REGULARIZATION METHODS FOR ILL-POSED INCLUSIONS AND VARIATIONAL INEQUALITIES WITH DOMAIN PERTURBATIONS

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ABSTRACT. We consider a class of Tychonov-type regularization methods for solving ill-posed inclusions requiring to find $x \in \Omega$ such that $f \in Ax$, or variational inequalities problems requiring to find $x \in \Omega$ such that $\langle Ax - f, x - z\rangle \le 0$, for all $z \in \Omega$, with maximal monotone operators A and closed convex sets Ω contained in $\mathrm{Dom}(A)$. The question we pose is whether the regularization methods we consider are stable under data perturbations involving modifications of the domain of A and of Ω. We show that in reflexive, strictly convex and smooth Banach spaces having Efimov's property the answer to this question is affirmative whenever the perturbations satisfy specific locally uniform boundedness conditions. Under these conditions, the regularization methods applied to the perturbed problems (instead of the initial problems) produce strong approximations of the minimal-norm solutions of the initial problems.

1. INTRODUCTION

In this paper we consider the following problems: Let X be a reflexive, smooth, strictly convex Banach space, $A : X \to X^*$ a maximal monotone point-to-set operator and Ω a nonempty closed convex subset of $Dom(A)$. Given $f \in X^*$,

$$\text{Find } x \in \Omega \text{ such that } f \in Ax, \tag{1}$$

or, alternatively,

$$\begin{gathered}\text{Find } x \in \Omega \text{ such that, for some } y \in Ax,\\ \langle y - f, x - z\rangle \le 0, \text{ for all } z \in \Omega,\end{gathered} \tag{2}$$

provided that such a vector $x \in \Omega$ exists, that is, the solution set is nonempty.

Problems like (1) and (2) may be ill-posed in the classical sense, that is, they may have infinitely many solutions and/or small data perturbations may lead to significant distortions of the solution sets. Examples of such problems, as well as surveys of various regularization techniques, can be found in [8] and [17]. A method

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of solving such problems consists of replacing the operator A by a *regularized* operator $A+\alpha J^{\mu}$ where $J^{\mu}: X \rightarrow X^{*}$ is the duality mapping of weight $\mu: \mathbf{R}_{+} \rightarrow \mathbf{R}_{+}$ and α is a positive real number. It is known, and it also follows from our results below, that, if the space X has Efimov's property, then, for small α, the solutions of the *regularized problems*

$$\text{Find } x \in \Omega \text{ such that } f \in A x+\alpha J^{\mu} x, \tag{3}$$

and, respectively,

$$\begin{gathered} \text{Find } x \in \Omega \text{ such that, for some } y \in A x, \\ \langle y+\alpha J^{\mu} x-f, x-z\rangle \leq 0, \text{ for all } z \in \Omega, \end{gathered} \tag{4}$$

are close to minimal norm solutions of the problems (1) and, respectively, (2). Recall that the space X is said to have *Efimov's property* if any sequence $\left\{x^{k}\right\}_{k \in \mathbf{N}}$ in X which converges weakly to some $x^{*} \in X$ converges strongly whenever $\lim _{k \rightarrow \infty}\left\|x^{k}\right\|=\left\|x^{*}\right\|$. Also, note that, since we presume that the problems (1) and (2) are consistent, their solution sets are nonempty, convex and closed (because A is maximal monotone) and, thus, these solutions sets contain unique elements of minimal norm (the metric projections onto them of the origin, which exists and are unique because X is strictly convex and smooth).

Observe that, since the space X is strictly convex and smooth, its dual X^{*} is smooth and strictly convex (cf. [6]). Therefore, no matter how the weight function μ is chosen, the duality mapping J^{μ} is single valued, the operator $A+\alpha J^{\mu}$ is maximal monotone with $\operatorname{Range}(A+\alpha J^{\mu})=X^{*}$ and its inverse $(A+\alpha J^{\mu})^{-1}: X^{*} \rightarrow X$ is maximal monotone and single valued too (cf. [15, Proposition 2.11, p. 122]). Due to these facts, each of the regularized problems (3) and (4) has unique solution by contrast to the original problems (1) and, respectively, (2), which may happen to have infinitely many solutions.

It often happens in practice that, either for computational or for measurement related reasons, the operator A, the vector f or both can not be precisely determined but approximations $\bar{A}$ and $\bar{f}$ of them are available. In such cases, the domains of A and of $\bar{A}$ may be quite different from each other and it may even happen that the set Ω, in which we are looking for solutions of the given problems, may not be contained in $\operatorname{Dom}(\bar{A})$ and a perturbed set $\bar{\Omega}$ should be considered instead. This naturally leads to the question whether, and under which conditions, determining solutions of the regularized problems associated to the perturbed problems (i.e., the problems (1) and (2) in which the data A, f, Ω are replaced by $\bar{A}$, $\bar{f}$ and $\bar{\Omega}$, respectively) still leads to approximations of minimal-norm solutions of the original problems. In other words, the question is whether the regularization methods described above are stable enough under data perturbations involving perturbations of $\operatorname{Dom}(A)$ and of the set Ω.

This question was studied before in a similar context by Mosco [14], Liskovets [12], [13], Ryasantseva [16], Alber [2], Alber and Notik [3] under the assumption that the perturbed operator $\bar{A}$ has $\operatorname{Dom}(\bar{A})=\operatorname{Dom}(A)$ and that $\bar{\Omega}$ is still contained in $\operatorname{Dom}(A)$. However, this condition is difficult to verify and, as illustrated by the example given in Section 2, it may not hold at all. The purpose of this paper is to provide answers to the question above without this or any other additional assumptions concerning the problem data and, in this way, to complete the knowledge on this topic. Theorem 2.2 and Theorem 3.2 show that, if X is reflexive, smooth, strictly convex and has Efimov's property and if the approximants $\bar{A}$, $\bar{f}$

and $\bar{\Omega}$ satisfy the conditions stated in Section 2.1 and, respectively, in Section 3.1, then by solving the regularized perturbed problem with a small enough relaxation parameter α one obtains norm-approximations of the minimal-norm solution of the original problem (1) and, respectively (2). In other words, we prove that the regularization methods we consider for solving inclusions in the form (1) and variational inequalities (2) are stable under certain data perturbations even if these perturbations affect the domain of the operator A and, implicitly, the set Ω in which the solutions should be contained.

A careful analysis of our proofs shows that the requirement that the space X has Efimov's property (a common feature of the most usual Banach spaces like the finite dimensional spaces, the Lebesgue spaces L^p and the Sobolev spaces $W^{m,p}$ with $p > 1$ - see [1]) is needed only when one wants to make sure that the solutions of the regularized perturbed problems are *strongly* approximating minimal norm solutions of the initial problem. The fact is that, if one contents himself with approximations in the weak topology, then the solutions of the regularized perturbed problems are weak approximations of the minimal norm solutions of the initial problems even if the space X does not have Efimov's property. In addition to that, one should observe that our proofs are based on a series of inequalities (see Lemma 2.3 and Lemma 3.4) which may be of use for studying other aspects of the inclusion (1) and variational inequality (2).

An interesting observation, for which we are indebted to the referees, is that, in some circumstances, one can represent the inclusion (1) as a particular variational inequality in the form (2) in which A is replaced by $A+N_\Omega$ where N_Ω is the normal cone of Ω and, then, one can apply Theorem 3.2 to this variational inequality and, thus, obtain a result similar to Theorem 2.2. However, the variant of Theorem 2.2 obtained in this way holds under much more restrictive conditions than Theorem 2.2 itself. This can be easily seen by comparing the hypothesis of Theorem 3.2 and Theorem 2.2. Needless to say that, replacing A by $A + N_\Omega$ in (2) and solving the variational inequality with the new data, implicitly means finding approximants of $A + N_\Omega$ and that is often difficult to do in practice.

The literature contains many results implicitly or explicitly establishing stability with respect to domain perturbations of regularization methods different from that described above (see, for instance, [8], [7] and the references therein). Usually, such methods are guaranteed to work for solving inclusions and/or variational inequalities whose data or geometric setting or both satisfy restrictive requirements which are not necessary functioning preconditions for the regularization methods we propose in this paper. However, the convergence of the methods we propose, when applied to a particular problem or another, may happen to be slower than that of methods specifically built for that particular type of problems and this may be a price to pay for the degree of generality of our regularization scheme. This leads to the open question of whether it is possible to determine rates of convergence for the regularization methods we propose and, if so, how are these rates dependent on the problem data and the geometry of the problem setting.

2. Regularization of ill-posed inclusions with domain perturbations

2.1 In this section X denotes a reflexive, smooth, strictly convex Banach space with Efimov's property. We consider the inclusion (1) where $A : X \to X^*$ is a maximal monotone point-to-set operator and $f \in A(\Omega)$. In these circumstances, the solution set of the inclusion (1) is nonempty, closed and convex and, therefore, it contains a (necessarily unique) minimal norm element (see, for instance, [10, p. 26]). We assume that there exist two positive real numbers h^* and δ^* and the functions $a, g, \zeta : \mathbf{R}_+ \to \mathbf{R}_+$ such that ζ is strictly increasing,

$$\lim_{t \searrow 0} \zeta(t) = 0 = \zeta(0) \tag{5}$$

and, for any pair $\gamma = (h, \delta) \in \Gamma := (0, h^*] \times (0, \delta^*]$, one can find a maximal monotone point-to-set operator $A^h : X \to X^*$ and a vector $f^\delta \in X^*$ with the properties that

$$\left\| f - f^\delta \right\| \leq \delta \tag{6}$$

and that for any $x \in Dom(A)$, there exists $x^h \in Dom(A^h)$ with

$$\left\| x - x^h \right\| \leq a(\|x\|)h; \tag{7}$$

and with

$$d_*(y, A^h x^h) := \inf \left\{ \|y - z\|_* : z \in A^h x^h \right\} \leq g(\|y\|_*)\zeta(h), \tag{8}$$

whenever $y \in Ax$. If, for some $\gamma = (h, \delta) \in \Gamma$, the maximal monotone operator $A^h : X \to X^*$ and the vector $f^\delta \in X^*$ satisfy (6), (7) and (8), then the pair (A^h, f^δ) is called a *γ-approximant of the pair (A, f)*.

2.2 It was noted in Section 1 that, under the assumptions made above, the spaces X and X^* are strictly convex and smooth and, therefore, for any weight function $\mu : \mathbf{R}_+ \to \mathbf{R}_+$, the duality mapping of weight μ is a single valued function $J^\mu : X \to X^*$. For any $\gamma = (h, \delta) \in \Gamma$, consider the inclusion

$$f^\delta \in A^h z + \alpha J^\mu z, \tag{9}$$

where α is a positive real number and (A^h, f^δ) is a γ-approximant of (A, f). This inclusion has an unique solution because $Range(A^h + \alpha J^\mu) = X^*$ and $(A^h + \alpha J^\mu)^{-1}$ is single valued (see, for instance, [6]). We denote by x_α^γ this solution, that is, we put

$$x_\alpha^\gamma = (A^h + \alpha J^\mu)^{-1} f^\delta. \tag{10}$$

Our purpose is to show that, if the perturbation of the data of the inclusion (1) is done by replacing the original data (A, f) by a γ-approximant with $\gamma = (\delta, h) \in \Gamma$ such that, for some small $\alpha > 0$, the number $(\delta + \zeta(h))\, \alpha^{-1}$ is small enough, then the corresponding x_α^γ is a reasonable good norm approximation of the unique minimal norm solution of (1). Precisely, we prove the following result.

Theorem. *If $\{(h_k, \delta_k)\}_{k \in \mathbf{N}} \subset \Gamma$ and $\{\alpha_k\}_{k \in \mathbf{N}} \subset (0, +\infty)$ are sequences satisfying the conditions*

$$\lim_{k \to \infty} \alpha_k = 0 \tag{11}$$

and

$$\lim_{k \to \infty} \frac{\delta_k + \zeta(h_k)}{\alpha_k} = 0, \tag{12}$$

then, for any sequence (A^{h_k}, f^{δ_k}) of γ_k-approximants of (A, f) with $\gamma_k = (h_k, \delta_k)$, the sequence $\left\{x^k\right\}_{k \in \mathbf{N}} \subset X$ which is defined by

$$x^k = (A^{h_k} + \alpha_k J^\mu)^{-1} f^{\delta_k}, \tag{13}$$

for any $k \in \mathbf{N}$*, converges strongly to the minimal norm solution of* (1)*, no matter how the weight function* $\mu : \mathbf{R}_+ \to \mathbf{R}_+$ *is chosen.*

The proof of this result, given below, consists of several lemmas. Some of these lemmas may be of interest by themselves because they emphasize properties of maximal monotone operators which, as far as we know, were not proved before.

2.3 We start the proof of the theorem above by establishing an inequality which relates the basic data we have about the approximants of (A, f) and the solutions of the inclusion (1).

Lemma. *If* $\gamma = (h, \delta) \in \Gamma$ *and if* $x \in X$ *is a solution of the inclusion* (1)*, then, for any positive real number* α*, we have*

$$\begin{aligned} \mu(\|x_\alpha^\gamma\|)\,\|x_\alpha^\gamma\| &\le \mu(\|x_\alpha^\gamma\|)\,[\|x\| + a(\|x\|)h] + \\ &\quad a(\|x\|)h\frac{\delta + g(\|f\|_*)\zeta(h)}{\alpha} + \frac{g(\|f\|_*)\zeta(h) + \delta}{\alpha}\,\|x_\alpha^\gamma - x\| \\ &\le \mu(\|x_\alpha^\gamma\|)\,[\|x\| + a(\|x\|)h] + \\ &\quad \frac{\delta + g(\|f\|_*)\zeta(h)}{\alpha}\,[\|x\| + a(\|x\|)h + \|x_\alpha^\gamma\|]\,. \end{aligned} \tag{14}$$

Proof. Let (A^h, f^δ) be a γ-approximant of (A, f) and let y_α^γ be a vector in $A^h x_\alpha^\gamma$ such that

$$y_\alpha^\gamma + \alpha J^\mu x_\alpha^\gamma = f^\delta. \tag{15}$$

Such a vector exists because x_α^γ is a solution of the inclusion (9). According to (15), we have

$$\left\langle y_\alpha^\gamma + \alpha J^\mu x_\alpha^\gamma - f^\delta, x_\alpha^\gamma - x^h \right\rangle = 0,$$

for all $x^h \in Dom(A^h)$. Hence,

$$\left\langle y_\alpha^\gamma - f, x_\alpha^\gamma - x^h \right\rangle + \alpha \left\langle J^\mu x_\alpha^\gamma, x_\alpha^\gamma - x^h \right\rangle = \left\langle f^\delta - f, x_\alpha^\gamma - x^h \right\rangle. \tag{16}$$

Note that, for any $y^h \in A^h x^h$, we have

$$\left\langle y_\alpha^\gamma - f, x_\alpha^\gamma - x^h \right\rangle = \left\langle y_\alpha^\gamma - y^h, x_\alpha^\gamma - x^h \right\rangle + \left\langle y^h - f, x_\alpha^\gamma - x^h \right\rangle,$$

where $\left\langle y_\alpha^\gamma - y^h, x_\alpha^\gamma - x^h \right\rangle \ge 0$, because A^h is monotone. Consequently,

$$\left\langle y_\alpha^\gamma - f, x_\alpha^\gamma - x^h \right\rangle \ge \left\langle y^h - f, x_\alpha^\gamma - x^h \right\rangle,$$

and, using (16), we get

$$\left\langle y^h - f, x_\alpha^\gamma - x^h \right\rangle + \alpha \left\langle J^\mu x_\alpha^\gamma, x_\alpha^\gamma - x^h \right\rangle \le \left\langle f^\delta - f, x_\alpha^\gamma - x^h \right\rangle. \tag{17}$$

Observe that, according to the definition of the duality mapping J^μ, we have

$$\begin{aligned} \left\langle J^\mu x_\alpha^\gamma, x_\alpha^\gamma - x^h \right\rangle &= \langle J^\mu x_\alpha^\gamma, x_\alpha^\gamma \rangle - \left\langle J^\mu x_\alpha^\gamma, x^h - x \right\rangle - \langle J^\mu x_\alpha^\gamma, x \rangle \\ &= \mu(\|x_\alpha^\gamma\|)\,\|x_\alpha^\gamma\| - \left\langle J^\mu x_\alpha^\gamma, x^h - x \right\rangle - \langle J^\mu x_\alpha^\gamma, x \rangle\,. \end{aligned}$$

Combining this and (17), we obtain

$$
\begin{aligned}
(18) \qquad \alpha\mu(\|x_\alpha^\gamma\|)\,\|x_\alpha^\gamma\| &\le \langle f^\delta - f, x_\alpha^\gamma - x^h\rangle - \langle y^h - f, x_\alpha^\gamma - x^h\rangle \\
&+ \alpha\left[\langle J^\mu x_\alpha^\gamma, x^h - x\rangle + \langle J^\mu x_\alpha^\gamma, x\rangle\right] \\
&\le \alpha\left[\mu(\|x_\alpha^\gamma\|)\left\|x^h - x\right\| + \mu(\|x_\alpha^\gamma\|)\,\|x\|\right] \\
&+ \langle f^\delta - f, x_\alpha^\gamma - x^h\rangle - \langle y^h - f, x_\alpha^\gamma - x^h\rangle \\
&\le \alpha\mu(\|x_\alpha^\gamma\|)\left(\left\|x^h - x\right\| + \|x\|\right) \\
&+ \left\|f^\delta - f\right\|_*\left\|x_\alpha^\gamma - x^h\right\| + \left\|y^h - f\right\|_*\left\|x_\alpha^\gamma - x^h\right\| \\
&\le \alpha\mu(\|x_\alpha^\gamma\|)\left(\left\|x^h - x\right\| + \|x\|\right) + \\
&+ \left\|y^h - f\right\|_*\left\|x_\alpha^\gamma - x^h\right\| + \delta\left\|x_\alpha^\gamma - x^h\right\|,
\end{aligned}
$$

where the last inequality holds because of (6). Now, let $x^h \in Dom(A^h)$ be a vector such (7) and (8) hold. Then, according to (18), we have

$$
\begin{aligned}
(19) \qquad \alpha\mu(\|x_\alpha^\gamma\|)\,\|x_\alpha^\gamma\| &\le \alpha\mu(\|x_\alpha^\gamma\|)\,(a(\|x\|)h + \|x\|) \\
&+ \left\|y^h - f\right\|_*\left\|x_\alpha^\gamma - x^h\right\| + \delta\left\|x_\alpha^\gamma - x^h\right\| \\
&\le \alpha\mu(\|x_\alpha^\gamma\|)\,(a(\|x\|)h + \|x\|) + \left\|y^h - f\right\|_*\left\|x_\alpha^\gamma - x^h\right\| \\
&+ \delta\left(\|x_\alpha^\gamma - x\| + \left\|x - x^h\right\|\right) \\
&\le \alpha\mu(\|x_\alpha^\gamma\|)\,(a(\|x\|)h + \|x\|) + \left\|y^h - f\right\|_*\left\|x_\alpha^\gamma - x^h\right\| \\
&+ \delta\,\|x_\alpha^\gamma - x\| + \delta a(\|x\|)h,
\end{aligned}
$$

for all $y^h \in A^h x^h$. Observe that $f \in Ax$ because x is a solution of (1). Hence, by (8), we have

$$
d_*(f, A^h x^h) \le g(\|f\|_*)\zeta(h).
$$

Note that, since A^h is maximal monotone, the set $A^h x^h$ is convex and closed. Also, $A^h x^h \neq \emptyset$ because $x^h \in Dom(A^h)$. Consequently, there exists an unique element $\bar{y}^h \in A^h x^h$ such that $d_*(f, A^h x^h) = \left\|\bar{y}^h - f\right\|_*$ (recall that X^* is strictly convex). Writing (19) with $\bar{y}^h$ instead of y^h and taking into account that

$$
\left\|\bar{y}^h - f\right\|_* = d_*(f, A^h x^h) \le g(\|f\|_*)\zeta(h),
$$

we deduce

$$
\begin{aligned}
\alpha\mu(\|x_\alpha^\gamma\|)\,\|x_\alpha^\gamma\| &\le \alpha\mu(\|x_\alpha^\gamma\|)\,(a(\|x\|)h + \|x\|) \\
&+ g(\|f\|_*)\zeta(h)\left\|x_\alpha^\gamma - x^h\right\| + \delta\,\|x_\alpha^\gamma - x\| + \delta a(\|x\|)h \\
&\le \alpha\mu(\|x_\alpha^\gamma\|)\,(a(\|x\|)h + \|x\|) \\
&+ g(\|f\|_*)\zeta(h)\left\|x_\alpha^\gamma - x^h\right\| + \delta\,\|x_\alpha^\gamma - x\| + \delta a(\|x\|)h \\
&+ \delta\,\|x_\alpha^\gamma - x\| + \delta a(\|x\|)h \\
&\le \alpha\mu(\|x_\alpha^\gamma\|)\,(a(\|x\|)h + \|x\|) + g(\|f\|_*)\zeta(h)a(\|x\|)h \\
&+ g(\|f\|_*)\zeta(h)\,\|x_\alpha^\gamma - x\| + \delta\,\|x_\alpha^\gamma - x\| + \delta a(\|x\|)h \\
&= \alpha\mu(\|x_\alpha^\gamma\|)\,(a(\|x\|)h + \|x\|) + a(\|x\|)h\,[\delta + g(\|f\|_*)\zeta(h)] \\
&+ [g(\|f\|_*)\zeta(h) + \delta]\,\|x_\alpha^\gamma - x\|
\end{aligned}
$$

Dividing this inequality by $\alpha > 0$ one obtains (14).

2.4 The relevance of Lemma 2.3 in our context comes from the following result.

Corollary. *For any real number $\varepsilon > 0$, there exists a positive real number $M(\varepsilon)$ such that, for all $\gamma = (h, \delta) \in \Gamma$ and $\alpha > 0$ which satisfy*

$$\frac{\delta + \zeta(h)}{\alpha} \leq \varepsilon, \tag{20}$$

we have $\|x_\alpha^\gamma\| \leq M(\varepsilon)$, no matter how the γ-approximants (A^h, f^δ) of (A, f) are chosen.

Proof. Observe that (20) implies

$$\frac{\delta + g(\|f\|_*)\zeta(h)}{\alpha} \leq \varepsilon \left[1 + g(\|f\|_*)\right]. \tag{21}$$

Let x be a solution of (1). According to Lemma 2.3, we have

$$\mu\left(\|x_\alpha^\gamma\|\right)\|x_\alpha^\gamma\| - C_1(\gamma)\mu\left(\|x_\alpha^\gamma\|\right) - C_2(\gamma)\|x_\alpha^\gamma\| - C_3(\gamma) \leq 0, \tag{22}$$

where

$$C_1(\gamma) := a\left(\|x\|\right)h + \|x\|,$$

$$C_2(\gamma) := \frac{\delta + g(\|f\|_*)\zeta(h)}{\alpha},$$

$$C_3(\gamma) := \frac{\delta + g(\|f\|_*)\zeta(h)}{\alpha}\left[a\left(\|x\|\right)h + \|x\|\right].$$

Denote $\varepsilon_0 := \varepsilon\left[1 + g(\|f\|_*)\right]$ and observe that $C_2(\gamma) \leq \varepsilon_0$ and $C_3(\gamma) \leq C_1(\gamma)\varepsilon_0$. Suppose, by contradiction, that for any integer $k > 0$ one can find real numbers h_k, δ_k and α_k such that $\gamma_k := (h_k, \delta_k) \in \Gamma$, $\alpha_k > 0$, $\left(\delta_k + \zeta(h_k)\right)/\alpha_k \leq \varepsilon$ and $\left\|x_{\alpha_k}^{\gamma_k}\right\| > k$. Taking into account (22), we obtain

$$\mu\left(\left\|x_{\alpha_k}^{\gamma_k}\right\|\right)\left\|x_{\alpha_k}^{\gamma_k}\right\| - C_1(\gamma_k)\mu\left(\left\|x_{\alpha_k}^{\gamma_k}\right\|\right) - \varepsilon_0\left\|x_{\alpha_k}^{\gamma_k}\right\| \leq C_1(\gamma_k)\varepsilon_0,$$

that is,

$$\mu\left(\left\|x_{\alpha_k}^{\gamma_k}\right\|\right)\left\|x_{\alpha_k}^{\gamma_k}\right\|\left[1 - \left(\frac{C_1(\gamma_k)}{\left\|x_{\alpha_k}^{\gamma_k}\right\|} + \frac{\varepsilon_0}{\mu\left(\left\|x_{\alpha_k}^{\gamma_k}\right\|\right)}\right)\right] \leq C_1(\gamma_k)\varepsilon_0. \tag{23}$$

Since μ is a weight function, it is strictly increasing and $\lim_{t\to\infty}\mu(t) = +\infty$. Hence,

$$\lim_{k\to\infty}\mu\left(\left\|x_{\alpha_k}^{\gamma_k}\right\|\right)\left\|x_{\alpha_k}^{\gamma_k}\right\| \geq \lim_{k\to\infty}\mu(k)\,k = +\infty.$$

Therefore, by letting $k \to \infty$ in (23), we get a contradiction.

2.5 Now, we are in position to establish weak convergence of the sequence $\left\{x^k\right\}_{k\in\mathbf{N}}$ given by (13).

Lemma. *If $\{(h_k, \delta_k)\}_{k\in\mathbf{N}} \subset \Gamma$ and $\{\alpha_k\}_{k\in\mathbf{N}} \subset (0, +\infty)$ are sequences satisfying the conditions* (11) *and* (12)*, then, for any sequence (A^{h_k}, f^{δ_k}) of γ_k-approximants of (A, f) with $\gamma_k = (h_k, \delta_k)$, the sequence $\left\{x^k\right\}_{k\in\mathbf{N}} \subset X$ defined by* (13)*, converges weakly to the minimal norm solution of* (1).

Proof. The sequence $\left\{\frac{\delta_k + \zeta(h_k)}{\alpha_k}\right\}_{k\in\mathbf{N}}$ converges to zero (cf. (12)). Therefore, this sequence is bounded by some $\varepsilon > 0$. Applying Corollary 2.4, it results that the sequence $\left\{x^k\right\}_{k\in\mathbf{N}}$ is bounded. Hence, the sequence $\left\{x^k\right\}_{k\in\mathbf{N}}$ has weak accumulation points because the space X is reflexive. We aim at showing that each weak accumulation point of $\left\{x^k\right\}_{k\in\mathbf{N}}$ is necessarily a minimal norm solution of (1). If this is true, then the sequence $\left\{x^k\right\}_{k\in\mathbf{N}}$ has an unique weak accumulation point [because there is an unique minimal norm solutions of (1) in the strictly convex space X] and, hence, the conclusion of the lemma is proved.

Let x^* be a weak accumulation point of $\{x^k\}_{k\in\mathbf{N}}$ and suppose that $\{x^{j_k}\}_{k\in\mathbf{N}}$ is a subsequence of $\{x^k\}_{k\in\mathbf{N}}$ which converges weakly to x^*. We claim that $f \in Ax^*$. Observe that, since A is maximal monotone, in order to prove this claim, it is sufficient to show that the following inequality holds

$$\langle f - y, x^* - w\rangle \geq 0, \tag{24}$$

for all $w \in Dom(A)$ and for all $y \in Aw$. Now, if $w \in Dom(A)$ and $y \in Aw$, then, for each $k \in N$, there exists a vector $w^{h_k} \in Dom(A^{h_k})$ such that

$$\left\|w - w^{h_k}\right\| \leq a(\|w\|)h_k \tag{25}$$

and such that, for some $y^{h_k} \in A^{h_k}w^{h_k}$,

$$\left\|y - y^{h_k}\right\|_* \leq g(\|y\|_*)\zeta(h_k). \tag{26}$$

Note that, for each $k \in N$, there exists $z^{h_k} \in A^{h_k}(x^k)$ such that

$$z^{h_k} + \alpha_k J^\mu x^k = f^{\delta_k}, \tag{27}$$

because of (13). From the monotonicity of A^{h_k} we deduce that

$$\left\langle z^{h_k} - y^{h_k}, x^k - w^{h_k}\right\rangle \geq 0. \tag{28}$$

Hence, we get

$$\begin{aligned} 0 &\leq \left\langle z^{h_k} - f, x^k - w^{h_k}\right\rangle + \left\langle f - y^{h_k}, x^k - w^{h_k}\right\rangle \\ &= \left\langle z^{h_k} - f, x^k - w^{h_k}\right\rangle + \left\langle f - y, x^k - w^{h_k}\right\rangle \\ &+ \left\langle y - y^{h_k}, x^k - w^{h_k}\right\rangle. \end{aligned} \tag{29}$$

Note that (5), (11) and (12) can not simultaneously hold unless

$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} \delta_k = 0. \tag{30}$$

Thus, $\{h_k\}_{k\in\mathbf{N}}$ is bounded and, according to (25), the sequence $\{w^{h_k}\}_{k\in\mathbf{N}}$ is bounded too. Taking into account (26), we deduce that the sequence $\{y^{h_k}\}_{k\in\mathbf{N}}$ converges strongly to y in X^*. Therefore,

$$\lim_{k\to\infty} \left\langle y - y^{h_k}, x^k - w^{h_k}\right\rangle = 0. \tag{31}$$

Observing that the sequence $\{f^{\delta_k}\}_{k\in\mathbf{N}}$ is strongly convergent to f in X^* because

$$\left\|f - f^{\delta_k}\right\|_* \leq \delta_k, \tag{32}$$

where $\{\delta_k\}_{k\in\mathbf{N}}$ converges to zero (see (6) and (30)) and taking into account that $\{J^\mu x^k\}_{k\in\mathbf{N}}$ is bounded (cf. [15, Lemma 3.6, p. 142]) we get $\lim_{k\to\infty} z^{h_k} = f$ because of (27) and because of (11). Consequently,

$$\lim_{k\to\infty} \left\langle z^{h_k} - f, x^k - w^{h_k}\right\rangle = 0. \tag{33}$$

Letting $k \to \infty$ in (29) and taking into account (31), (32) and (33) the inequality (24) results and it shows that $f \in Ax^*$, i.e., x^* is a solution of (1).

It remains to show that, no matter how the weak accumulation point x^* of $\{x^k\}_{k\in\mathbf{N}}$ is chosen, it is a minimal norm solution of (1), that is $\|x^*\| \leq \|x\|$, for any solution x of (1). This clearly holds when $x^* = 0$. Suppose that $x^* \neq 0$. Observe that the first inequality in (14) implies that

$$\mu(\|x^k\|)\|x^k\| \leq \mu(\|x^k\|)\left(a(\|x\|)h_k + \|x\|\right) \tag{34}$$
$$+a(\|x\|)h_k\frac{\delta_k + g(\|f\|_*)\zeta(h_k)}{\alpha_k} + \frac{g(\|f\|_*)\zeta(h_k) + \delta_k}{\alpha_k}\left\|x^k - x\right\|,$$

for all $k \in \mathbf{N}$ and any solution x of (1). Let $\{x^{j_k}\}_{k\in\mathbf{N}}$ be a subsequence of $\{x^k\}_{k\in\mathbf{N}}$ such that $\{x^{j_k}\}_{k\in\mathbf{N}}$ converges weakly to x^*. Obviously,

$$0 < \frac{\|x^*\|}{2} < \|x^*\| \leq \liminf_{k\to\infty}\left\|x^{j_k}\right\|. \tag{35}$$

Therefore, there exists a positive integer k_0 such that, for all $k \geq k_0$, we have $\left\|x^{j_k}\right\| \geq \|x^*\|/2$ and, then,

$$\mu(\left\|x^{j_k}\right\|) \geq \mu(\frac{1}{2}\|x^*\|) > 0,$$

because μ is increasing. This and (34) lead to the following inequality for each $k \geq k_0$:

$$\begin{aligned}\left\|x^{j_k}\right\| &\leq a(\|x\|)h_{j_k} + \|x\| + a(\|x\|)h_{j_k}\frac{\delta_{j_k} + g(\|f\|_*)\zeta(h_{j_k})}{\alpha_{j_k}\mu(\|x^{j_k}\|)} \\ &+ \frac{g(\|f\|_*)\zeta(h_{j_k}) + \delta_{j_k}}{\alpha_{j_k}\mu(\|x^{j_k}\|)}\left\|x^{j_k} - x\right\| \\ &\leq a(\|x\|)h_{j_k} + \|x\| + a(\|x\|)h_{j_k}\frac{\delta_{j_k} + g(\|f\|_*)\zeta(h_{j_k})}{\alpha_{j_k}\mu(\frac{1}{2}\|x^*\|)} \\ &+ \frac{g(\|f\|_*)\zeta(h_{j_k}) + \delta_{j_k}}{\alpha_{j_k}\mu(\frac{1}{2}\|x^*\|)}\left\|x^{j_k} - x\right\|.\end{aligned}$$

Since (11), (12) and (30) hold, by letting $k \to \infty$ on both sides of this inequality, we obtain

$$\limsup_{k\to\infty}\left\|x^{j_k}\right\| \leq \|x\|. \tag{36}$$

for any solution x of (1). From (35) and (36) we get

$$\|x^*\| \leq \liminf_{k\to\infty}\left\|x^{j_k}\right\| \leq \limsup_{k\to\infty}\left\|x^{j_k}\right\| \leq \|x\|, \tag{37}$$

for any solution x of (1). This implies that x^* is a minimal norm solution of (1).

2.6 The following lemma completes the proof of Theorem 2.2 by showing that the sequence $\{x^k\}_{k\in\mathbf{N}}$ is strongly convergent. Observe that, until now, we did not make use of the assumption that the Banach space X has Efimov's property. However, this assumption is essentially involved in our proof of strong convergence given below.

Lemma. *If $\{(h_k, \delta_k)\}_{k\in\mathbf{N}} \subset \Gamma$ and $\{\alpha_k\}_{k\in\mathbf{N}} \subset (0, +\infty)$ are sequences satisfying the conditions* (11) *and* (12), *then, for any sequence (A^{h_k}, f^{δ_k}) of γ_k-approximants of (A, f) with $\gamma_k = (h_k, \delta_k)$, the sequence $\{x^k\}_{k\in\mathbf{N}} \subset X$ defined by* (13) *converges strongly.*

Proof. According to Lemma 2.5, we have that $\{x^k\}_{k\in\mathbf{N}}$ converges weakly to a minimal norm solution x^* of (1). We distinguish two possible situations. Assume first that $x^* = 0$. Then, the second inequality in (14) written for $x = x^*$ implies that

$$\mu(\|x^k\|)\,\|x^k\| \leq \mu(\|x^k\|)a(\|x^*\|)h_k + \frac{\delta_k + g(\|f\|_*)\zeta(h_k)}{\alpha_k}\left[a(\|x^*\|)h_k + \|x^k\|\right],$$

for all $k \in N$. Letting $k \to \infty$ on both sides of this inequality we obtain

$$0 \leq \limsup_{k\to\infty} \mu(\|x^k\|)\,\|x^k\| \leq 0,$$

which can not hold unless $\lim_{k\to\infty}\|x^k\| = 0 = \|x^*\|$. Hence, by Efimov's property, we deduce that $\lim_{k\to\infty} x^k = 0 = x^*$ in this case. Now, assume that $x^* \neq 0$. Repeating the argument leading to (37) in the proof of Lemma 2.5 with $\{x^k\}_{k\in\mathbf{N}}$ instead of $\{x^{j_k}\}_{k\in\mathbf{N}}$ we deduce that

$$\|x^*\| \leq \liminf_{k\to\infty}\|x^k\| \leq \limsup_{k\to\infty}\|x^k\| \leq \|x\|,$$

for any solution x of (1). In particular, this inequality holds for $x = x^*$ and it shows that $\lim_{k\to\infty}\|x^k\| = \|x^*\|$. Applying Efimov's property again, we obtain that $\lim_{k\to\infty} x^k = x^*$ in this case too.

2.7 The regularization method presented above can be applied for solving ill-posed differential equations, feasibility and optimization problems. In order to illustrate how the method works we consider the inclusion (1) in $X = \mathbf{R}^2$ with $\Omega \subset \mathbf{R}^2$ being the closed convex cone determined by the lines of equations $x_2 = q_1x_1$ and $x_2 = q_2x_1$, where $0 < q_1 < q_2$ are constants (see Figure 1) and the operator $A : \Omega \to \mathbf{R}^2$ is given by

$$(38)\qquad Ax = \begin{cases} Bx & \text{if } x \in Int(\Omega), \\ \{Bx + \lambda u^1 : \lambda \geq 0\} & \text{if } x_2 = q_1x_1,\ x \neq 0, \\ \{Bx + \lambda u^2 : \lambda \geq 0\} & \text{if } x_2 = q_2x_1,\ x \neq 0, \\ \{\lambda\left[tu^1 + (1-t)u^2\right] : \lambda \geq 0,\ t \in [0,1]\} & \text{if } x = 0, \end{cases}$$

where $u^1 = (q_1, -1)$, $u^2 = (-q_2, 1)$ and $B : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear operator

$$(39)\qquad Bx = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} x.$$

Note that the operator A is monotone and its domain is closed, convex and has nonempty interior. Hence, A has an unique maximal monotone extension with domain $\Omega = Dom(A)$ which is exactly A itself because of the way in which A is given on the boundary (see [5]).

For each $h \in (0,1]$ consider a closed convex cone $\Omega_h \subset \mathbf{R}^2$ determined by the lines $x_2 = q_1(h)x_1$ and $x_2 = q_2(h)x_2$, where $q_1 < q_1(h) < q_2 < q_2(h) < \infty$. Define the operator $B^h : \Omega_h \to \mathbf{R}^2$ by

$$(40)\qquad B^hx = \begin{pmatrix} (1+h)^2 & 2+h \\ 2+h & 4+h \end{pmatrix} x,$$

FIGURE 1. The domains of the given and of the perturbed operators

and note that B^h is positively definite. Therefore, the operator $A^h : \Omega_h \to \mathbf{R}^2$ given by

$$A^h x = \begin{cases} B^h x & \text{if } x \in Int(\Omega_h), \\ \left\{B^h x + \lambda u^{1,h} : \lambda \geq 0\right\} & \text{if } x_2 = q_1(h)x_1,\ x \neq 0, \\ \left\{B^h x + \lambda u^{2,h} : \lambda \geq 0\right\} & \text{if } x_2 = q_2(h)x_1,\ x \neq 0, \\ \left\{\lambda\left[tu^{1,h} + (1-t)u^{2,h}\right] : \lambda \geq 0,\ t \in [0,1]\right\} & \text{if } x = 0, \end{cases}$$

where $u^{1,h} = (q_1(h), -1)$ and $u^{2,h} = (-q_2(h), 1)$ is maximal monotone too (for similar reasons as A).

We show next that, if there exists a constant $c_0 > 0$ such that for any $h \in (0, 1]$ we have

$$\max\left(\left|q_1 - q_1(h)\right|, \left|q_2 - q_2(h)\right|\right) \leq c_0 h,$$

then the operators A and A^h defined above have the property that there exist the functions $a, g, \zeta : \mathbf{R}_+ \to \mathbf{R}_+$ such that (7) and (8) hold for any $h \in (0, 1]$. To this end, we denote by $\|\cdot\|_0$ the maximum norm on $\mathbb{R}^2$ and observe that for the linear operators B and B^h we have

$$\left\|Bx - B^h x\right\|_0 \leq \left\|B - B^h\right\|_0 \cdot \|x\|_0 \leq 3h\,\|x\|_0\,.$$

Since the norm $\|\cdot\|_0$ and the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^2$ are equivalent, it results that for some positive constant c_1 we have

$$\left\|Bx - B^h x\right\| \leq \left\|B - B^h\right\| \cdot \|x\| \leq c_1 h\,\|x\|\,. \tag{41}$$

We distinguish two possible situations:

Case 1: $x \in Int(\Omega)$. In this case we have $\|Ax\| = \|Bx\| \geq \sqrt{5}\,\|x\|$ and, according to (41), we deduce that

$$\left\|Bx - B^h x\right\| \leq c_2\,\|Ax\|\,h, \tag{42}$$

where $c_2 = \frac{c_1}{\sqrt{5}}$. Let x^h be any of the point where the circle of radius $\|x\|$ intersects the boundary of Ω_h and denote

$$\phi_i(h) = \arctan\frac{\left|q_i - q_i(h)\right|}{1 + q_i q_i(h)},\ i = 1, 2.$$

Then

$$\begin{aligned} \left\|x - x^h\right\| &\leq \widehat{(x, x^h)} \leq \|x\| \max\left(\phi_1(h), \phi_2(h)\right) \\ &\leq \|x\| \max\left(\left|q_1 - q_1(h)\right|, \left|q_2 - q_2(h)\right|\right) \\ &\leq c_0\,\|x\|\,h \leq \frac{c_0}{\sqrt{5}}\,\|Ax\|\,h \end{aligned} \tag{43}$$

and, since $B^h x^h \in A^h x^h$, we get that, for all $h \in (0,1]$,

$$
\begin{aligned}
d_*(Ax, A^h x^h) &\leq d_*(Ax, B^h x^h) = \left\| Bx - B^h x^h \right\| \\
&\leq \left\| Bx - B^h x \right\| + \left\| B^h x - B^h x^h \right\| \\
&\leq c_2 \left\| Ax \right\| h + \left\| B^h \right\| \cdot \left\| x - x^h \right\| \\
&\leq c_2 \left\| Ax \right\| h + \left\| B^h \right\| \frac{c_0}{\sqrt{5}} \left\| Ax \right\| h \leq c_3 \left\| Ax \right\| h,
\end{aligned} \tag{44}
$$

where c_3 is a positive constant.

Case 2: $x \in \partial\Omega$. In this case, for some $i \in \{1,2\}$, we have $x_2 = q_i x_1$ and we chose $x^h := (x_1^h, x_2^h)$ such that $\left\| x^h \right\| = \|x\|$ and $x_2^h = q_i(h) x_1^h$. If $y \in Ax$, then for some $\lambda \geq 0$ we have

$$
y = Bx + \lambda u^i
$$

and, consequently, there exists a constant $c_4 > 0$ such that

$$
\begin{aligned}
d_*(y, A^h x^h) &\leq \left\| \left[Bx + \lambda u^i \right] - \left[B^h x^h + \lambda u^{i,h} \right] \right\| \\
&\leq \left\| Bx - B^h x^h \right\| + \lambda \left| q_i - q_i(h) \right| \\
&\leq c_1 \left\| Bx \right\| h + \lambda c_0 h \leq c_4 \left\| y \right\| h,
\end{aligned} \tag{45}
$$

for all $h \in (0,1]$.

Summarizing the results in (43), (44) and (45), we obtain that conditions (7) and (8) hold for the functions $\zeta(t) = t$, $a(t) = \frac{c_0}{\sqrt{5}} t$ and $g(t) = t \max(c_3, c_4)$.

Computational experiments with the method (13) show that the speed convergence of the method is very much dependent on the choice of the weight function μ and on the sizes of perturbations of data (i.e., the sequences $\{h_k\}_{k\in\mathbf{N}}$, $\{\delta_k\}_{k\in\mathbf{N}}$ and $\{\alpha_k\}_{k\in\mathbf{N}}$) and it may happen to be slow. For instance, consider the example discussed above with $q_1 = 0.25$, $q_2 = 0.5$, $\Omega_h = B^h(\Omega)$. We are trying to approximate via (13) the minimal norm solution in Ω of the linear system $Ax = f$ in which $f = (3,6)^T$ by using the weight function $\mu(t) = 2t$, i.e., the normalized duality mapping $J^\mu x = x$. In these circumstances,

$$
(A^h + \alpha J^\mu)^{-1} x =
$$

$$
\begin{pmatrix}
\frac{4+h+\alpha}{5h+5\alpha+5h^2+3\alpha h+h^3+h^2\alpha+\alpha^2} & -\frac{2+h}{5h+5\alpha+5h^2+3\alpha h+h^3+h^2\alpha+\alpha^2} \\
-\frac{2+h}{5h+5\alpha+5h^2+3\alpha h+h^3+h^2\alpha+\alpha^2} & \frac{1+2h+h^2+\alpha}{5h+5\alpha+5h^2+3\alpha h+h^3+h^2\alpha+\alpha^2}
\end{pmatrix} x,
$$

whenever $x \in \mathbf{R}^2$. The minimal norm solution of the problem is $x^* = \left(\frac{9}{5}, \frac{3}{5}\right)^T$ whose norm is $\|x\| \approx 1.894$. We take $f^\delta = (3 + 0.5\delta, 6 - 0.5\delta)^T$ which satisfies

$$
\left\| f - f^\delta \right\| = \frac{\sqrt{2}}{2} \delta \leq \delta.
$$

The procedure is then

$$
x^k = (A^{h_k} + \alpha_k J^\mu)^{-1} \begin{pmatrix} 3 + 0.5\delta_k \\ 6 - 0.5\delta_k \end{pmatrix}.
$$

We take $h_k = 1/k^2 = \delta_k$ and $\alpha_k = 1/k$ and we get $x^1 = (.21426, .80954)^T$, $x^{10} = (.49803, 1.217)^T$, $x^{100} = (.59592, 1.199)^T$, whose distances to x^* are 1.5995, 1.4408 and 1.3448, respectively. One may observe that the distances of x^k to x^* are decreasing slowly in this case. Doing the same computations with the duality mapping J^μ corresponding to the weight function $\mu(t) = rt^{r-1}$ with $r > 2$ instead the normalized duality mapping, one can see that the iterates x^k converge much faster to x^*.

3. Regularization of ill-posed variational inequalities with domain perturbations

3.1 In this section we consider the variational inequality (2) where $A : X \to X^*$ is a maximal monotone point-to-set operator and $\Omega \subseteq Dom(A)$ is a closed convex set with $Int(\Omega) \neq \emptyset$. We presume that the given variational inequality has solutions and that there exist the functions $a_1, a_2, g_1 : \mathbf{R}_+ \to \mathbf{R}_+$ and the function $\beta : \mathbf{R}_+^2 \to \mathbf{R}_+$ and three positive real numbers h^*, δ^* and σ^* such that a_2 is nondecreasing,

$$\lim_{t\to\infty} \frac{a_2(t)}{\mu(t)t} = 0, \tag{46}$$

and for any triple $\xi = (h, \delta, \sigma) \in \Delta := (0, h^*] \times (0, \delta^*] \times (0, \sigma^*]$ one can find a maximal monotone operator $A^h : X \to X^*$, a vector $f^\delta \in X^*$ and a closed convex subset $\Omega_\sigma \subseteq Dom(A^h)$ which has nonempty interior and satisfy the following properties:

(I) The triple $(A^h, f^\delta, \Omega_\sigma)$ is a (h, δ, σ)-approximant of (A, f, Ω), that is,

$$\left\| f - f^\delta \right\| \leq \delta,$$

and for any $x \in \Omega$, there exists $z(\sigma) \in \Omega_\sigma$ such that

$$\|x - z(\sigma)\| \leq a_1(\|x\|)\sigma, \tag{47}$$

and such that, for all $u \in Ax$, we have

$$d_*(u, A^h z(\sigma)) := \inf \left\{ \|u - u'\|_* : u' \in A^h z(\sigma) \right\} < g_1(\|u\|_*)\beta(h, \sigma). \tag{48}$$

(II) For any $w \in \Omega_\sigma$, there exists $v \in \Omega$ such that

$$\|w - v\| \leq a_2(\|w\|)\sigma. \tag{49}$$

Note that, in these circumstances, the solution set of (2) is nonempty, convex and it contains an unique minimal norm element. Also, for any real number $\alpha > 0$ and for each $\xi = (h, \delta, \sigma) \in \Delta$, the operator $A^h + \alpha J^\mu$ is strictly and maximal monotone and, therefore, the problem of finding $w \in \Omega_\sigma$ such that, for some $y \in A^h w$,

$$\left\langle y + \alpha J^\mu w - f^\delta, w - z \right\rangle \leq 0, \text{ for all } z \in \Omega_\sigma, \tag{50}$$

has unique solution which we denote by w_α^ξ. The question is whether by solving (50) we determine vectors w_α^ξ which are close to the minimal norm solution of (2). The fact that this question has an affirmative answer in the case that, for all $h \in (0, h^*]$ and $\sigma \in (0, \sigma^*]$ we have

$$\Omega_\sigma \subseteq Dom(A) = Dom(A^h), \tag{51}$$

is known. Mosco [14] and Liskovets [12], [13] have considered this problem and have shown that w_α^ξ approximate solutions for (2) in a weaker topology (Mosco's topology on the family of closed convex subsets of X). Also under the assumption (51), the problem was studied in [16], [2], [3] where strong convergence of w_α^ξ was established. We would like to know what happens in the more general case in which (51) is not necessarily satisfied, but conditions (I)-(II) above hold. The relevance of this question comes from the fact that, when one replaces the operator A by a perturbed operator A^h having $Dom(A^h) \neq Dom(A)$, it may happen that the set

Ω itself is perturbed and replaced by a set Ω_σ which is contained in $Dom(A^h)$ but not in $Dom(A)$.

3.2 Our answer to that question consists of the following result which shows that the solutions w_α^ξ of the equations (50) approximate strongly the minimal norm solution of (2) when the errors h, σ, δ and α are small enough and comply with specific rules.

Theorem. *Suppose that the conditions in Subsection 3.1 hold and that the sequences* $\{\xi^k := (\delta_k, h_k, \sigma_k)\}_{k\in\mathbf{N}} \subset \Delta$ *and* $\{\alpha_k\}_{k\in\mathbf{N}} \subset (0,+\infty)$ *are such that*

$$\lim_{k\to\infty} \alpha_k = 0 \tag{52}$$

and

$$\lim_{k\to\infty} \frac{\delta_k + \sigma_k + \beta(h_k,\sigma_k)}{\alpha_k} = 0. \tag{53}$$

Then, the sequence $w^k := w_{\alpha_k}^{\xi^k}$, $k \in \mathbf{N}$, *of solutions of the corresponding variational inequalities* (50) *converges strongly to the minimal norm solution of* (2), *no matter how the weight function* $\mu : \mathbf{R}_+ \to \mathbf{R}_+$ *is chosen.*

The proof of this result, given below, consists of several lemmas which first establish weak convergence and, then, strong convergence of $\{w^k\}_{k\in\mathbf{N}}$ to a solution of (2) and finally show that the solution to which $\{w^k\}_{k\in\mathbf{N}}$ converges has minimal norm.

3.3 We start the proof of Theorem 3.2 by showing that the sequence $\{w^k\}_{k\in\mathbf{N}}$ is bounded. This is an immediate implication of the following result.

Lemma. *For any real number* $\varepsilon > 0$, *there exists a positive real number* $M(\varepsilon)$ *such that, whenever* $\xi := (h,\delta,\sigma) \in \Delta$ *and* $\alpha > 0$ *satisfy*

$$\frac{\delta + \sigma + \beta(h,\sigma)}{\alpha} < \varepsilon, \tag{54}$$

we have $\|w_\alpha^\xi\| \le M(\varepsilon)$.

Proof. Let $x \in \Omega$ be a solution of (2) and let $y \in Ax$ be a vector such that (2) is satisfied. Let $z(\sigma) \in \Omega_\sigma$ be such that (47) and (48) hold. Then, for some $u' \in A^h z(\sigma)$ we have

$$\|y - u'\|_* \le g_1(\|y\|_*)\beta(h,\sigma). \tag{55}$$

Since w_α^ξ is the solution of (50), there exists an element $y_\alpha^\xi \in A^h w_\alpha^\xi$ such that

$$\langle y_\alpha^\xi + \alpha J^\mu w_\alpha^\xi - f^\delta, w_\alpha^\xi - z\rangle \le 0, \text{ for all } z \in \Omega_\sigma. \tag{56}$$

Also, $w_\alpha^\xi \in \Omega_\sigma$ and, therefore, according to condition (II) above, there exists $v_\alpha^\xi \in \Omega$ such that

$$\|w_\alpha^\xi - v_\alpha^\xi\| \le a_2(\|w_\alpha^\xi\|)\sigma. \tag{57}$$

The inequality in (2) holds for $z = v_\alpha^\xi$, that is,

$$\langle y - f, x - v_\alpha^\xi\rangle \le 0. \tag{58}$$

Also, the inequality in (56) holds for $z = z(\sigma)$, that is,

$$\langle y_\alpha^\xi + \alpha J^\mu w_\alpha^\xi - f^\delta, w_\alpha^\xi - z(\sigma)\rangle \le 0. \tag{59}$$

Adding the last two inequalities we get

$$
\begin{aligned}
0 &\geq \left\langle y_\alpha^\xi - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle + \alpha\left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle \\
&= \alpha\left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle y_\alpha^\xi - u', w_\alpha^\xi - z(\sigma)\right\rangle + \\
&\quad \left\langle u' - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle \\
&= \alpha\left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle y_\alpha^\xi - u', w_\alpha^\xi - z(\sigma)\right\rangle + \\
&\quad \left\langle u' - f^\delta, w_\alpha^\xi - x\right\rangle + \left\langle u' - f^\delta, x - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle .
\end{aligned} \tag{60}
$$

Taking into account the fact that A^h is monotone and that $y_\alpha^\xi \in A^h w_\alpha^\xi$ and $u' \in A^h z(\sigma)$, we deduce that

$$\left\langle y_\alpha^\xi - u', w_\alpha^\xi - z(\sigma)\right\rangle \geq 0.$$

Hence, according to (60), we get

$$
\begin{aligned}
0 \geq{}& \alpha\left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle u' - f^\delta, w_\alpha^\xi - x\right\rangle + \\
& \left\langle u' - f^\delta, x - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle .
\end{aligned}
$$

Consequently, using the definition of the duality mapping, we have

$$
\begin{aligned}
0 &\geq \alpha\left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi\right\rangle - \alpha\left\langle J^\mu w_\alpha^\xi, z(\sigma)\right\rangle + \\
&\quad \left\langle u' - f^\delta, w_\alpha^\xi - x\right\rangle + \left\langle u' - f^\delta, x - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle \\
&= \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left\|w_\alpha^\xi\right\| - \alpha\left\langle J^\mu w_\alpha^\xi, z(\sigma) - x\right\rangle - \alpha\left\langle J^\mu w_\alpha^\xi, x\right\rangle + \\
&\quad \left\langle u' - y, w_\alpha^\xi - x\right\rangle + \left\langle y - f, w_\alpha^\xi - x\right\rangle + \left\langle f - f^\delta, w_\alpha^\xi - x\right\rangle + \\
&\quad \left\langle u' - f^\delta, x - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle .
\end{aligned}
$$

Using the fact that $(A^h, f^\delta, \Omega_\sigma)$ is an approximant of (A, f, Ω) we get

$$
\begin{aligned}
\left\langle f - f^\delta, w_\alpha^\xi - x\right\rangle &\geq -\left\|f - f^\delta\right\|_* \left\|w_\alpha^\xi - x\right\| \\
&\geq -\delta\left\|w_\alpha^\xi - x\right\| \geq -\delta\left(\left\|w_\alpha^\xi\right\| + \|x\|\right).
\end{aligned}
$$

Also, from (55), we have

$$
\begin{aligned}
\left\langle u' - y, w_\alpha^\xi - x\right\rangle &\geq -\|y - u'\|_* \left\|w_\alpha^\xi - x\right\| \\
&\geq -g_1\left(\|y\|_*\right)\beta(h,\sigma)\left(\left\|w_\alpha^\xi\right\| + \|x\|\right).
\end{aligned}
$$

Combining the last three inequalities, applying the definition of the duality mapping and taking into account (47), we obtain

$$
\begin{aligned}
0 &\geq \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left\|w_\alpha^\xi\right\| - \alpha\left\langle J^\mu w_\alpha^\xi, z(\sigma) - x\right\rangle - \alpha\left\langle J^\mu w_\alpha^\xi, x\right\rangle \\
&\quad - \delta\left(\left\|w_\alpha^\xi\right\| + \|x\|\right) - g_1\left(\|y\|\right)\beta(h,\sigma)\left(\left\|w_\alpha^\xi\right\| + \|x\|\right) + \\
&\quad \left\langle y - f, w_\alpha^\xi - x\right\rangle + \left\langle u' - f^\delta, x - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle \\
&\geq \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left(\left\|w_\alpha^\xi\right\| - \|x\|\right) - \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\|z(\sigma) - x\| \\
&\quad - \left(\delta + g_1\left(\|y\|_*\right)\beta(h,\sigma)\right)\left(\left\|w_\alpha^\xi\right\| + \|x\|\right) + \\
&\quad \left\langle y - f, w_\alpha^\xi - v_\alpha^\xi\right\rangle + \left\langle u' - f^\delta, x - z(\sigma)\right\rangle
\end{aligned}
$$

$$\begin{aligned}
&\geq \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left(\left\|w_\alpha^\xi\right\|-\|x\|\right)-\alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)a_1\left(\|x\|\right)\sigma \\
&-\left(\delta+g_1\left(\|y\|_*\right)\beta(h,\sigma)\right)\left(\left\|w_\alpha^\xi\right\|+\|x\|\right)+ \\
&\left\langle y-f,w_\alpha^\xi-v_\alpha^\xi\right\rangle+\left\langle u'-f^\delta,x-z(\sigma)\right\rangle.
\end{aligned}$$

Since, according to (57), we have

$$\left\langle y-f,w_\alpha^\xi-v_\alpha^\xi\right\rangle\geq-\|y-f\|_*\,a_2\left(\left\|w_\alpha^\xi\right\|\right)\sigma,$$

we get

$$\begin{aligned}
0 &\geq \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left(\left\|w_\alpha^\xi\right\|-\|x\|\right)-\alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)a_1\left(\|x\|\right)\sigma \\
&-\left[\delta+g_1\left(\|y\|_*\right)\beta(h,\sigma)\right]\left(\left\|w_\alpha^\xi\right\|+\|x\|\right) \\
&-\sigma a_2\left(\left\|w_\alpha^\xi\right\|\right)\|y-f\|_*+\left\langle u'-f^\delta,x-z(\sigma)\right\rangle.
\end{aligned}$$

Now, using the fact that

$$\begin{aligned}
\left\langle u'-f^\delta,x-z(\sigma)\right\rangle &= \left\langle u'-y,x-z(\sigma)\right\rangle-\left\langle f-y,x-z(\sigma)\right\rangle \\
&-\left\langle f^\delta-f,x-z(\sigma)\right\rangle \\
&\geq-\left\|u'-y\right\|_*\|x-z(\sigma)\|-\|f-y\|_*\|x-z(\sigma)\| \\
&-\left\|f^\delta-f\right\|_*\|x-z(\sigma)\| \\
&\geq-\sigma a_1\left(\|x\|\right)\left[g_1\left(\|y\|_*\right)\beta(h,\sigma)+\|f-y\|_*+\delta\right],
\end{aligned}$$

we deduce

$$\begin{aligned}
0 &\geq \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left(\left\|w_\alpha^\xi\right\|-\|x\|\right)-\alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)a_1\left(\|x\|\right)\sigma \\
&-\left[\delta+g_1\left(\|y\|_*\right)\beta(h,\sigma)\right]\left(\left\|w_\alpha^\xi\right\|+\|x\|+\sigma a_1\left(\|x\|\right)\right) \\
&-\sigma\left[a_1\left(\|x\|\right)+a_2\left(\left\|w_\alpha^\xi\right\|\right)\right]\|y-f\|_*.
\end{aligned}$$

Dividing this inequality by α and taking into account (54) we obtain

$$\begin{aligned}
0 &\geq \mu\left(\left\|w_\alpha^\xi\right\|\right)\left(\left\|w_\alpha^\xi\right\|-\|x\|\right)-\mu\left(\left\|w_\alpha^\xi\right\|\right)a_1\left(\|x\|\right)\varepsilon\alpha \\
&-\left[\varepsilon+g_1\left(\|y\|_*\right)\varepsilon\right]\left(\left\|w_\alpha^\xi\right\|+\|x\|+\varepsilon\alpha a_1\left(\|x\|\right)\right) \\
&-\varepsilon\left[a_1\left(\|x\|\right)+a_2\left(\left\|w_\alpha^\xi\right\|\right)\right]\|y-f\|_*.
\end{aligned}$$

Hence,

$$0\geq\mu\left(\left\|w_\alpha^\xi\right\|\right)\left\|w_\alpha^\xi\right\|\left[1-\sum_{i=1}^{4}C_i\left(\left\|w_\alpha^\xi\right\|\right)\right], \tag{61}$$

where

$$C_1\left(\left\|w_\alpha^\xi\right\|\right):=\frac{\|x\|}{\left\|w_\alpha^\xi\right\|}+\frac{a_1\left(\|x\|\right)\varepsilon\alpha}{\left\|w_\alpha^\xi\right\|},$$

$$C_2\left(\left\|w_\alpha^\xi\right\|\right):=\frac{\varepsilon+g_1\left(\|y\|_*\right)\varepsilon}{\mu\left(\left\|w_\alpha^\xi\right\|\right)},$$

$$C_3\left(\left\|w_\alpha^\xi\right\|\right):=C_2\left(\left\|w_\alpha^\xi\right\|\right)\frac{\|x\|+\varepsilon\alpha a_1\left(\|x\|\right)}{\left\|w_\alpha^\xi\right\|}$$

and

$$C_4\left(\left\|w_\alpha^\xi\right\|\right):=\frac{\varepsilon\left[a_1\left(\|x\|\right)+a_2\left(\left\|w_\alpha^\xi\right\|\right)\right]\|y-f\|_*}{\mu\left(\left\|w_\alpha^\xi\right\|\right)\left\|w_\alpha^\xi\right\|}.$$

Suppose, by contradiction, that the set of those w_α^ξ for which (54) holds is unbounded. Then, for some sequence $w_{\alpha_k}^{\xi_k}$, $k \in \mathbf{N}$, in this set we have that

$$\lim_{k\to\infty} \left\| w_{\alpha_k}^{\xi_k} \right\| = +\infty$$

and, thus,

$$\lim_{k\to\infty} C_i \left(\left\| w_{\alpha_k}^{\xi_k} \right\| \right) = 0, \ i = 1,2,3,4,$$

because of (46). By (61), this implies

$$0 \geq \lim_{k\to\infty} \mu \left(\left\| w_{\alpha_k}^{\xi_k} \right\| \right) \left\| w_{\alpha_k}^{\xi_k} \right\| \left[1 - \sum_{i=1}^{4} C_i \left(\left\| w_{\alpha_k}^{\xi_k} \right\| \right) \right] = +\infty,$$

that is, a contradiction.

3.4 The following technical result is essential in the sequel.

Lemma. *For any $x \in \Omega$ which is a solution of the variational inequality* (2) *and for any real number $\alpha^* > 0$, there exists a positive real number $M = M_{x,\alpha^*}$ such that for each for any $\xi = (h,\delta,\sigma) \in \Delta$ and $\alpha \in (0,\alpha^*]$ satisfying*

$$\frac{\delta + \sigma + \beta(h,\sigma)}{\alpha} \leq 1, \tag{62}$$

we have

$$\mu \left(\left\| w_\alpha^\xi \right\| \right) \left\| w_\alpha^\xi \right\| \leq \mu \left(\left\| w_\alpha^\xi \right\| \right) \|x\| + M \frac{\delta + \sigma + \beta(h,\sigma)}{\alpha}. \tag{63}$$

Proof. Let $x \in \Omega$ be a solution of (2) and let $y \in Ax$ be such that

$$\langle y - f, x - z \rangle \leq 0, \tag{64}$$

for all $z \in \Omega$. Fix $z(\sigma) \in \Omega_\sigma$ be such that (47) is satisfied. Denote by u' a vector in $A^h z(\sigma)$ such that

$$\|y - u'\|_* \leq g_1 \left(\|y\|_* \right) \beta(h,\sigma). \tag{65}$$

Such a u' exists because of condition (I). With the solution w_α^ξ of (50) we associate the vector $y_\alpha^\xi \in A^h w_\alpha^\xi$ such that (56) is satisfied. According to condition (II), there exists $v_\alpha^\xi \in \Omega$ such that (57) is true. Then, we also have

$$\left\langle y - f, x - v_\alpha^\xi \right\rangle \leq 0,$$

because of (64). Summing up this inequality and (56) we obtain again (60). From (60) we get

$$\begin{aligned}
0 &\geq \alpha \left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi - z(\sigma) \right\rangle + \left\langle y_\alpha^\xi - u', w_\alpha^\xi - z(\sigma) \right\rangle + \\
&\left\langle u' - f^\delta, w_\alpha^\xi - z(\sigma) \right\rangle + \left\langle y - f, x - v_\alpha^\xi \right\rangle \\
&\geq \alpha \left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi - z(\sigma) \right\rangle + \left\langle u' - f^\delta, w_\alpha^\xi - z(\sigma) \right\rangle + \left\langle y - f, x - v_\alpha^\xi \right\rangle,
\end{aligned}$$

because $\left\langle y_\alpha^\xi - u', w_\alpha^\xi - z(\sigma) \right\rangle \geq 0$ due to the monotonicity of A^h. Thus, we deduce

$$0 \geq \alpha \left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi - z(\sigma) \right\rangle + \left\langle u' - y, w_\alpha^\xi - x \right\rangle \tag{66}$$

$$+ \langle u' - y, x - z(\sigma) \rangle + \left\langle y - f^\delta, w_\alpha^\xi - z(\sigma) \right\rangle + \left\langle y - f, x - v_\alpha^\xi \right\rangle$$

$$\begin{aligned}
&= \alpha \left\langle J^\mu w_\alpha^\xi, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle u' - y, w_\alpha^\xi - x\right\rangle \\
&+ \langle u' - f, x - z(\sigma)\rangle + \langle f - y, x - z(\sigma)\rangle \\
&+ \left\langle y - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle .
\end{aligned}$$

Note that

$$\langle f - y, x - z(\sigma)\rangle + \left\langle y - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle$$

$$\begin{aligned}
&= \left\langle y - f, z(\sigma) - v_\alpha^\xi\right\rangle + \left\langle y - f, v_\alpha^\xi - x\right\rangle \\
&+ \left\langle y - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle y - f, x - v_\alpha^\xi\right\rangle \\
&= \left\langle y - f, z(\sigma) - v_\alpha^\xi\right\rangle + \left\langle y - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle \\
&= \left\langle y - f, z(\sigma) - v_\alpha^\xi\right\rangle + \left\langle f - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle + \left\langle y - f, w_\alpha^\xi - z(\sigma)\right\rangle \\
&= \left\langle y - f, w_\alpha^\xi - v_\alpha^\xi\right\rangle + \left\langle f - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle .
\end{aligned}$$

This and (66) imply

$$\begin{aligned}
0 \geq{}& \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left\|w_\alpha^\xi\right\| - \alpha\left\langle J^\mu w_\alpha^\xi, z(\sigma)\right\rangle \\
&+ \left\langle u' - y, w_\alpha^\xi - x\right\rangle + \langle u' - f, x - z(\sigma)\rangle \\
&+ \left\langle y - f, w_\alpha^\xi - v_\alpha^\xi\right\rangle + \left\langle f - f^\delta, w_\alpha^\xi - z(\sigma)\right\rangle ,
\end{aligned}$$

due to the properties of the duality mapping J^μ. Hence, taking into account (47), (56), (64) and (65), we obtain

$$\begin{aligned}
\alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left\|w_\alpha^\xi\right\| &\leq \alpha\left\langle J^\mu w_\alpha^\xi, z(\sigma) - x\right\rangle + \alpha\left\langle J^\mu w_\alpha^\xi, x\right\rangle \\
&+ \|u' - y\|_* \left\|w_\alpha^\xi - x\right\| + \|u' - f\|_* \|x - z(\sigma)\| \\
&+ \|y - f\|_* \left\|w_\alpha^\xi - v_\alpha^\xi\right\| + \left\|f - f^\delta\right\|_* \left\|w_\alpha^\xi - z(\sigma)\right\| \\
&\leq \alpha\left\|J^\mu w_\alpha^\xi\right\|_* \|z(\sigma) - x\| + \alpha\left\|J^\mu w_\alpha^\xi\right\|_* \|x\| \\
&+ \|u' - y\|_* \left\|w_\alpha^\xi - x\right\| + \|u' - f\|_* \|x - z(\sigma)\| \\
&+ \|y - f\|_* \left\|w_\alpha^\xi - v_\alpha^\xi\right\| + \left\|f - f^\delta\right\|_* \left\|w_\alpha^\xi - z(\sigma)\right\| \\
&= \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\|x\| + \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right) a_1(\|x\|)\,\sigma \\
&+ g_1(\|y\|_*)\,\beta(h,\sigma)\left\|w_\alpha^\xi - x\right\| \\
&+ a_1(\|x\|)\,\sigma\,\|u' - f\|_* \\
&+ a_2\left(\left\|w_\alpha^\xi\right\|\right)\sigma\,\|y - f\|_* \\
&+ \delta\left(\left\|w_\alpha^\xi - x\right\| + \|x - z(\sigma)\|\right) .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\left\|w_\alpha^\xi\right\| &\leq \alpha\mu\left(\left\|w_\alpha^\xi\right\|\right)\|x\| \\
+\sigma\left[\alpha\mu\left(\left\|w_\alpha^\xi\right\|\right) a_1(\|x\|) + a_1(\|x\|)\|u' - f\|_* + a_2\left(\left\|w_\alpha^\xi\right\|\right)\|y - f\|_*\right] & \\
+g_1(\|y\|_*)\,\beta(h,\sigma)\left\|w_\alpha^\xi - x\right\| & \\
+\delta\left[\left\|w_\alpha^\xi - x\right\| + a_1(\|x\|)\,\sigma\right] . &
\end{aligned} \tag{67}$$

According to Lemma 3.3, if (62) holds, then $\left\|w_\alpha^\xi\right\| \leq M(1)$. This shows that there exists a real number $M > 0$ such that the coefficients of σ, δ and $\beta(h,\sigma)$ on the right hand side of (67) are at most equal to M whenever $\alpha \in (0,1]$ and (62) holds.

Taking into account this fact and dividing the inequality (67) by $\alpha > 0$ we obtain (63).

3.5 The next result completes the proof of Theorem 3.2.

Lemma. *The sequence* $\{w^k\}_{k\in\mathbf{N}}$, *converges strongly to a minimal norm solution* w^* *of the variational inequality* (2).

Proof. Let $\alpha^* > 0$ be an upper bound of the sequence $\{\alpha_k\}_{k\in\mathbf{N}}$. Also, let $\varepsilon^* > 0$ be an upper bound of the sequence $\left\{\frac{\delta_k+\sigma_k+\beta(h_k,\sigma_k)}{\alpha_k}\right\}_{k\in\mathbf{N}}$. Lemma 3.3 implies that the sequence $\{w^k\}_{k\in\mathbf{N}}$ is bounded by some number $M(\varepsilon^*)$. Since X is reflexive, this implies that the sequence $\{w^k\}_{k\in\mathbf{N}}$ has a weak accumulation point w^*. We show first that if w^* is a weak accumulation point of $\{w^k\}_{k\in\mathbf{N}}$, then w^* is a solution of (2), that is, that $w^* \in \Omega$ and there exists a point $y^* \in Aw^*$ such that

$$\langle y^* - f, w^* - z\rangle \le 0, \text{ for all } z \in \Omega. \tag{68}$$

To this end, note that according to condition (II), for each $k \in \mathbf{N}$, there exists a point $v^k \in \Omega$ such that

$$\left\|w^k - v^k\right\| \le a_2(\left\|w^k\right\|)\sigma_k. \tag{69}$$

Since the function a_2 is nondecreasing, the sequence $\left\{\left\|w^k\right\|\right\}_{k\in\mathbf{N}}$ is bounded and $\sigma_k \to 0$ as $k \to \infty$, it results that

$$\lim_{k\to\infty} a_2(\left\|w^k\right\|)\sigma_k = 0$$

and, therefore, w^* is also the weak limit of a subsequence of $\{v^k\}_{k\in\mathbf{N}} \subset \Omega$. Since Ω is convex and closed it is also weakly closed and, thus, $w^* \in \Omega$.

It remains to show that (68) holds for some $y^* \in Aw^*$. To this end, note that, for each $k \in \mathbf{N}$, there exists $y^k \in A^{h_k}w^k$ such that

$$\left\langle y^k + \alpha_k J^\mu w^k - f^{\delta_k}, w^k - z\right\rangle \le 0, \text{ for all } z \in \Omega_{\sigma_k}. \tag{70}$$

Taking into account that $A^{h_k} + \alpha_k J^\mu$ is monotone, we deduce that, for any $z \in \Omega_{\sigma_k}$ and any $y \in A^{h_k}z$, we have

$$\begin{aligned} 0 &\le \left\langle \left(y^k + \alpha_k J^\mu w^k\right) - \left(y + \alpha_k J^\mu z\right), w^k - z\right\rangle \\ &= \left\langle y^k + \alpha_k J^\mu w^k - f^{\delta_k}, w^k - z\right\rangle - \left\langle y + \alpha_k J^\mu z - f^{\delta_k}, w^k - z\right\rangle, \end{aligned}$$

that is,

$$\left\langle y + \alpha_k J^\mu z - f^{\delta_k}, w^k - z\right\rangle \le \left\langle y^k + \alpha_k J^\mu w^k - f^{\delta_k}, w^k - z\right\rangle \le 0.$$

In other words, for each $z \in \Omega_{\sigma_k}$ we have

$$\left\langle y + \alpha_k J^\mu z - f^{\delta_k}, w^k - z\right\rangle \le 0, \text{ for all } y \in A^{h_k}z. \tag{71}$$

Fix $z \in \Omega$, $\tilde{y} \in Az$ and let $z(\sigma_k) \in \Omega_{\sigma_k}$ be the corresponding element whose existence is guaranteed by condition (I). Then, there exists $\tilde{y}^k \in A^{h_k}z(\sigma_k)$ such that

$$\left\|\tilde{y} - \tilde{y}^k\right\|_* \le g_1(\|\tilde{y}\|_*)\beta(h_k, \sigma_k) \tag{72}$$

and

$$\left\langle \tilde{y}^k + \alpha_k J^\mu z(\sigma_k) - f^{\delta_k}, w^k - z(\sigma_k)\right\rangle \le 0, \tag{73}$$

because of (48) and (71), respectively. Clearly, $\lim_{k\to\infty} z(\sigma_k) = z$ - see (47). Also, since the sequence $\{z(\sigma_k)\}_{k\in\mathbf{N}}$ is bounded so is the sequence $\{J^\mu z(\sigma_k)\}_{k\in\mathbf{N}}$. Consequently, $\lim_{k\to\infty} \alpha_k J^\mu z(\sigma_k) = 0$ since $\alpha_k \to 0$ as $k \to \infty$. The sequence $\{\tilde{y}^k\}_{k\in\mathbf{N}}$ converges strongly to $\tilde{y}$ because (72). Hence, from (73) we deduce that

$$0 \geq \langle \tilde{y}^k - \tilde{y}, w^k - z(\sigma_k)\rangle + \langle \tilde{y} + \alpha_k J^\mu z(\sigma_k) - f^{\delta_k}, w^k - z(\sigma_k)\rangle$$

$$= \langle \tilde{y}^k - \tilde{y}, w^k - z(\sigma_k)\rangle + \langle \tilde{y} - f, w^k - z(\sigma_k)\rangle +$$
$$\langle f + \alpha_k J^\mu z(\sigma_k) - f^{\delta_k}, w^k - z(\sigma_k)\rangle,$$

where

$$\lim_{k\to\infty} \langle \tilde{y}^k - \tilde{y}, w^k - z(\sigma_k)\rangle = 0,$$

and

$$\lim_{k\to\infty} \langle f + \alpha_k J^\mu z(\sigma_k) - f^{\delta_k}, w^k - z(\sigma_k)\rangle = 0.$$

By consequence, for any subsequence $\{w^{i_k}\}_{k\in\mathbf{N}}$ of $\{w^k\}_{k\in\mathbf{N}}$ which converges weakly to w^* we have

$$0 \geq \lim_{k\to\infty} \langle \tilde{y} - f, w^{i_k} - z(\sigma_{i_k})\rangle = \langle \tilde{y} - f, w^* - z\rangle.$$

Thus, for any $z \in \Omega$ and for any $\tilde{y} \in Az$ we have that

$$\langle \tilde{y} - f, z - w^*\rangle \geq 0. \tag{74}$$

which, according to Lemma Minty (see [11, p. 324]), cannot hold unless (68) is true for some $y^* \in Aw^*$.

We prove next that weak accumulation points of $\{w^k\}_{k\in\mathbf{N}}$ are minimal norm solutions of (2). This is obviously true when $w^* = 0$. Suppose that $w^* \neq 0$ and that $x \in \Omega$ is another solution of (2). According to (53), there exists a positive integer k_0 such that, for all integers $k \geq k_0$, we have

$$\frac{\delta_k + \sigma_k + \beta(h_k, \sigma_k)}{\alpha_k} \leq 1.$$

Therefore, Lemma 3.4 implies that for any solution x of (2) there exists a constant $M = M_{x,\alpha^*} > 0$ such that

$$\mu\left(\|w^k\|\right)\|w^k\| \leq \mu\left(\|w^k\|\right)\|x\| + M\frac{\delta_k + \sigma_k + \beta(h_k, \sigma_k)}{\alpha_k}, \tag{75}$$

for all $k \geq k_0$. Since $w^* \neq 0$, there exists a subsequence $\{w^{i_k}\}_{k\in\mathbf{N}}$ of $\{w^k\}_{k\in\mathbf{N}}$ which converges weakly to w^* and such that $w^{i_k} \neq 0$ for all $k \in \mathbf{N}$. Clearly, we have

$$0 < \frac{\|w^*\|}{2} < \|w^*\| \leq \liminf_{k\to\infty} \|w^{i_k}\|. \tag{76}$$

Hence, there exists an integer $m_0 \geq k_0$ such that, for all $k \geq m_0$, we have $\|w^{i_k}\| \geq \frac{\|w^*\|}{2} > 0$ and, consequently,

$$0 < \mu\left(\frac{\|w^*\|}{2}\right) \leq \mu\left(\|w^{i_k}\|\right) \leq \mu\left(M(\varepsilon^*)\right). \tag{77}$$

From (75) we get that

$$\|w^{i_k}\| \leq \|x\| + \frac{M}{\mu\left(\frac{1}{2}\|w^*\|\right)}\frac{\delta_{i_k} + \sigma_{i_k} + \beta(h_{i_k}, \sigma_{i_k})}{\alpha_{i_k}}, \tag{78}$$

for all $k \geq m_0$. Taking the limit for $k \to \infty$ on both sides of this inequality we deduce that $\|w^*\| \leq \|x\|$ because of (53). Since x was an arbitrarily chosen solution of (2), this shows that w^* is a minimal norm solution of (2).

The fact is that there is only one minimal norm solution of (2) because the solution set is convex and the space is strictly convex and reflexive. Hence, since any weak accumulation point of $\{w^k\}_{k\in\mathbf{N}}$ has to be a minimal norm solution, it results that $\{w^k\}_{k\in\mathbf{N}}$ has a single weak accumulation point w^*. This implies that $\{w^k\}_{k\in\mathbf{N}}$ is weakly convergent to w^*. Therefore, (76), (77) and (78) hold with k instead of i_k and we have

$$\begin{aligned}\|w^*\| \leq \liminf_{k\to\infty} \left\|w^k\right\| \leq & \limsup_{k\to\infty} \left\|w^k\right\| \\ & \leq \|w^*\| + \limsup_{k\to\infty} \frac{M}{\mu\left(\frac{1}{2}\|w^*\|\right)} \frac{\delta_k + \sigma_k + \beta(h_k, \sigma_k)}{\alpha_k} = \|w^*\|,\end{aligned}$$

Thus, we obtain that $\lim_{k\to\infty}\left\|w^k\right\| = \|w^*\|$ and, since X has Efimov's property, this implies that the sequence $\{w^k\}_{k\in\mathbf{N}}$ converges strongly to the minimal norm solution w^*.

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