\in\mathbb{N}$, there exists a number $h_k\geq 0$ such that A_k is *strongly* h_k*-hypomonotone on* Ω_k, that is, for any $x,y\in\Omega_k$, the following inequality holds

$$\langle A_k x - A_k y, x-y\rangle \geq -h_k\left(\|x\|-\|y\|\right)^2; \tag{5}$$

(**A3**) There exist sequences of positive real numbers $\{\alpha_k\}_{k\in\mathbb{N}}$, $\{\delta_k\}_{k\in\mathbb{N}}$, $\{\omega_k\}_{k\in\mathbb{N}}$ and $\{\tau_k\}_{k\in\mathbb{N}}$ and there exist three bounded on bounded sets functions $a,b:X\to\mathbb{R}_+$ and $c:X^*\to\mathbb{R}_+$ such that $\lim_{k\to\infty}\alpha_k=0$, $h_k<\alpha_k$ for all $k\in\mathbb{N}$,

$$\lim_{k\to\infty}\frac{\delta_k+\omega_k+\tau_k+h_k}{\alpha_k}=0, \tag{6}$$

$$\lim\sup_{\|x\|\to\infty}\frac{b(x)}{\|x\|^2}<\infty, \tag{7}$$

and such that, for each $k\in\mathbb{N}$, the following proximity requirements are satisfied:

(**A3-1**) $f^k\in X^*$ and $\left\|f-f^k\right\|_*\leq\delta_k$;

(**A3-2**) For any $x\in\Omega$, there exists $z^k\in\Omega_k$ such that

$$\left\|x-z^k\right\|\leq a(x)\omega_k, \tag{8}$$

and such that, for all $\zeta\in Ax$, we have

$$\operatorname{dist}_*(\zeta, A_k z^k)\leq c(\zeta)\tau_k; \tag{9}$$

(**A3-3**) For any $v^k \in \Omega_k$, there exists $v \in \Omega$ such that

$$\left\| v^k - v \right\| \leq b(v^k)\omega_k. \tag{10}$$

Assumption (**A1**) is commonly used in the study of the stability of the regularization method we discuss here. If the operators A_k are maximal monotone, as they are often presumed to be, and if $\Omega_k \subseteq \operatorname{Int}(\operatorname{Dom} A_k)$, then the other requirements of (**A1**) are automatically satisfied. The strong hypomonotonicity, defined by (5) in assumption (**A2**), is a strenghtened form of the hypomonotonicity concept used by Rockafellar [44] (see also [43, p. 548]). Its importance in our context steams from the fact that if A_k are strongly h_k-hypomonotone with $h_k < \alpha_k$, then the operators $A_k + \alpha_k J$ are strictly monotone and coercive – see Lemma 2.1 below. Assumption (**A3**) is a weaker version of the often used requirement (see, for instance, [13], [10], [11]) that $\operatorname{Dom} A_k = \operatorname{Dom} A$ for all $k \in \mathbb{N}$ and the operators A_k converge to A uniformly on bounded subset of $\operatorname{Dom} A$. Similar conditions have been previously used in [4], and [8]. It implies graphical convergence to $A + N_\Omega$ of the operators $A_k + N_{\Omega_k}$ as shown in [7, Section 2].

We prove (see Theorem 2.1 below) that the assumptions (**A1**)-(**A3**) are sufficient to ensure well definedness, as well as convergence, of the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ to a solution of the original variational inequality (1). Using this fact, we establish (see Theorem 4.1) convergence of an iterative algorithm for finding solutions of the variational inequality (1). The algorithm we consider here (see Section 4) is the product of analyzing and refining a series of procedures for solving equations and variational inequalities due to Alber [1], Bruck [20], [21], Bakushinskii [15], Vainikko [52], Ryazantseva [45], [46], [47], [48], Alber and Reich [14], Alber, Kartsatos and Litsyn [10], and Alber and Nashed [11]. It has the important feature that it involves in computations the approximative data only. The information on the original data needed for the application of the algorithm is exclusively qualitative (knowledge of the monotonicity and demiclosedness of A and of the existence of solutions to (1)). Convergence of variants of this algorithm was proved before for monotone operators A_k in uniformly convex and uniformly smooth Banach spaces whose moduli of uniform convexity were subjected to some restrictive requirement and under proximity assumptions somewhat stronger than those we make in this paper (compare with [10, Theorem 1] and [11, Theorem 3.3]). We prove that the algorithm we propose in Section 4 converges in any smooth uniformly convex space and, by contrast to previously studied variants, our convergence result does not require monotonicity of A_k, but strong hypomonotonicity – see Theorem 4.1. Assumptions (**A1**)-(**A3**) by themselves are not sufficient for ensuring convergence of our algorithm. In addition to these conditions we had to impose conditions (**A4**) and (**A5**). Condition (**A4**) is a uniform boundedness on bounded sets requirement for the operators A_k. Condition (**A5**) requires that the relaxation parameters ε_k involved in the algorithm be chosen in a manner which is compatible with the geometric structure of the underlaying space X.

The stability and convergence results (Theorems 2.1 and 4.1) proved in this paper were made possible by several newly established properties of the strongly h-hypomonotone operators (see Lemmas 2.1, 2.2 and 2.3) and by the improved "proximity theorem" (see Theorem 3.1) which may be of interest by themselves. In the context, the concept of epi-quasi-inverse of a real function, introduced and studied by Penot and Volle [40], proved to be very useful.

2 Operator Regularization of Variational Inequalities

Let X be a reflexive, smooth, strictly convex Banach space and let $A : X \to 2^{X^*}$ be a monotone operator which is demiclosed and convex valued on the interior of its domain. Suppose that Ω be a nonempty, closed, convex subset of $\operatorname{Int}(\operatorname{Dom} A)$ and let $f \in X^*$. In this section we prove that, in these circumstances, the Tichonov-Browder regularization method is stable, that is, the following result holds:

Theorem 2.1. *Consider the variational inequality* (1) *whose data* A, f *and* Ω *are given by sequences of approximations* $\{A_k\}_{k\in\mathbb{N}}$, $\{f^k\}_{k\in\mathbb{N}}$ *and* $\{\Omega_k\}_{k\in\mathbb{N}}$, *respectively.*

(i) *If conditions* (**A1**) *and* (**A2**) *are satisfied, and if* $h_k < \alpha_k$ *for all* $k \in \mathbb{N}$, *then, for each* $k \in \mathbb{N}$, *the variational inequality* (4) *has a unique solution* x^k.

(ii) *If conditions* (**A1**)-(**A3**) *are satisfied, then the variational inequality* (1) *has solutions if and only if the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *is bounded. In this case, the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *converges weakly to the minimal norm solution of* (1). *If, in addition to the previous requirements, the space* X *has the Kadeč-Klee property, then the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *converges strongly.*

The proof of this result consists of a sequence of lemmas presented below. We start our proof by establishing the following technical result:

Lemma 2.1. *If the operator* $B : X \to 2^{X^*}$ *is strongly* h*-hypomonotone on the nonempty subset* Λ *of* $\operatorname{Dom} B$ *and if* $\alpha \in (h, \infty)$, *then the operator*

$B + \alpha J$ is strictly monotone on Λ. In these circumstances, the operator $B + \alpha J$ is also coercive on Λ in the sense that, if $z \in \Lambda$, then

$$\lim_{n\to\infty} \frac{\langle \zeta^n + \alpha J y^n, y^n - z\rangle}{\|y^n\|} = \infty,$$

for any sequence $\{(y^n, \zeta^n)\}_{n\in\mathbb{N}}$ with $y^n \in \Lambda$, $\zeta^n \in By^n$ for all $n \in \mathbb{N}$ and such that $\|y^n\| \to \infty$.

Proof: If $x, y \in \Lambda$, then

$$\begin{aligned} &\langle (B+\alpha J)\,x - (B+\alpha J)\,y, x-y\rangle \\ &= \langle Bx - By, x-y\rangle + \alpha\,\langle Jx - Jy, x-y\rangle \\ &\geq -h\,(\|x\| - \|y\|)^2 + \alpha\,\langle Jx - Jy, x-y\rangle \\ &\geq -h\,(\|x\| - \|y\|)^2 + \alpha\,(\|x\| - \|y\|)^2 . \end{aligned} \tag{11}$$

Suppose that $x \neq y$. If $\|x\| = \|y\|$, then the first inequality in (11) implies that

$$\langle (B+\alpha J)\,x - (B+\alpha J)\,y, x-y\rangle \geq \alpha\,\langle Jx - Jy, x-y\rangle > 0,$$

where the strict inequality results from the strict monotonicity of J. If $\|x\| \neq \|y\|$, then the last inequality in (11) implies that

$$\langle (B+\alpha J)\,x - (B+\alpha J)\,y, x-y\rangle \geq (\alpha - h)\,(\|x\| - \|y\|)^2 > 0 \tag{12}$$

because $\alpha > h$. Hence, $B + \alpha J$ is strictly monotone.

In order to prove that $B+\alpha J$ is coercive, fix $z \in \Lambda$ and $\xi \in Bz$. Suppose that $y \in \Lambda$ and $\zeta \in By$. Then we have that

$$\begin{aligned} \langle \zeta + \alpha J y, y - z\rangle &= \langle \zeta + \alpha J y - \xi - \alpha J z, y - z\rangle \\ &\quad + \langle \xi + \alpha J z, y - z\rangle \\ &\geq (h - \alpha)\,(\|y\| - \|z\|)^2 \\ &\quad - \|\xi + \alpha J z\|_* \,\|y - z\| , \end{aligned}$$

where the inequality is results from (12). This implies that, whenever $y^n \neq 0$ and $\zeta^n \in By^n$, we have

$$\begin{aligned} \frac{\langle \zeta^n + \alpha J y^n, y^n - z\rangle}{\|y^n\|} &\geq (h - \alpha)\,\|y^n\|\left(1 - \frac{\|z\|}{\|y^n\|}\right)^2 \\ &\quad - \|\xi + \alpha J z\|_* \left(1 + \frac{\|z\|}{\|y^n\|}\right). \end{aligned}$$

Letting $n \to \infty$ in this inequality, the coercivity condition results. $\square$

The following lemma shows that maximal monotone extensions of a demiclosed, convex valued, monotone operator B may differ at points of the boundary of $\operatorname{Dom} B$ only.

Lemma 2.2: *Let $B : X \to 2^{X^*}$ be a monotone demiclosed operator. If $x \in \operatorname{Int}(\operatorname{Dom} B)$ and if Bx is convex, then for any maximal monotone extension $\bar{B}$ of B we have $\bar{B}x = Bx$.*

Proof. The operator B is monotone and, therefore, it is locally bounded at x. Consequently, if $\{x^n\}_{k\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} x^n = x$, and if $\{\zeta^n\}_{k\in\mathbb{N}}$ is a sequence such that $\zeta^n \in Bx^n$ for all $n \in \mathbb{N}$, then $\{\zeta^n\}_{n\in\mathbb{N}}$ is bounded. Since X^* is reflexive, it results that $\{\zeta^n\}_{n\in\mathbb{N}}$ has weak accumulation points. From the demiclosedness of B it follows that any such weak accumulation point belongs to Bx. Denote by Rx the closed convex hull of the set of weak accumulation points of all sequences $\{\zeta^n\}_{n\in\mathbb{N}}$ as described above. The set Bx is convex. Due to the demiclosedness of B, the set Bx is weakly closed and, therefore, closed. Hence, we have that $Rx \subseteq Bx$. Let $\bar{B}$ be a maximal monotone extension of B. Obviously, $Rx \subseteq Bx \subseteq \bar{B}x$. We claim that the inclusion $Rx \supseteq \bar{B}x$ holds too. Suppose by contradiction that this is not the case. Then, there exists $\eta \in \bar{B}x$ such that $\eta \notin Rx$. According to the strong separation theorem (see, for instance, [26, p. 64]), there exists a vector $u \in X$ such that

$$\langle \zeta - \eta, u \rangle < 0, \quad \forall \zeta \in Rx. \tag{13}$$

Let $\{t_n\}_{k\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero such that, for any $n \in \mathbb{N}$, we have that $z^n := x + t_n u \in \operatorname{Dom} B$. For each $n \in \mathbb{N}$ let η^n be an arbitrary element of Bz^n. The sequence $\{\eta^n\}_{n\in\mathbb{N}}$ is bounded because $\lim_{n\to\infty} z^n = x$ and B is locally bounded at x. Thus, there exists a subsequence $\left\{\eta^{i_n}\right\}_{n\in\mathbb{N}}$ of $\{\eta^n\}_{n\in\mathbb{N}}$ which converges weakly to some $\bar{\eta}$ in X^*. Clearly, $\bar{\eta} \in Rx$. By the monotonicity of $\bar{B}$ we have

$$\left\langle \eta^{i_n} - \eta, z^{i_n} - x \right\rangle \geq 0, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\left\langle \eta^{i_n} - \eta, u \right\rangle \geq 0, \quad \forall n \in \mathbb{N}.$$

Letting here $n \to \infty$ we obtain that $\langle \bar{\eta} - \eta, u \rangle \geq 0$ and this contradicts (13). $\square$

Observe that, under the assumptions of Theorem 2.1(i), the operators A_k are demiclosed, strongly h_k-hypomonotone and convex valued on the closed, convex, nonempty subset Ω_k of $\operatorname{Int}(\operatorname{Dom} A_k)$. Therefore, application of the following lemma completes the proof of Theorem 2.1(i).

Lemma 2.3. *Suppose that the operator $B : X \to 2^{X^*}$ is demiclosed, convex valued and strongly h-hypomonotone on the nonempty, closed, convex set Λ contained in* $\operatorname{Int}(\operatorname{Dom} B)$ *and that $\alpha \in (h, \infty)$. Then the variational inequality*

$$\langle (B + \alpha J)\, x - f, y - x \rangle \geq 0, \quad \forall y \in \Lambda, \tag{14}$$

has a unique solution.

Proof. According to Lemma 2.1, the operator $B+\alpha J$ is strictly monotone on Λ. The following standard argument shows that, since $B + \alpha J$ is strictly monotone on Λ, the variational inequality (14) can not have more than one solution. Suppose by contradiction that x, $x' \in \Lambda$ are different solutions of (14). Then, for some $\upsilon \in Bx'$, we have

$$\left\langle \upsilon + \alpha J x' - f, y - x' \right\rangle \geq 0, \ \forall y \in \Lambda.$$

In particular, we have

$$\left\langle \upsilon + \alpha J x' - f, x - x' \right\rangle \geq 0.$$

Similarly, for some $\xi \in Bx$ we have

$$\left\langle \xi + \alpha J x - f, x' - x \right\rangle \geq 0.$$

Summing up the two inequalities we deduce that

$$\left\langle (\xi + \alpha J x) - \left(\upsilon + \alpha J x'\right), x' - x \right\rangle \geq 0.$$

Since the operator $B + \alpha J$ is monotone, this implies

$$\left\langle (\xi + \alpha J x) - \left(\upsilon + \alpha J x'\right), x - x' \right\rangle = 0.$$

The operator $B+\alpha J$ is strictly monotone and, consequently, the last equality cannot hold unless $x = x'$, hence, a contradiction.

We are going to prove now that the variational inequality (14) has at least one solution. To this end, observe that the solution set of (14) and the solution set of the inclusion

$$f \in (B + \alpha J + N_\Lambda)\, x, \tag{15}$$

where N_Λ stands for the normal cone operator of the set Λ, coincide. The operator $B + \alpha J$ is demiclosed on $\Lambda \subseteq \operatorname{Int}(\operatorname{Dom} B) = \operatorname{Int}(\operatorname{Dom} B + \alpha J)$.

Clearly, $B + \alpha J$ is also convex valued on Λ because B is so. Let $\bar{B}$ be a maximal monotone extension of B. Then $\bar{B} + \alpha J$ is a maximal extension of $B + \alpha J$. According to Lemma 2.2, we have that $(B + \alpha J)\,x = \left(\bar{B} + \alpha J\right) x$ for all $x \in \Lambda$. Hence, the inclusion (15) is equivalent to the inclusion

$$f \in \left(\bar{B} + \alpha J + N_\Lambda\right) x \tag{16}$$

where the operator $\bar{B}+\alpha J+N_\Lambda$ is maximal monotone. As shown above, the operator $B+\alpha J$ is coercive on Λ. This implies that $\bar{B}+\alpha J+N_\Lambda$ is coercive too. Hence, Corollary 32.27 in [53] applies to the inclusion (16) and it shows that this inclusion has solution. Clearly, that implies that the equivalent inclusion (15) has solution and, consequently, the variational inequality (14) has solution too. □

From now and until the end of this section, we assume that the variational inequality (1) has solutions and that conditions (**A1**)-(**A3**) are satisfied. By the already proved Theorem 2.1(i), under these conditions, the variational inequality (4) has a unique solution x^k. The following lemma is a first step in the proof of Theorem 2.1(ii).

Lemma 2.4. *If the variational inequality* (1) *has at least one solution, then the sequence* $\left\{x^k\right\}_{k\in\mathbb{N}}$ *is bounded.*

Proof. Since x^k is a solution of (4), there exists a vector $\xi^k \in A_k x^k$ such that

$$\left\langle \xi^k + \alpha_k J x^k - f^k, x - x^k \right\rangle \geq 0, \quad \forall x \in \Omega_k. \tag{17}$$

Let x^* be a solution of (1) and let $\xi^* \in A x^*$ be such that (2) is satisfied. According to condition (**A3**) there exists $v^k \in \Omega_k$ and $\zeta^k \in A_k v^k$ such that

$$\left\| x^* - v^k \right\| \leq a(x^*)\omega_k, \tag{18}$$

and

$$\left\| \xi^* - \zeta^k \right\|_* \leq \tau_k c(\xi^*). \tag{19}$$

Also, for some $w^k \in \Omega$, we have that

$$\left\| x^k - w^k \right\| \leq b(x^k)\omega_k. \tag{20}$$

Taking $x = v^k$ in (17) and $y = w^k$ in (2) and adding the resulting inequalities we obtain

$$\left\langle \xi^* - f, w^k - x^* \right\rangle + \left\langle \xi^k + \alpha_k J x^k - f^k, v^k - x^k \right\rangle \geq 0.$$

This implies that

$$
\begin{aligned}
0 &\le \alpha_k \left\langle Jx^k, v^k - x^k \right\rangle - \left\langle \xi^k - \zeta^k, x^k - v^k \right\rangle + \left\langle \zeta^k - \xi^*, v^k - x^k \right\rangle \\
&+ \left\langle \xi^* - f, v^k - x^k + w^k - x^* \right\rangle + \left\langle f - f^k, v^k - x^k \right\rangle .
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left\langle Jx^k, x^k - v^k \right\rangle &= \left\langle Jx^k, x^k - x^* \right\rangle + \left\langle Jx^k, x^* - v^k \right\rangle \\
&\ge \left\|x^k\right\|^2 - \left\|x^k\right\| \|x^*\| - \left\|x^k\right\| \left\|x^* - v^k\right\| \\
&\ge \left\|x^k\right\|^2 - \left\|x^k\right\| \left(\|x^*\| + a(x^*)\omega_k\right),
\end{aligned}
$$

where the last inequality follows from (18). From the strong h_k-hypomonotonicity of A_k combined with the fact that $\xi^k \in A_k x^k$ and $\zeta^k \in A_k v^k$ we deduce that

$$
-\left\langle \xi^k - \zeta^k, x^k - v^k \right\rangle \le h_k \left\|x^k - v^k\right\|^2 .
$$

The last three relations combined give

$$
\begin{aligned}
&\alpha_k \left[\left\|x^k\right\|^2 - \left\|x^k\right\| \left(\|x^*\| + a(x^*)\omega_k\right) \right] \\
&\le h_k \left\|x^k - v^k\right\|^2 + \left\langle \zeta^k - \xi^*, v^k - x^k \right\rangle \\
&+ \left\langle \xi^* - f, v^k - x^k + w^k - x^* \right\rangle + \left\langle f - f^k, v^k - x^k \right\rangle
\end{aligned}
$$

By (19) and (20) we deduce that

$$
\begin{aligned}
&\alpha_k \left[\left\|x^k\right\|^2 - \left\|x^k\right\| \left(\|x^*\| + a(x^*)\omega_k\right) \right] \qquad (21) \\
&\le h_k \left\|x^k - v^k\right\|^2 + \tau_k c(\xi^*) \left(\left\|v^k - x^*\right\| + \left\|x^* - x^k\right\| \right) \\
&+ \left\langle \xi^* - f, v^k - x^k + w^k - x^* \right\rangle + \left\langle f - f^k, v^k - x^k \right\rangle \\
&\le h_k \left\|x^k - v^k\right\|^2 + \tau_k c(\xi^*) \left(a(x^*)\omega_k + \|x^*\| + \left\|x^k\right\| \right) \\
&+ \|\xi^* - f\|_* \left(\left\|v^k - x^*\right\| + \left\|w^k - x^k\right\| \right) + \left\|f - f^k\right\|_* \left\|v^k - x^k\right\| \\
&\le h_k \left\|x^k - v^k\right\|^2 + \tau_k c(\xi^*) \left(a(x^*)\omega_k + \|x^*\| + \left\|x^k\right\| \right) \\
&+ \|\xi^* - f\|_* \left(a(x^*) + b(x^k) \right) \omega_k + \delta_k \left(a(x^*)\omega_k + \|x^*\| + \left\|x^k\right\| \right).
\end{aligned}
$$

Suppose, by contradiction, that the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is unbounded. Then, for some unbounded subsequence of $\left\{x^k\right\}_{k\in\mathbb{N}}$, still denoted $\left\{x^k\right\}_{k\in\mathbb{N}}$, and for $k \in \mathbb{N}$ sufficiently large, we deduce from (21) that

$$
\begin{aligned}
& 1-\frac{\|x^*\|+a(x^*)\omega_k}{\|x^k\|} \\
& \leq \frac{h_k}{\alpha_k}\left(1+\frac{\left\|v^k\right\|^2}{\left\|x^k\right\|^2}+2\frac{\left\|v^k\right\|}{\left\|x^k\right\|}\right) \\
& +\frac{\tau_k}{\alpha_k}\frac{c(\xi^*)}{\left\|x^k\right\|^2}\left(a(x^*)\omega_k+\|x^*\|+\left\|x^k\right\|\right) \\
& +\|\xi^*-f\|_*\left(\frac{a(x^*)}{\left\|x^k\right\|^2}+\frac{b(x^k)}{\left\|x^k\right\|^2}\right)\frac{\omega_k}{\alpha_k} \\
& +\frac{\delta_k}{\alpha_k}\left(\frac{a(x^*)}{\left\|x^k\right\|^2}\omega_k+\frac{\|x^*\|}{\left\|x^k\right\|^2}+\left\|x^k\right\|^{-1}\right).
\end{aligned}
\tag{22}
$$

Taking lim sup on both sides of (22), observing that the sequence $\left\{v^k\right\}_{k\in\mathbb{N}}$ is bounded and using (7), we obtain a contradiction. □

Lemma 2.4 shows that, if the variational inequality (1) has solutions, then the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is bounded. Since the space X is reflexive, if the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is bounded, then it has weakly convergent subsequences. Therefore, the following lemma shows that the variational inequality (1) has solutions if and only if the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is bounded.

Lemma 2.5: *If the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is bounded, then any weak accumulation point of the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is a solution of* (1).

Proof. Let $\left\{x^{i_k}\right\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ and let $\bar{x}$ be the weak limit of this subsequence. According to (20), for each $k \in \mathbb{N}$, there exists $w^{i_k} \in \Omega$ such that $\left\|x^{i_k}-w^{i_k}\right\| \leq b(x^{i_k})\omega_{i_k}$, where b is bounded on bounded sets and the sequence $\left\{\omega_{i_k}\right\}_{k\in\mathbb{N}}$ converges to zero (cf. (6)). Hence, the sequences $\left\{x^{i_k}\right\}_{k\in\mathbb{N}}$ and $\left\{w^{i_k}\right\}_{k\in\mathbb{N}}$ have the same weak limit $\bar{x}$. Since $\left\{w^{i_k}\right\}_{k\in\mathbb{N}}$ is a sequence in Ω and Ω is weakly closed as being closed and convex, it follows that $\bar{x} \in \Omega$. Taking into account that $\Omega_k \subseteq \operatorname{Int}(\operatorname{Dom} A_k)$ one can apply the version of Minty's lemma presented in [9, Lemma 2.3] and deduce that for any $k \in \mathbb{N}$, for any $y \in \Omega_k$ and for any $\varphi^k \in A_k y$ we have

$$
\left\langle \varphi^k+\alpha_k Jy-f^k, y-x^k\right\rangle \geq 0. \tag{23}
$$

Let z be an arbitrary vector in Ω and let $\zeta \in Az$. By assumption (**A3-2**), there exists a sequence $\{z^k\}_{k\in\mathbb{N}}$ in X and a sequence $\{\zeta^k\}_{k\in\mathbb{N}}$ in X^* such that $\lim_{k\to\infty} z^k = z$, $\lim_{k\to\infty} \zeta^k = \zeta$, $z^k \in \Omega_k$ and $\zeta^k \in A_k z^k$ for all $k \in \mathbb{N}$. According to (23), we have

$$\left\langle \zeta^k + \alpha_k J z^k - f^k, z^k - x^k \right\rangle \geq 0,$$

for all $k \in \mathbb{N}$. Replacing k by i_k in this inequality and letting $k \to \infty$ we obtain

$$\langle \zeta - f, z - \bar{x} \rangle \geq 0.$$

Since the last inequality holds for any $z \in \Omega$ and $\zeta \in Az$, application of the Minty type lemma from [9, Lemma 2.3] implies that $\bar{x}$ is a solution of (1).□

Suppose from now that the variational inequality (1) has at least one solution and, hence, that the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded. Note that the Minty type lemma quoted above also implies that the solutions set $\mathcal{S}(A, f, \Omega)$ is convex and closed. Since $\mathcal{S}(A, f, \Omega)$ is nonempty, it follows that $\mathcal{S}(A, f, \Omega)$ has a unique minimal norm element.

Lemma 2.6: *The sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to the minimal norm element of $\mathcal{S}(A, f, \Omega)$. If the space has the Kadeč-Klee property, then $\{x^k\}_{k\in\mathbb{N}}$ converges strongly.*

Proof: By Lemma 2.3, since $\mathcal{S}(A, f, \Omega)$ is nonempty, the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded. We show that the unique weak accumulation point of $\{x^k\}_{k\in\mathbb{N}}$ is exactly the minimal norm element of the set $\mathcal{S}(A, f, \Omega)$. Let $\{x^{i_k}\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of the sequence $\{x^k\}_{k\in\mathbb{N}}$ and let $\bar{x}$ be its weak limit. By Lemma 2.5, $\bar{x} \in \mathcal{S}(A, f, \Omega)$. Clearly, if $\bar{x} = 0$, then $\bar{x}$ is the minimal norm solution of (1). Suppose that $\bar{x} \neq 0$. Then, by eventually taking a subsequence, we may assume without loss of generality that $x^{i_k} \neq 0$ for all $k \in \mathbb{N}$. Now, replace k by i_k in (21), divide by $\|x^{i_k}\| \alpha_{i_k}$ the resulting inequality and take lim sup on both sides. In this way one obtains that, for any $x^* \in \mathcal{S}(A, f, \Omega)$, we have that $\limsup_{k\to\infty} \|x^{i_k}\| \leq \|x^*\|$ and, consequently,

$$\|\bar{x}\| \leq \liminf_{k\to\infty} \|x^{i_k}\| \leq \limsup_{k\to\infty} \|x^{i_k}\| \leq \|x^*\|. \tag{24}$$

Hence, $\bar{x}$ is the minimal norm element of $\mathcal{S}(A, f, \Omega)$ in this case too.

Suppose now that the space X has the Kadeč-Klee property. In this situation, for proving that the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges strongly, it is sufficient to show that the sequence $\{\|x^k\|\}_{k\in\mathbb{N}}$ converges to $\|\bar{x}\|$. We distinguish again two situations. Suppose that $\bar{x} = 0$, but $\limsup_{k\to\infty} \|x^k\| = q > 0$. Then there exists a subsequence $\{x^{i_k}\}_{k\in\mathbb{N}}$ of the sequence $\{x^k\}_{k\in\mathbb{N}}$ consisting of non-null vectors and such that $\lim_{k\to\infty} \|x^{i_k}\| = q$. For this subsequence formula (24) still holds and it also holds true when $x^* = \bar{x}$ because the vector x^* involved there is an arbitrary element of $\mathcal{S}(A, f, \Omega)$. Hence, we get

$$0 < q = \lim_{k\to\infty} \|x^{i_k}\| \leq \|\bar{x}\| = 0,$$

that is, a contradiction. Consequently, $\limsup_{k\to\infty} \|x^k\| = 0$ and this implies $\lim_{k\to\infty} \|x^k\| = 0 = \|\bar{x}\|$. If $\bar{x} \neq 0$, then all but eventually finitely many terms of the sequence $\{x^k\}_{k\in\mathbb{N}}$ are not zero. Hence, there exists a positive integer k_0 such that, for any $k \geq k_0$, one can divide (21) by $\|x^k\| \alpha_k$. Taking lim sup as $k \to \infty$ on both sides of the resulting inequality one deduces

$$\|\bar{x}\| \leq \liminf_{k\to\infty} \left\|x^k\right\| \leq \limsup_{k\to\infty} \left\|x^k\right\| \leq \|x^*\|,$$

for any $x^* \in \mathcal{S}(A, f, \Omega)$. Writing the previous inequality for $x^* = \bar{x}$ we deduce that $\lim_{k\to\infty} \|x^k\| = \|\bar{x}\|$ in this case too. □

3 A Proximity Theorem

In this section we prove a preliminary result which is needed in the build up of the algorithm for solving variational inequalities we are going to present in the next section. This result is a development of a series of previous similar theorems known as "proximity lemmas". Our proximity theorem is a generalization and improvement of the proximity lemmas due to Alber, Kartsatos and Litsyn [10] and to Alber and Nashed [11]. It improves upon these results in the sense that our proximity theorem applies to operators which are not necessarily monotone. Also, by contrast with the proximity lemmas mentioned above, which are proved under restrictive conditions on the moduli of convexity and smoothness of the underlying space X, our proximity theorem only requires that the space X is uniformly convex, that is, the modulus of convexity of X, $\delta_X : \mathbb{R} \to [0, +\infty]$ given by[1]

$$\delta_X(t) := \inf \left\{1 - \tfrac{1}{2}\|x + y\| : \|x\| = \|y\| = 1,\ \|x - y\| \geq t\right\},$$

is positive for any $t > 0$. In that follows we assume that the space X is uniformly convex.

[1] We make here the ususal assumtion that $\inf \varnothing = +\infty$ and $\sup \varnothing = -\infty$.

Let $\bar{g}_X : \mathbb{R} \to [0, +\infty]$ be the function defined by

$$\bar{g}_X(t) := \begin{cases} \frac{\delta_X(t)}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases} \tag{25}$$

This function is nondecreasing (cf. [25, Proposition 3]), vanishes when $t \leq 0$ and is positive whenever $t > 0$ because X is uniformly convex. From now on, we assume that $g_X : \mathbb{R} \to [0, +\infty]$ is a nondecreasing function which vanishes on $(-\infty, 0]$ and such that

$$0 < g_X(t) \leq \bar{g}_X(t), \quad \forall t > 0. \tag{26}$$

Clearly, one can choose $g_X = \bar{g}_X$. The relevance of choosing a function g_X other than $\bar{g}_X$ is related with the possibility of using lower evaluations of δ_X instead of δ_X itself which may be hard to precisely compute. This aspect will became clear in the next section. To the function g_X we associate its *epi-quasi-inverse* $g_X^\sharp : \mathbb{R} \to [-\infty, +\infty]$ given by (cf. [40, p. 126])

$$g_X^\sharp(s) = \sup\{t \in \mathbb{R} : g_X(t) < s\}. \tag{27}$$

Clearly, the function $g_X^\sharp$ is nondecreasing too. If the restriction of g_X to $[0, 2]$ is invertible, then $g_X^\sharp(s) = g_X^{-1}(s)$ for any $s \in g_X([0, 2])$ (cf. [40]).

In order to state our proximity theorem, let $T_i : X \to 2^{X^*}$ be an operator, let $\varphi^i \in X^*$ and let Λ_i be a nonempty subset of $\operatorname{Dom} T_i$, where $i \in \{1, 2\}$. Suppose that, for some positive real numbers α and β, the variational inequalities

$$\left\langle (T_1 + \alpha J)x - \varphi^1, y - x \right\rangle \geq 0, \ \forall y \in \Lambda_1 \tag{28}$$

and

$$\left\langle (T_2 + \beta J)x - \varphi^2, y - x \right\rangle \geq 0, \ \forall y \in \Lambda_2, \tag{29}$$

have solutions x_α and x_β, respectively. Then there exist $\xi_\alpha \in T_1 x_\alpha$ and $\xi_\beta \in T_2 x_\beta$ such that

$$\left\langle \xi_\alpha + \alpha J x_\alpha - \varphi^1, y - x_\alpha \right\rangle \geq 0, \ \forall y \in \Lambda_1 \tag{30}$$

and

$$\left\langle \xi_\beta + \beta J x_\beta - \varphi^2, y - x_\beta \right\rangle \geq 0, \ \forall y \in \Lambda_2. \tag{31}$$

With these notations we state the following result which gives an evaluation of the distance between x_α and x_β. In the statement we use a constant L which occurs in Figiel's paper [25, Proposition 10]. It is a lower bound for the speed of variation of the function $\rho_X(t)/t^2$, where $\rho_X(t)$ is the modulus of smoothness of the space X. It follows from [49] and [5] that $L \in (1, 1.7)$.

Theorem 3.1 (The Proximity Theorem): *Suppose that the space X is uniformly convex and the operator T_2 is strongly h-hypomonotone. Let $\bar{a}$, $\bar{b}$, $\bar{c}$, δ, τ and ω be nonnegative real numbers such that*

$$\left\|\varphi^1 - \varphi^2\right\|_* \leq \delta \tag{32}$$

and the following conditions hold:

(i) There exists $\bar{x}_\alpha \in \Lambda_2$ such that

$$\|x_\alpha - \bar{x}_\alpha\| \leq \bar{a}\omega \text{ and } \operatorname{dist}_*(\xi_\alpha, T_2\bar{x}_\alpha) \leq \bar{c}\tau; \tag{33}$$

(ii) There exists $\bar{x}_\beta \in \Lambda_1$ is such that

$$\|x_\beta - \bar{x}_\beta\| \leq \bar{b}\omega. \tag{34}$$

If

$$M \geq \max\{\|x_\alpha\|, \|x_\beta\|\} \text{ and } M_1 \geq \max\left\{\|\xi_\alpha\|_*, \|\xi_\beta\|_*\right\}, \tag{35}$$

then

$$\|x_\alpha - x_\beta\| \leq K_0 g_X^\sharp\left(K_1\frac{|\alpha-\beta|}{\alpha} + K_2\frac{\tau+\delta}{\alpha} + K_3\sqrt{\frac{\omega+h}{\alpha}}\right), \tag{36}$$

where

$$K_0 = 2\max\{1, M\}, \quad K_1 = 2LMK_0, \quad K_2 = 2LK_0\max\{\bar{c}, 1\}, \tag{37}$$

and

$$K_3 = \max\{1, 4L(K_4 + K_5)\}, \tag{38}$$

with

$$K_4 = (M_1 + \left\|\varphi^1\right\|_* + \alpha M)\bar{b} + (3M_1 + \left\|\varphi^2\right\|_* + \tau\bar{c} + \beta M)\bar{a}, \tag{39}$$

$$K_5 = (\omega\bar{a} + 2M)^2, \tag{40}$$

and L is the Figiel constant.

Proof: The inequality (36) clearly holds when $x_\alpha = x_\beta$. In that follows we assume that $x_\alpha \neq x_\beta$. Condition (i) implies that there exists $\bar{\xi}_\alpha \in T_2\bar{x}_\alpha$ such that

$$\left\|\xi_\alpha - \bar{\xi}_\alpha\right\|_* \le \tau\bar{c}. \tag{41}$$

Denote

$$D = \left\langle \xi_\alpha + \alpha J x_\alpha - \varphi^1 - \xi_\beta - \beta J x_\beta + \varphi^2, x_\alpha - x_\beta \right\rangle.$$

Then

$$\begin{aligned} D &= \alpha \left\langle J x_\alpha - J x_\beta, x_\alpha - x_\beta \right\rangle + (\alpha - \beta) \left\langle J x_\beta, x_\alpha - x_\beta \right\rangle \\ &\quad + \left\langle \xi_\alpha - \bar{\xi}_\alpha, x_\alpha - x_\beta \right\rangle + \left\langle \bar{\xi}_\alpha - \xi_\beta, x_\alpha - \bar{x}_\alpha \right\rangle \\ &\quad + \left\langle \bar{\xi}_\alpha - \xi_\beta, \bar{x}_\alpha - x_\beta \right\rangle - \left\langle \varphi^1 - \varphi^2, x_\alpha - x_\beta \right\rangle. \end{aligned} \tag{42}$$

The strong h-hypomonotonicity of T_2 implies that

$$\left\langle \bar{\xi}_\alpha - \xi_\beta, \bar{x}_\alpha - x_\beta \right\rangle \ge -h \left(\|\bar{x}_\alpha\| - \|x_\beta\|\right)^2. \tag{43}$$

According to [13, Lemma 2.1] we have that

$$\left\langle J x_\alpha - J x_\beta, x_\alpha - x_\beta \right\rangle \ge (2L)^{-1} \delta_X \left(\frac{\|x_\alpha - x_\beta\|}{C} \right), \tag{44}$$

where $L > 0$ is the Figiel constant and

$$C := 2 \max \left\{1, \|x_\alpha\|, \|x_\beta\|\right\}.$$

Combining (41), (42), (43) and (44) and taking into account (32) and (33), we obtain that

$$\begin{aligned} D &\ge \alpha (2L)^{-1} \delta_X \left(\frac{\|x_\alpha - x_\beta\|}{C} \right) \\ &\quad - |\alpha - \beta| \, \|x_\beta\| \, \|x_\alpha - x_\beta\| - \tau\bar{c} \, \|x_\alpha - x_\beta\| \\ &\quad - \delta \, \|x_\alpha - x_\beta\| - \omega\bar{a} \left\|\bar{\xi}_\alpha - \xi_\beta\right\|_* - h \left(\|\bar{x}_\alpha\| - \|x_\beta\|\right)^2 \\ &\ge \alpha (2L)^{-1} \delta_X \left(\frac{\|x_\alpha - x_\beta\|}{C} \right) \\ &\quad - |\alpha - \beta| \, \|x_\beta\| \, \|x_\alpha - x_\beta\| - \tau\bar{c} \, \|x_\alpha - x_\beta\| \\ &\quad - \delta \, \|x_\alpha - x_\beta\| - \omega\bar{a} \left\|\bar{\xi}_\alpha - \xi_\beta\right\|_* - h \left\|\bar{x}_\alpha - x_\beta\right\|^2. \end{aligned} \tag{45}$$

Note that, by (41) and (35), we have

$$\begin{aligned}\left\|\bar{\xi}_\alpha-\xi_\beta\right\|_* &\le \left\|\bar{\xi}_\alpha-\xi_\alpha\right\|_*+\|\xi_\alpha-\xi_\beta\|_* \\ &\le \left\|\bar{\xi}_\alpha-\xi_\alpha\right\|_*+\|\xi_\alpha\|_*+\|\xi_\beta\|_* \le \tau\bar{c}+2M_1.\end{aligned}$$

Similarly, by (33) and (35), we obtain that

$$\|\bar{x}_\alpha\|_* \le \|\bar{x}_\alpha-x_\alpha\|+\|x_\alpha\| \le \omega\bar{a}+M.$$

Therefore, inequality (45) leads to

$$\begin{aligned}D \ge\ &\alpha(2L)^{-1}\delta_X\left(\frac{\|x_\alpha-x_\beta\|}{C}\right)-\omega\bar{a}(2M_1+\tau\bar{c}) \\ &-(M\,|\alpha-\beta|+\tau\bar{c}+\delta)\,\|x_\alpha-x_\beta\|-hK_5,\end{aligned} \tag{46}$$

where K_5 is given by (40). Observe that

$$\begin{aligned}\left\langle\xi_\alpha+\alpha Jx_\alpha-\varphi^1,x_\alpha-x_\beta\right\rangle &= \left\langle\xi_\alpha+\alpha Jx_\alpha-\varphi^1,x_\alpha-\bar{x}_\beta\right\rangle \\ &\quad+\left\langle\xi_\alpha+\alpha Jx_\alpha-\varphi^1,\bar{x}_\beta-x_\beta\right\rangle \\ &\le \left\langle\xi_\alpha+\alpha Jx_\alpha-\varphi^1,\bar{x}_\beta-x_\beta\right\rangle \\ &\le \left(\left\|\xi_\alpha-\varphi^1\right\|_*+\alpha\,\|x_\alpha\|\right)\|\bar{x}_\beta-x_\beta\| \\ &\le \left(M_1+\left\|\varphi^1\right\|_*+\alpha M\right)\omega\bar{b},\end{aligned} \tag{47}$$

where the first inequality follows from (30) and the last inequality follows from (35) and (34). Analogously, using (31), we deduce that

$$\left\langle\xi_\beta+\beta Jx_\beta-\varphi^2,x_\beta-x_\alpha\right\rangle \le \left(M_1+\left\|\varphi^2\right\|_*+\beta M\right)\omega\bar{a}. \tag{48}$$

From (42), (47) and (48) taken together, we obtain that

$$D \le \omega\left[\left(M_1+\left\|\varphi^1\right\|_*+\alpha M\right)\bar{b}+\left(M_1+\left\|\varphi^2\right\|_*+\beta M\right)\bar{a}\right]. \tag{49}$$

Denote by K_6 be the quantity occurring in (49) between square brackets. By (39) we have

$$K_4 = K_6+(2M_1+\tau\bar{c})\,\bar{a}.$$

Then, according to (46) and (49), we obtain that

$$\begin{aligned}&\omega K_4+K_5h+(M\,|\alpha-\beta|+\tau\bar{c}+\delta)\,\|x_\alpha-x_\beta\| \\ &\ge \alpha(2L)^{-1}\delta_X\left(\frac{\|x_\alpha-x_\beta\|}{K_0}\right),\end{aligned} \tag{50}$$

where K_0 is given by (37). Dividing this inequality by $\|x_\alpha - x_\beta\| > 0$ we deduce that

$$\begin{aligned}
&\frac{\omega K_4 + K_5 h}{\|x_\alpha - x_\beta\|} + M\,|\alpha - \beta| + \tau\bar{c} + \delta \qquad (51)\\
&\geq \alpha(2LK_0)^{-1}\delta_X\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right)\frac{K_0}{\|x_\alpha - x_\beta\|}\\
&= \alpha(2LK_0)^{-1}\bar{g}_X\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right)\\
&> \alpha(4LK_0)^{-1}\bar{g}_X\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right)\\
&\geq \alpha(4LK_0)^{-1}g_X\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right),
\end{aligned}$$

where $\bar{g}_X$ is the function defined by (25) and g_X is the nondecreasing function satisfying (26). Note that, according to (27), if $u, v \in \mathbb{R}$, then

$$u > g_X(v) \Rightarrow g_X^\sharp(u) \geq v. \qquad (52)$$

Thus, using (51), we obtain that

$$\|x_\alpha - x_\beta\| \leq K_0 g_X^\sharp\left(\frac{4LK_0}{\alpha}\frac{\omega K_4 + K_5 h}{\|x_\alpha - x_\beta\|} + K_1\frac{|\alpha - \beta|}{\alpha} + K_2\frac{\tau + \delta}{\alpha}\right), \qquad (53)$$

where K_1 and K_2 are given by (37). Now we distinguish the following complementary situations.

Case 1: Suppose that

$$\frac{\|x_\alpha - x_\beta\|}{K_0} \leq g_X^\sharp\left(\sqrt{\frac{\omega + h}{\alpha}} + K_1\frac{|\alpha - \beta|}{\alpha} + K_2\frac{\tau + \delta}{\alpha}\right).$$

In this case, we have that

$$\begin{aligned}
&g_X^\sharp\left(\sqrt{\frac{\omega + h}{\alpha}} + K_1\frac{|\alpha - \beta|}{\alpha} + K_2\frac{\tau + \delta}{\alpha}\right)\\
&\leq g_X^\sharp\left(K_3\sqrt{\frac{\omega + h}{\alpha}} + K_1\frac{|\alpha - \beta|}{\alpha} + K_2\frac{\tau + \delta}{\alpha}\right)
\end{aligned}$$

because $g_X^\sharp$ is nondecreasing and $K_3 \geq 1$ (see (38)). Thus, combining the last two inequalities, we deduce (36).

Case 2: Suppose that

$$\frac{\|x_\alpha - x_\beta\|}{K_0} > g_X^\sharp\left(\sqrt{\frac{\omega+h}{\alpha}} + K_1\frac{|\alpha-\beta|}{\alpha} + K_2\frac{\tau+\delta}{\alpha}\right). \tag{54}$$

Recall (see, for instance, [27, Theorem 2.7.8]) that, since X is a uniformly convex Banach space, for any Hilbert space H we have that

$$\delta_X(t) \le \delta_H(t) \le \frac{t^2}{4}, \quad \forall t \in [0,2].$$

Consequently,

$$\frac{1}{4}\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right)^2 \ge \delta_X\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right),$$

because $K_0 \ge 2M \ge \|x_\alpha - x_\beta\|$ as follows from (35) and (37). This implies that

$$\frac{1}{4}\frac{\|x_\alpha - x_\beta\|}{K_0} \ge \bar{g}_X\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right) \ge g_X\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right). \tag{55}$$

Observe that, by (27), if $u, v \in \mathbb{R}$, then

$$u > g_X^\sharp(v) \Rightarrow g_X(u) \ge v. \tag{56}$$

From (54) and (56) we deduce that

$$g_X\left(\frac{\|x_\alpha - x_\beta\|}{K_0}\right) \ge \sqrt{\frac{\omega+h}{\alpha}} + K_1\frac{|\alpha-\beta|}{\alpha} + K_2\frac{\tau+\delta}{\alpha} > \sqrt{\frac{\omega+h}{\alpha}}.$$

This and (55) combined imply

$$\frac{\|x_\alpha - x_\beta\|}{K_0} > 4\sqrt{\frac{\omega+h}{\alpha}}. \tag{57}$$

Taking into account (38), (57), the fact that $g_X^\sharp$ is nondecreasing and the inequality (53), we deduce that

$$\begin{aligned}
&K_0 g_X^\sharp\left[K_3\frac{\omega+h}{\alpha}\left(4\sqrt{\frac{\omega+h}{\alpha}}\right)^{-1} + K_1\frac{|\alpha-\beta|}{\alpha} + K_2\frac{\tau+\delta}{\alpha}\right] \\
&\ge K_0 g_X^\sharp\left[4L\left(K_4+K_5\right)\frac{\omega+h}{\alpha}\frac{K_0}{\|x_\alpha - x_\beta\|} + K_1\frac{|\alpha-\beta|}{\alpha} + K_2\frac{\tau+\delta}{\alpha}\right] \\
&\ge K_0 g_X^\sharp\left[4L\frac{K_4\omega+K_5 h}{\alpha}\frac{K_0}{\|x_\alpha - x_\beta\|} + K_1\frac{|\alpha-\beta|}{\alpha} + K_2\frac{\tau+\delta}{\alpha}\right] \\
&\ge \|x_\alpha - x_\beta\|
\end{aligned}$$

and this implies (36). □

4 An Iterative Algorithm for Solving Variational Inequalities

In this section we present an iterative algorithm for approximating solutions of the variational inequality (1), presuming that such a solution exists. All over this section we assume that the space X is smooth and uniformly convex and that the assumptions (**A1**)-(**A3**), as well as the following assumption, are satisfied:

(**A4**) There exists a continuous nondecreasing function $\Phi : [0, \infty) \to \mathbb{R}$ such that, for any $k \in \mathbb{N}$, we have that

$$x \in \Omega_k \text{ and } \zeta \in A_k x \Rightarrow \|\zeta\|_* \leq \Phi(\|x\|). \tag{58}$$

In order to describe our algorithm in a consistent manner, we need some notations and preparations. Note that, since the space X is uniformly convex, it is also strictly convex and has the Kadeč-Klee property. This guarantees applicability of Theorem 2.1(ii) in this space. Since the variational inequality (1) is presumed to have solutions, Theorem 2.1 guarantees that the sequence $\{x^k\}_{k\in\mathbb{N}}$ of solutions of the variational inequalities (4) exists and converges strongly to the minimal norm solution of (1). Second, recall that each operator A_k is strongly h_k-hypomonotone (by condition (**A2**)). Consider the variational inequalities

$$\langle (A + \alpha_k J)x - f, y - x\rangle \geq 0, \ \forall y \in \Omega. \tag{59}$$

According to Lemma 2.3 (which is applicable to (59) because A is monotone and, therefore, strongly 0-hypomonotone), the variational inequality (59) has a unique solution which we denote by u^k in that follows. Applying Theorem 2.1(ii) to the particular situation when $A_k = A$, $\Omega_k = \Omega$ and $f^k = f$ for all $k \in \mathbb{N}$, we deduce that the sequence $\{u^k\}_{k\in\mathbb{N}}$ converges to the same solution as $\{x^k\}_{k\in\mathbb{N}}$, that is, to the minimal norm solution $\bar{x}$ of (1). Since both sequences $\{x^k\}_{k\in\mathbb{N}}$ and $\{u^k\}_{k\in\mathbb{N}}$ are bounded (as being convergent), there exists a positive real number M such that

$$M \geq \max\left\{\left\|u^k\right\|, \left\|x^k\right\|\right\}, \quad \forall k \in \mathbb{N}. \tag{60}$$

For each $k \in \mathbb{N}$, let $\xi^k \in A_k x^k$ be such that (17) holds. According to (**A4**), we have that $\|\xi^k\|_* \leq \Phi(\|x^k\|)$, for all $k \in \mathbb{N}$. Since Φ is bounded on bounded sets, it results that the sequence $\{\xi^k\}_{k\in\mathbb{N}}$ is bounded. Let $\zeta^k \in Au^k$ be such that

$$\left\langle \zeta^k + \alpha_k J u^k - f, y - u^k \right\rangle \geq 0, \quad \forall y \in \Omega. \tag{61}$$

Since $\{u^k\}_{k\in\mathbb{N}}$ is a sequence in $\operatorname{Int}(\operatorname{Dom} A)$ which converges to $\bar{x} \in \Omega \subseteq \operatorname{Int}(\operatorname{Dom} A)$ and A is locally bounded at $\bar{x}$ (as being monotone), it results that $\{\zeta^k\}_{k\in\mathbb{N}}$ is bounded too. Consequently, there exists a positive real number M_1 such that

$$M_1 \geq \max\left\{\left\|\zeta^k\right\|_*, \left\|\xi^k\right\|_*\right\}, \quad \forall k \in \mathbb{N}. \tag{62}$$

Finally, observe that, according to (**A3-1**), the sequence $\{f^k\}_{k\in\mathbb{N}}$ is convergent. Therefore, there exists a positive number M_2 such that

$$M_2 \geq \left\|f^k\right\|_*, \quad \forall k \in \mathbb{N}. \tag{63}$$

The functions a, b and c being those involved in condition (**A3**), we denote

$$\bar{a} := \sup\{a(x) : \|x\| \leq M\}, \tag{64}$$
$$\bar{b} := \sup\{b(x) : \|x\| \leq M\}, \tag{65}$$
$$\bar{c} := \sup\{c(\zeta) : \|\zeta\|_* \leq M_1\}, \tag{66}$$

where M and M_1 are numbers satisfying (60) and (62), respectively. Clearly, $\bar{a}$, $\bar{b}$ and $\bar{c}$ are finite because of the boundedness on bounded sets of a, b and c. In what follows, we denote by K_0, K_1, K_2 and K_3 the numbers defined by (37) and (38) where, instead of K_4 and K_5 given by (39) and, respectively, (40), we take

$$K_4 := (M_1 + M_2 + \bar{\delta} + \bar{\alpha}M)\bar{b} + (3M_1 + M_2 + \bar{\tau}\bar{c} + \bar{\alpha}M)\bar{a}, \tag{67}$$
$$K_5 := (M + \bar{\omega}\bar{a})^2 + M^2, \tag{68}$$

with $\bar{\alpha}$, $\bar{\delta}$, $\bar{\tau}$, $\bar{\omega}$ being positive upper bounds of the sequences $\{\alpha_k\}_{k\in\mathbb{N}}$, $\{\delta_k\}_{k\in\mathbb{N}}$, $\{\tau_k\}_{k\in\mathbb{N}}$, $\{\omega_k\}_{k\in\mathbb{N}}$, respectively (such upper bounds exists because, by (**A3**), these sequences are convergent).

Let g_X be the nonnegative, nondecreasing function which satisfies (26) and vanishes on $(-\infty, 0)$ and let $g_X^\sharp$ be its epi-quasi-inverse defined by (27). For each $k \in \mathbb{N}$, we denote

$$G_k := K_0 g_X^\sharp\left(K_2\frac{\tau_k + \delta_k}{\alpha_k} + K_3\sqrt{\frac{\omega_k + h_k}{\alpha_k}}\right), \tag{69}$$

$$\bar{G}_k := K_0 g_X^\sharp\left(K_1\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k}\right), \tag{70}$$

where the numbers $K_0, ..., K_3$ are those defined above (with K_4 and K_5 given by (67) and (68)). Let $\rho_{X^*} : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of the dual space X^*, i.e., the function which, according to a theorem of J. Lindenstrauss (see [27, Theorem 2.7.5]), is given by

$$\rho_{X^*}(t) = \sup \left\{ \frac{ts}{2} - \delta_X(s) : s \in [0, 2] \right\}. \tag{71}$$

Since the space X is uniformly convex, its dual X^* is uniformly smooth, that is, $\lim_{t \to 0^+} t^{-1} \rho_{X^*}(t) = 0$. We assume that the problem data and the geometry of the space X in which our variational inequality (1) is set are interconnected in the sense of the following condition:

(**A5**) The sequence of regularization parameters $\{\alpha_k\}_{k \in \mathbb{N}}$ has the property that there exists a bounded sequence of positive real numbers $\{\varepsilon_k\}_{k \in \mathbb{N}}$ such that $\alpha_k \varepsilon_k < 1$ for all $k \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \alpha_k \varepsilon_k = \infty \text{ and } \lim_{k \to \infty} (\alpha_k \varepsilon_k)^{-1} \left[\rho_{X^*}(\varepsilon_k) + G_k + G_{k+1} + \bar{G}_k \right] = 0. \tag{72}$$

Condition (**A5**) represents the rule of choosing the relaxation parameters in the algorithm described below when the sequences involved in (**A3**) are a priori given. Implementation of this rule requires evaluations of ρ_{X^*} as well as of $g_X^\sharp$. In some spaces such evaluations are readily available. That is the case of the spaces which are p-convex for some $p > 1$ as, for instance, the Lebesgue spaces, the Sobolev spaces, the Orlicz spaces (cf. [12, p. 613]). Recall that the space X is called p-convex if there exists a positive real number c such that $\delta_X(t) \geq ct^p$ for all $t > 0$. If X is p-convex for some $p > 1$, then, according to (71), we obtain that

$$\rho_{X^*}(t) \leq \sup \left\{ \frac{ts}{2} - cs^p : s \in [0, 2] \right\} = \bar{c} t^q,$$

where $\bar{c}$ is a positive constant and $q = p/(p-1)$. In this case, one can define $g_X(t) = ct^{p-1}$ if $t \geq 0$, and $g_X(t) = 0$, otherwise. This implies that $g_X^\sharp(t) = \left(c^{-1} t\right)^{q/p}$ for any $t \geq 0$.

To any closed convex nonempty subset Λ of X we associate the operator $\Gamma_\Lambda : X^* \to \Lambda$ given by

$$\Gamma_\Lambda \xi := \arg\min \left\{ W(\xi, u) : u \in \Lambda \right\},$$

where, for any $\xi \in X^*$ and for any $u \in X$, the function $W : X^* \times X \to \mathbb{R}$ is defined by

$$W(\xi, u) = \|u\|^2 - 2\langle \xi, u\rangle + \|\xi\|_*^2 .$$

As shown in [5, p. 31], where this operator was introduced and studied, Γ_Λ is well defined. The algorithm for finding solutions of variational inequalities we present below requires computing values of Γ_Λ (see (78)) for various nonempty convex closed sets Λ. In this respect, recall (see [5]) that

$$\Gamma_\Lambda = (\Gamma_\Lambda \circ J) \circ J^* = P_\Lambda \circ J^*, \tag{73}$$

where J^* is the normalized duality mapping of X^* and P_Λ denotes the Bregman projection onto the set Λ with respect to the function $\theta(x) := \|x\|^2$. It results from (73) that, if the values of J^* are computable (as happens in many usual Banach spaces as, for instance, in Lebesgue spaces L^p, in Sobolev spaces $W^{m,p}$), then computing values of Γ_Λ amounts to determining values of P_Λ. If Λ is a closed hyperplane or a closed half-space, then values of P_Λ can be determined by formulae established in [6] and [22]. If Λ is an arbitrary nonempty closed convex subset of X, then values of P_Λ can be calculated using the algorithms presented in [17].

With these facts in mind, we proceed to the description of the iterative procedure we propose for solving the monotone variational inequality (1).

THE ALGORITHM

Step 0 (Initialization).

(a) Fix three numbers M, M_1 and M_2 such that conditions (60), (62) and (63), respectively, are satisfied, let R_0 be a positive real number and put

$$K := M + \sqrt{R_0}. \tag{74}$$

(b) Define the functions μ, $\varkappa$, $r : [0, \infty) \to \mathbb{R}$ and $\Upsilon : [0, \infty) \times \mathbb{N} \to \mathbb{R}$ by

$$\begin{aligned} \mu(t) &= \max\{\Phi(t) + \bar{\alpha}t + M_2, 1\}, \qquad (75)\\ \varkappa(t) &= \bar{\varepsilon}(\Phi(t) + M_2) + (\bar{\alpha} + 1)t,\\ r(t) &= \sqrt{\mu^2(t) + \varkappa^2(t)},\\ \Upsilon(t, k) &:= 2\left[2Lr^2(t)\rho_{X^*}\left(4r(t)^{-1}\mu(t)\varepsilon_k\right) + \varepsilon_k h_k\left(t^2 + M^2\right)\right.\\ &\quad \left. + \left(\varepsilon_k \mu(t) + t + M\right)\left(G_k + G_{k+1} + \bar{G}_k\right)\right], \end{aligned}$$

where $\bar{\alpha}$ and $\bar{\varepsilon}$ are upper bounds of the sequences $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{\varepsilon_k\}_{k\in\mathbb{N}}$, respectively.

(c) Let $n_0 \in \mathbb{N}$ be a nonnegative integer such that for any $k \geq n_0$ we have

$$\Upsilon(K, k) \leq R_0 \alpha_k \varepsilon_k, \tag{76}$$

put $p(0) = n_0$ and choose $z^0 \in \Omega_{p(0)}$ such that

$$R_0 \geq W(Jz^0, x^{p(0)}). \tag{77}$$

Step 1 (Iteration).
Given $n \in \mathbb{N}$ and $z^n \in \Omega_{p(n)}$, put $p(n+1) = n_0 + n + 1$, choose $\chi^n \in A_{p(n)} z^n$ and compute

$$z^{n+1} = \Gamma_{\Omega_{p(n+1)}} \left[Jz^n - \varepsilon_{p(n)} \left(\chi^n + \alpha_{p(n)} Jz^n - f^{p(n)} \right) \right]. \tag{78}$$

Step 2 (Loop).
Let $n \to n+1$ and go to Step 1.

The following result describes the convergence behavior of our algorithm.

Theorem 4.1. *If the variational inequality* (1) *has at least one solution and if the assumptions* (**A1**)-(**A5**) *are satisfied, then the sequence* $\{z^k\}_{k\in\mathbb{N}}$ *generated by the algorithm described above is well defined and is strongly convergent to the minimal norm solution of* (1).

The proof of this theorem is presented below as a succession of lemmas. The basic idea of the proof is to show that the sequence $\{z^k\}_{k\in\mathbb{N}}$ generated by the algorithm and the sequence $\{x^k\}_{k\in\mathbb{N}}$, whose existence and convergence is ensured by Theorem 2.1(ii), have the same limit. One should observe that the algorithm does not require computing the sequence $\{x^k\}_{k\in\mathbb{N}}$, but only to have an evaluation of an upper bound of the set $\{\|x^k\|\}_{k\in\mathbb{N}}$. Once such an evaluation is established, one can use Lemma 4.2 below in order to estimate the number M required by the algorithm because, as follows from Lemma 4.1, the sequence $\{G_k\}_{k\in\mathbb{N}}$ occurring in (82) is convergent and, hence, bounded. In order to estimate the number M_1 required by the algorithm one should note that, by virtue of (**A4**), the number $\Phi(M)$ is an upper bound of the sequence $\{\|\xi^k\|_*\}_{k\in\mathbb{N}}$, where ξ^k is defined by (17). This fact and (**A3-2**) allow for determining M_1. These remarks show that implementation of the algorithm does not require computations with the original problem data, but involves computations with the approximative data only.

Now, we start our proof of Theorem 4.1 by establishing well definedness of the algorithm. The following result implies that the number n_0 required in Step 0 of the algorithm (see (76)) exists and, hence, the algorithm is well defined.

Lemma 4.1: *The sequence* $\{\gamma_k\}_{k\in\mathbb{N}}$ *defined by*

$$\gamma_k = (\alpha_k\varepsilon_k)^{-1}\Upsilon(K,k), \tag{79}$$

where K *is the real number given by* (74)*, converges to zero.*

Proof: According to (75) we have

$$\begin{aligned}\tfrac{1}{2}\gamma_k &= 2(\alpha_k\varepsilon_k)^{-1}L\bar{r}^2\rho_{X^*}\left(4\bar{r}^{-1}\bar{\mu}\varepsilon_k\right) + \varepsilon_k h_k\left(K^2+M^2\right)\\ &\quad + (\alpha_k\varepsilon_k)^{-1}\left(\varepsilon_k\bar{\mu}+K+M\right)\left(G_k+G_{k+1}+\bar{G}_k\right),\end{aligned}$$

where

$$\bar{r} = r(K) \text{ and } \bar{\mu} = \mu(K). \tag{80}$$

By (72), the last term of this sum converges to zero as $k\to\infty$. According to (**A5**), the sequence $\{\alpha_k\varepsilon_k\}_{k\in\mathbb{N}}$ is bounded from above by 1. Therefore, we have that $\varepsilon_k h_k \le \alpha_k^{-1}h_k$. Since, by (6), the sequence $\left\{\alpha_k^{-1}h_k\right\}_{k\in\mathbb{N}}$ converges to zero, we deduce that the second term of the sum converges to zero as $k\to\infty$. It remains to show that

$$\lim_{k\to\infty}(\alpha_k\varepsilon_k)^{-1}\rho_{X^*}\left(4\bar{r}^{-1}\bar{\mu}\varepsilon_k\right) = 0. \tag{81}$$

To this end, observe that, according to (75), we have that $\bar{r}^{-1}\bar{\mu}\le 1$. Since the function ρ_{X^*} is nondecreasing (cf. [27, Lemma 2.7.4]) we deduce that

$$\frac{\rho_{X^*}\left(4\bar{r}^{-1}\bar{\mu}\varepsilon_k\right)}{\alpha_k\varepsilon_k} \le \frac{\rho_{X^*}\left(4\varepsilon_k\right)}{\alpha_k\varepsilon_k}.$$

Applying twice Lemma 8 in [25] we deduce

$$\begin{aligned}\frac{\rho_{X^*}\left(4\varepsilon_k\right)}{\alpha_k\varepsilon_k} &\le 4(1+\varepsilon_k)\frac{\rho_{X^*}\left(2\varepsilon_k\right)}{\alpha_k\varepsilon_k}\\ &\le 16(1+\varepsilon_k)(1+\tfrac{1}{2}\varepsilon_k)\tfrac{\rho_{X^*}(\varepsilon_k)}{\alpha_k\varepsilon_k}.\end{aligned}$$

The last two inequalities and (72) imply (81). □

We are going to use Theorem 3.1 in order to obtain an evaluation of the distance between the vectors x^k and u^k.

Lemma 4.2: *For any* $k\in\mathbb{N}$*, we have that*

$$\left\|x^k-u^k\right\| \le G_k. \tag{82}$$

Proof: We apply Theorem 3.1 to the variational inequalities (59) and (4) by taking $T_1 = A$, $T_2 = A_k$, $\varphi^1 = f$, $\varphi^2 = f^k$, $\omega = \omega_k$, $\tau = \tau_k$, and $\delta = \delta_k$. Condition (**A3-3**) ensures that hypothesis (*ii*) of Theorem 3.1 is valid in this case. By conditions (**A3-1**) and (**A3-2**) we deduce that hypothesis (*i*) of Theorem 3.1 is also satisfied. Thus, we deduce the following particular version of (36) with $\alpha = \beta = \alpha_k$:

$$\left\| x^k - u^k \right\| \le K_0 g_X^\sharp \left(K_2 \frac{\tau_k + \delta_k}{\alpha_k} + K_3 \sqrt{\frac{\omega_k + h_k}{\alpha_k}} \right), \tag{83}$$

where the numbers $K_1, ..., K_5$ are defined by (37)-(40) with α, ω, τ and δ as above. By direct comparison one can see that the numbers K_4 and K_5, defined by (67) and (68), are at least equal to their homonymous numbers defined by (39) and (40) in our current circumstances. This implies that the corresponding values of K_0, K_1, K_2 and K_3 obtained when K_4 and K_5 are given by (67) and (68) are at least equal to those of their homonymous counterparts obtained when K_4 and K_5 are given by (39) and (40). Since the function $g_X^\sharp$ in nondecreasing, it follows that by replacing on the right hand side of (83) the numbers K_i by their larger counterparts, the inequality still stands and this is exactly (82). □

The following lemma is a consequence of Theorem 3.1 and of Lemma 4.2. It is the key result for our proof of convergence of the sequences $\left\{z^k\right\}_{k \in \mathbb{N}}$ generated by the algorithm described above.

Lemma 4.3: *For each $n \in \mathbb{N}$, we have that*

$$W(Jz^{n+1}, x^{p(n+1)}) \le (1 - \varepsilon_{p(n)} \alpha_{p(n)}) W(Jz^n, x^{p(n)}) + \Upsilon(\|z^n\|, p(n)). \tag{84}$$

Proof. Denote

$$w^n := Jz^n - \varepsilon_{p(n)} \left(\chi^n + \alpha_{p(n)} Jz^n - f^{p(n)} \right), \tag{85}$$

$$\varkappa_n := \varkappa(\|z^n\|), \; r_n := r(\|z^n\|), \; \mu_n := \mu(\|z^n\|),$$

and note that, according to (78), we have

$$z^{n+1} = \Gamma_{\Omega_{p(n+1)}} w^n. \tag{86}$$

Observe that, whenever $u \in X$, the function $W(\cdot, u)$ is convex and (Gâteaux) differentiable at any point of X^* because the space X^* is smooth (since X is uniformly convex). Moreover, for any $\zeta \in X^*$ and for any $u \in X$, we have that

$$\left[W(\cdot, u)\right]'(\zeta) = 2(J^* \zeta - u),$$

where $J^* = J^{-1}$ is the normalized duality mapping of the space X^* (cf. [5, Lemma 6.1]). Therefore, we obtain that

$$W(Jz^n, x^{p(n+1)}) - W(w^n, x^{p(n+1)}) \tag{87}$$
$$\geq \left\langle Jz^n - w^n, \left[W(\cdot, x^{p(n+1)})\right]' (w^n) \right\rangle$$
$$= 2 \left\langle Jz^n - w^n, J^* w^n - x^{p(n+1)} \right\rangle .$$

Applying [5, Property 6h] and taking into account (86) and the fact that $x^{p(n+1)} \in \Omega_{p(n+1)}$ we obtain that

$$W(w^n, z^{n+1}) \leq W(w^n, x^{p(n+1)}). \tag{88}$$

Combining this and (87) we deduce that

$$W(Jz^n, x^{p(n+1)}) - W(Jz^{n+1}, x^{p(n+1)}) \tag{89}$$
$$\geq 2 \left\langle Jz^n - w^n, J^* w^n - x^{p(n+1)} \right\rangle .$$

For any $\zeta \in X^*$, the function $W(\zeta, \cdot)$ is also convex and differentiable and we have (cf. [5, Lemma 6.1]) that

$$[W(\zeta, \cdot)]' (u) = 2(Ju - \zeta). \tag{90}$$

Thus, we deduce that

$$W(Jz^n, x^{p(n)}) - W(Jz^n, x^{p(n+1)})$$
$$\geq \left\langle [W(Jz^n, \cdot)]' (x^{p(n+1)}), x^{p(n)} - x^{p(n+1)} \right\rangle$$
$$= 2 \left\langle Jx^{p(n+1)} - Jz^n, x^{p(n)} - x^{p(n+1)} \right\rangle .$$

Adding this and inequality (89) one gets

$$W(Jz^n, x^{p(n)}) - W(Jz^{n+1}, x^{p(n+1)})$$
$$\geq 2 \left\langle Jz^n - w^n, J^* w^n - x^{p(n+1)} \right\rangle$$
$$+ 2 \left\langle Jx^{p(n+1)} - Jz^n, x^{p(n)} - x^{p(n+1)} \right\rangle .$$

Thus, we have

$$\begin{aligned} W(Jz^{n+1}, x^{p(n+1)}) \leq\; & W(Jz^n, x^{p(n)}) \\ & + 2 \left\langle Jx^{p(n+1)} - Jz^n, x^{p(n+1)} - x^{p(n)} \right\rangle \\ & + 2 \left\langle w^n - Jz^n, J^* w^n - x^{p(n+1)} \right\rangle . \end{aligned}$$

By consequence, we obtain

$$W(Jz^{n+1}, x^{p(n+1)}) \leq W(Jz^n, x^{p(n)}) + 2\left\langle Jx^{p(n+1)} - Jz^n, x^{p(n+1)} - x^{p(n)}\right\rangle + 2\left(U_n + V_n + Z_n\right). \tag{91}$$

where

$$\begin{aligned} U_n &:= \langle w^n - Jz^n, J^*w^n - z^n\rangle, \\ V_n &:= \left\langle w^n - Jz^n, z^n - x^{p(n)}\right\rangle, \\ Z_n &:= \left\langle w^n - Jz^n, x^{p(n)} - x^{p(n+1)}\right\rangle. \end{aligned} \tag{92}$$

Recall (see [5, Theorem 7.5]) that, if $\xi', \xi'' \in X^*$ and $R \geq \sqrt{\frac{1}{2}\left(\|\xi'\|_*^2 + \|\xi''\|_*^2\right)}$, then we have that

$$\left\langle \xi' - \xi'', J^*\xi' - J^*\xi''\right\rangle \leq 2LR^2\rho_{X^*}\left(4R^{-1}\left\|\xi' - \xi''\right\|_*\right), \tag{93}$$

where ρ_{X^*} is the modulus of smoothness of X^*. Let $\vartheta^n = Jz^n$ and observe that $z^n = J^*(Jz^n) = J^*\vartheta^n$. Note that, according to (75) and (**A4**), we have that

$$\begin{aligned} \|w^n\|_* &\leq \|z^n\| + \varepsilon_{p(n)}\left(\|\chi^n\|_* + \alpha_{p(n)}\|z^n\| + M_2\right) \\ &\leq (1 + \bar{\alpha})\|z^n\| + \varepsilon_{p(n)}\left(\Phi(\|z^n\|) + M_2\right) \leq \varkappa_n. \end{aligned}$$

Thus, we deduce that

$$\begin{aligned} r_n &\geq \sqrt{\varkappa_n^2 + \|z^n\|^2} \geq \sqrt{\|w^n\|_*^2 + \|z^n\|^2} \\ &\geq \sqrt{\tfrac{1}{2}\left(\|w^n\|_*^2 + \|z^n\|^2\right)} = \sqrt{\tfrac{1}{2}\left(\|w^n\|_*^2 + \|\vartheta^n\|_*^2\right)}, \end{aligned}$$

because of (75). Then, by (93), we obtain that

$$\begin{aligned} U_n &= \langle w^n - \vartheta^n, J^*w^n - J^*\vartheta^n\rangle \\ &\leq 2Lr_n^2\rho_{X^*}\left(4r_n^{-1}\|w^n - Jz^n\|_*\right). \end{aligned} \tag{94}$$

Note that, by (85), we have

$$\|w^n - Jz^n\|_* = \varepsilon_{p(n)}\left\|\chi^n + \alpha_{p(n)}Jz^n - f^{p(n)}\right\|_*, \tag{95}$$

where, according to (58), $\|\chi^n\|_* \le \Phi(\|z^n\|)$. Thus, we deduce that

$$\|w^n - Jz^n\|_* \le \varepsilon_{p(n)} \left(\|\chi^n\|_* + \alpha_{p(n)} \|z^n\| + \left\| f^{p(n)} \right\|_* \right) \le \varepsilon_{p(n)} \left(\Phi(\|z^n\|) + \alpha_{p(n)} \|z^n\| + M_2 \right) \le \varepsilon_{p(n)} \mu_n, \tag{96}$$

where μ_n is given by (85). This and (94) imply

$$U_n \le 2L r_n^2 \rho_{X^*} \left(4 r_n^{-1} \mu_n \varepsilon_{p(n)} \right). \tag{97}$$

Now we are going to estimate V_n. Observe that, since $x^{p(n)}$ is a solution of (4) with $k = p(n)$, there exists $\phi^{p(n)} \in A_{p(n)} x^{p(n)}$ such that

$$\left\langle \phi^{p(n)} + \alpha_{p(n)} J x^{p(n)} - f^{p(n)}, y - x^{p(n)} \right\rangle \ge 0, \quad \forall y \in \Omega_{p(n)}. \tag{98}$$

According to (85) and (98), we have

$$\begin{aligned} V_n &= -\varepsilon_{p(n)} \left\langle \chi^n + \alpha_{p(n)} J z^n - f^{p(n)}, z^n - x^{p(n)} \right\rangle \\ &= -\varepsilon_{p(n)} \left[\left\langle \chi^n - \phi^{p(n)}, z^n - x^{p(n)} \right\rangle \right. \\ &+ \alpha_{p(n)} \left\langle J z^n - J x^{p(n)}, z^n - x^{p(n)} \right\rangle \\ &+ \left. \left\langle \phi^{p(n)} + \alpha_{p(n)} J x^{p(n)} - f^{p(n)}, z^n - x^{p(n)} \right\rangle \right] \\ &\le -\varepsilon_{p(n)} \left[\left\langle \chi^n - \phi^{p(n)}, z^n - x^{p(n)} \right\rangle \right. \\ &+ \left. \alpha_{p(n)} \left\langle J z^n - J x^{p(n)}, z^n - x^{p(n)} \right\rangle \right]. \end{aligned} \tag{99}$$

The operator $A_{p(n)}$ is strongly $h_{p(n)}$-hypomonotone on $\Omega_{p(n)}$, and this implies

$$\left\langle \chi^n - \phi^{p(n)}, z^n - x^{p(n)} \right\rangle \ge -h_{p(n)} \left(\|z^n\| - \left\| x^{p(n)} \right\| \right)^2.$$

Using (90) and the convexity of $W(Jz^n, \cdot)$, we deduce that

$$W(Jz^n, x^{p(n)}) - W(Jz^n, z^n) \le 2 \left\langle J z^n - J x^{p(n)}, z^n - x^{p(n)} \right\rangle,$$

where $W(Jz^n, z^n) = 0$. Thus, by (99), we get

$$V_n \le \varepsilon_{p(n)} h_{p(n)} \left(\|z^n\| - \left\| x^{p(n)} \right\| \right)^2 - \frac{\varepsilon_{p(n)} \alpha_{p(n)}}{2} W(Jz^n, x^{p(n)}). \tag{100}$$

For evaluating Z_n we take into account that

$$Z_n \leq \|w^n - Jz^n\|_* \left\|x^{p(n)} - x^{p(n+1)}\right\|$$
$$\leq \varepsilon_{p(n)}\mu_n \left\|x^{p(n)} - x^{p(n+1)}\right\|,$$

where the last inequality follows from (96). Consequently,

$$Z_n \leq \varepsilon_{p(n)}\mu_n \left(\left\|x^{p(n)} - u^{p(n)}\right\| + \left\|u^{p(n+1)} - x^{p(n+1)}\right\| + \left\|u^{p(n)} - u^{p(n+1)}\right\|\right). \tag{101}$$

Applying Lemma 4.2 successively, first for $k = p(n)$ and next for $k = p(n+1)$, we obtain

$$\left\|x^{p(n)} - u^{p(n)}\right\| + \left\|u^{p(n+1)} - x^{p(n+1)}\right\| \leq G_{p(n)} + G_{p(n+1)}. \tag{102}$$

Consider the variational inequalities

$$\left\langle (A + \alpha_{p(n)}J)x - f, y - x\right\rangle \geq 0, \ \forall y \in \Omega,$$

and

$$\left\langle (A + \alpha_{p(n+1)}J)x - f, y - x\right\rangle \geq 0, \ \forall y \in \Omega,$$

which have the solutions $u^{p(n)}$ and $u^{p(n+1)}$, respectively. These variational inequalities satisfy the requirements of Theorem 3.1 when $\bar{a}$, $\bar{b}$, $\bar{c}$, h, δ, τ and ω are all zero. Therefore, application of Theorem 3.1 in this particular case, leads to the inequality

$$\left\|u^{p(n)} - u^{p(n+1)}\right\| \leq K_0 g_X^\sharp \left(K_1 \frac{\left|\alpha_{p(n)} - \alpha_{p(n+1)}\right|}{\alpha_{p(n)}}\right), \tag{103}$$

which, in conjunction with (102), implies

$$\left\|x^{p(n)} - x^{p(n+1)}\right\| \leq G_{p(n)} + G_{p(n+1)} + \bar{G}_{p(n)}. \tag{104}$$

So, from (101) and (104), we get

$$Z_n \leq \varepsilon_{p(n)}\mu_n \left(G_{p(n)} + G_{p(n+1)} + \bar{G}_{p(n)}\right), \tag{105}$$

where $\bar{G}_{p(n)}$ is defined by (70). Combining (91), (92), (97), (100) and (105) we deduce that

$$\begin{aligned} W(Jz^{n+1}, x^{p(n+1)}) \leq & W(Jz^n, x^{p(n)}) - \varepsilon_{p(n)}\alpha_{p(n)}W(Jz^n, x^{p(n)}) \\ & + 2\left\langle Jx^{p(n+1)} - Jz^n, x^{p(n+1)} - x^{p(n)}\right\rangle \\ & + 2\left[2Lr_n^2\rho_{X^*}\left(4r_n^{-1}\varepsilon_{p(n)}\mu_n\right)\right. \\ & + \varepsilon_{p(n)}h_{p(n)}\left(\|z^n\| - \left\|x^{p(n)}\right\|\right)^2 \\ & \left.+\varepsilon_{p(n)}\mu_n\left(G_{p(n)} + G_{p(n+1)} + \bar{G}_{p(n)}\right)\right]. \end{aligned} \tag{106}$$

Note that

$$\begin{aligned} & \left\langle Jx^{p(n+1)} - Jz^n, x^{p(n+1)} - x^{p(n)}\right\rangle \\ & \leq \left\|Jx^{p(n+1)} - Jz^n\right\|_* \left\|x^{p(n+1)} - x^{p(n)}\right\| \\ & \leq \left(\left\|Jx^{p(n+1)}\right\|_* + \|Jz^n\|_*\right)\left\|x^{p(n+1)} - x^{p(n)}\right\| \\ & = \left(\left\|x^{p(n+1)}\right\| + \|z^n\|\right)\left\|x^{p(n+1)} - x^{p(n)}\right\| \\ & \leq (M + \|z^n\|)\left\|x^{p(n+1)} - x^{p(n)}\right\| \\ & \leq (M + \|z^n\|)(G_{p(n)} + G_{p(n+1)} + \bar{G}_{p(n)}), \end{aligned}$$

where the last two inequalities result from (60) and (104), respectively. This and (106) imply

$$\begin{aligned} & W(Jz^{n+1}, x^{p(n+1)}) \\ & \leq W(Jz^n, x^{p(n)}) - \varepsilon_{p(n)}\alpha_{p(n)}W(Jz^n, x^{p(n)}) \\ & + 2\left[2Lr_n^2\rho_{X^*}\left(4r_n^{-1}\mu_n\varepsilon_{p(n)}\right)\right. \\ & + \varepsilon_{p(n)}h_{p(n)}\left(\|z^n\| - \left\|x^{p(n)}\right\|\right)^2 \\ & \left.+\left(\varepsilon_{p(n)}\mu_n + M + \|z^n\|\right)\left(G_{p(n)} + G_{p(n+1)} + \bar{G}_{p(n)}\right)\right]. \end{aligned} \tag{107}$$

According to (60) we have that

$$\left(\|z^n\| - \left\|x^{p(n)}\right\|\right)^2 \leq \|z^n\|^2 + M^2.$$

Consequently, we deduce that

$$
\begin{aligned}
&2Lr_n^2\rho_{X^*}\left(4r_n^{-1}\mu_n\varepsilon_{p(n)}\right)+\varepsilon_{p(n)}h_{p(n)}\left(\|z^n\|-\left\|x^{p(n)}\right\|\right)^2\\
&+\left(\varepsilon_{p(n)}\mu_n+M+\|z^n\|\right)\left(G_{p(n)}+G_{p(n+1)}+\bar{G}_{p(n)}\right)\\
&\le 2Lr_n^2\rho_{X^*}\left(4r_n^{-1}\mu_n\varepsilon_{p(n)}\right)+\varepsilon_{p(n)}h_{p(n)}\left(\|z^n\|^2+M^2\right)\\
&+\left(\varepsilon_{p(n)}\mu_n+M+\|z^n\|\right)\left(G_{p(n)}+G_{p(n+1)}+\bar{G}_{p(n)}\right)\\
&=\tfrac{1}{2}\Upsilon(\|z^n\|,p(n)).
\end{aligned}
$$

This and (107) implies (84). □

The following result ensures boundedness of the sequence $\left\{z^k\right\}_{k\in\mathbb{N}}$ generated by the algorithm.

Lemma 4.4: *For any $n\in N$ we have that*

$$\|z^n\|\le K \quad \text{and} \quad W(Jz^n,x^{p(n)})\le R_0,$$

where R_0 and K are defined in Step 0 of the algorithm (*see* (74)).

Proof. It is sufficient to show that

$$W(Jz^n,x^{p(n)})\le R_0,\quad \forall n\in\mathbb{N}. \tag{108}$$

Indeed, if (108) is true, then we have

$$\left(\|z^n\|-\left\|x^{p(n)}\right\|\right)^2\le W(Jz^n,x^{p(n)})\le R_0,$$

where the first inequality follows from the definition of W. This implies

$$\|z^n\|\le\sqrt{R_0}+\left\|x^{p(n)}\right\|\le\sqrt{R_0}+M=K.$$

We prove (108) by induction upon n. If $n=0$, then (108) holds because of (77). Now, assume that (108) holds for some nonnegative integer n. Suppose by contradiction that

$$W(Jz^{n+1},x^{p(n+1)})>R_0. \tag{109}$$

Then, according to (76), we deduce that

$$W(Jz^{n+1},x^{p(n+1)})>R_0\ge\Upsilon\left(K,p(n)\right)\left(\alpha_{p(n)}\varepsilon_{p(n)}\right)^{-1}$$

because $p(n) = n + n_0 \geq n_0$. As noted above, the assumption that (108) holds for n implies that $\|z^n\| \leq K$. The function $\Upsilon(\cdot, p(n))$, defined at (75), is nondecreasing. Thus, we have

$$W(Jz^{n+1}, x^{p(n+1)}) \geq \Upsilon(K, p(n)) \left(\alpha_{p(n)}\varepsilon_{p(n)}\right)^{-1} \geq \Upsilon(\|z^n\|, p(n)) \left(\alpha_{p(n)}\varepsilon_{p(n)}\right)^{-1}.$$

This and Lemma 4.3 imply that

$$\begin{aligned}&(1-\alpha_{p(n)}\varepsilon_{p(n)})W(Jz^{n+1}, x^{p(n+1)}) \\ &\leq W(Jz^{n+1}, x^{p(n+1)}) - \Upsilon(\|z^n\|, p(n)) \\ &\leq (1-\varepsilon_{p(n)}\alpha_{p(n)})W(Jz^n, x^{p(n)}).\end{aligned}$$

According to (**A5**), we have that $0 < \varepsilon_{p(n)}\alpha_{p(n)} < 1$. Hence, we obtain that

$$W(Jz^{n+1}, x^{p(n+1)}) \leq W(Jz^n, x^{p(n)}) \leq R_0,$$

and this contradicts (109). □

The following result completes the proof of Theorem 4.1. Recall that, as already shown in Section 2, in the current circumstances, the solution set $\mathcal{S}(A, f, \Omega)$ is convex and has a minimal norm element.

Lemma 4.5: *The sequence* $\{z^k\}_{k\in\mathbb{N}}$ *generated by the algorithm converges strongly to the minimal norm solution of the variational inequality* (1).

Proof. Lemma 4.4 ensures that the sequence of nonnegative real numbers $\{W(Jz^k, x^{p(k)})\}_{k\in\mathbb{N}}$ is bounded. Lemma 4.3 and Lemma 4.4 show that the numbers $\lambda_n = W(Jz^n, x^{p(n)})$ satisfy the inequality

$$\lambda_{n+1} \leq \lambda_n - \alpha_{p(n)}\varepsilon_{p(n)}\lambda_n + \Upsilon(K, p(n)),$$

where, according to (**A5**) and Lemma 4.1, we have

$$\lim_{n\to\infty} \alpha_{p(n)}\varepsilon_{p(n)} = 0 = \lim_{n\to\infty} \Upsilon(K, p(n)) \left(\alpha_{p(n)}\varepsilon_{p(n)}\right)^{-1}.$$

These allow us to apply Lemma 1 in [3] in order to deduce that

$$\lim_{n\to\infty} W(Jz^n, x^{p(n)}) = 0. \tag{110}$$

The space X being uniformly convex and smooth, the function $\theta(x) = \|x\|^2$ is uniformly convex on bounded sets and differentiable. According to [23, Proposition 4.2], in these circumstances, the modulus of total convexity of the function $\theta(x)$ on the bounded set $E = \{z^n\}_{n\in\mathbb{N}}$, denoted $\nu_\theta(E,\cdot)$, has $\nu_\theta(E,t) > 0$ when $t > 0$, and also satisfies

$$\nu_\theta\left(E, \left\|z^n - x^{p(n)}\right\|\right) \leq W(Jz^n, x^{p(n)}), \quad \forall n \in \mathbb{N}.$$

This and (110) imply that

$$\lim_{n\to\infty} \nu_\theta\left(E, \left\|z^n - x^{p(n)}\right\|\right) = 0.$$

This cannot hold unless $\lim_{n\to\infty} \left\|z^n - x^{p(n)}\right\| = 0$ because the function $\nu_\theta(E,\cdot)$ is (strictly) increasing on $[0,\infty)$ – cf. [24, Lemma 2.4]. By Theorem 2.1(ii), the sequence $\left\{x^{p(n)}\right\}_{n\in\mathbb{N}}$ converges strongly to the minimal norm solution of the variational inequality (1). Consequently, the sequence $\{z^n\}_{n\in\mathbb{N}}$ has the same limit and the proof is complete. □

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