8. R. LOWEN, Initial and final fuzzy topologies and the fuzzy Tychonoff Theorem, *J. Math. Anal. Appl.* **58** (1977), 11–21.
9. R. LOWEN, Convergence in fuzzy topological spaces, *Gen. Topology Appl.* **10** (1979), 147–160.
10. R. LOWEN, Compactness notions in fuzzy neighbourhood spaces, *Manuscripta Math.* **38** (1982), 265–287.
11. R. LOWEN, Compactness properties in the fuzzy real line, *Fuzzy Sets and Systems* **13** (1984), 193–200.
12. PU PAO-MING AND LIU YING-MING, Fuzzy topology, I. Neighbourhood structure of a fuzzy point and Moore Smith convergence, *J. Math. Anal. Appl.* **76** (1980), 571–599.
13. WANG GUOJUN, A new fuzzy compactness defined by fuzzy nets, *J. Math. Anal. Appl.* **94** (1983), 1–23.
14. L. A. ZADEH, Fuzzy sets, *Information and Control* **8** (1965), 338–353.
15. ZHAO DONGSHENG, The N-compactness in L-fuzzy topological spaces, *J. Math. Anal. Appl.* **128** (1987), 64–79.

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **162**, 111–143 (1991)

Triangular Norm-Based Measures and Their Markov Kernel Representation

DAN BUTNARIU

Department of Mathematics and Computer Science, University of Haifa, 31905 Haifa, Israel

AND

ERICH PETER KLEMENT

Institut für Mathematik, Johannes Kepler Universität, A-4040 Linz, Austria

Submitted by Ulrich Höhle

Received October 23, 1989

We approach the problem whether left-continuous triangular norm-based valuations (called T-measures or T-probability measures) defined on triangular norm-based tribes of the unit cube can be disintegrated by Markov kernels. We prove that each T-measure based on a "fundamental" triangular norm (these triangular norms T, together with their corresponding triangular conorms S, satisfy the functional equation $T(x, y) + S(x, y) = x + y$) can be uniquely represented as a sum of a "disintegrable" T-measure and a "hard core" which is either identically zero or which is monotonically irreducible (i.e., cannot be disintegrated).

INTRODUCTION

The concept of a *triangular norm* is due to Menger [28], and it was studied from algebraic and topological points of view in fields like *Probabilistic Metric Spaces* (Wald [39], Schweizer and Sklar [34–37]), *Multivalued Logic* (Rose and Rosser [30], Hamacher [17]), and *Semigroups* (Climescu [11], Schweizer and Sklar [36], Paalman-de Miranda [29], Ling [27], Kimberling [19]).

Frank [15] has shown that the class of continuous triangular norms T, which together with their corresponding triangular conorms S satisfy the equation $T(x, y) + S(x, y) = x + y$, consists of the ordinal sums of sequences of "fundamental" triangular norms and conorms.

Triangular norm-based measures (*T-measures*) appear under various names, and in specific analytical forms, in fields ranging from *Mathematical Statistics* (Dvoretzki, Wald, and Wolfowitz [13]. Aczel and Alsina [2]), to *Capacity Theory* (Frank [15]), *Probability and Measure Theory* (Schmidt [31], Klement *et al.* [25, 26], Klement [22–24], Butnariu [5, 6]), *Pattern Recognition* (Sugeno [38]), *Game Theory* (Aumann and Shapley [4], Aubin [3], Butnariu [7, 9, 10]), etc. In this paper we study the triangular norm-based measures in their proximal context, namely T-measures defined on subsets of the unit cube $[0, 1]^X$, which are triangular norm-based tribes (*T-tribes*). The main purpose is to find out whether, or under which conditions, T-measures can be represented as integrals of specific Markov kernels. We concentrate on fundamental triangular norm-based T-measures mainly because this class of T-measures is of interest in most of the applications mentioned above. One may also note that there are classes of nonfundamental triangular norms, on which no nontrivial T-measure is based (cf. Klement [24]).

We first deal with T-tribes, the main result being Theorem 2.1 showing that any fundamental triangular norm based T-tribe $\mathscr{T}$ consists of functions, which are measurable with respect to the intrinsic σ-algebra $\mathscr{T}^{\vee}$ corresponding to $\mathscr{T}$ (i.e., with respect to the σ-algebra of those sets whose characteristic functions belong to $\mathscr{T}$). In this context, we give a characterization of the generated T-tribes introduced by Klement [22]– see Theorem 2.1 and Remark 2.3. Theorem 2.1 allows the deduction (see Section 5) that on a fundamental triangular norm based tribe $\mathscr{T}$ any function $\mathbf{m}$ of the form

$$\mathbf{m}(A) = \int_{\{A>0\}} (g + h \cdot A)\, d\mathbf{p} \tag{$*$}$$

is a well-defined monotone T-measure, provided g, h are nonnegative $\mathscr{T}^{\vee}$-measurable functions, and $\mathbf{p}$ is a probability measure on $\mathscr{T}^{\vee}$. The question is whether any monotone T-measure on $\mathscr{T}$ is of the form $(*)$. In fact, this is equivalent to the question whether any T-measure is disintegrable by a Markov kernel, and it is not essentially new. It arises implicitly in many works dealing with T-measures, and it was already known that for fundamental triangular norm-based measures on generated T-tribes $\mathscr{T}$ the answer is affirmative (cf. Klement [24]). Also, it was previously known that, even if $\mathscr{T}$ is nongenerated, T_∞-based measures are necessarily of the form $(*)$ (cf. Butnariu [10]—see also Theorem 4.1). The main result of the paper is Theorem 5.3 showing that, in general, each finite monotone fundamental triangular norm-based measure can be uniquely decomposed into a sum of a T-measure of the form $(*)$ and a monotonically irreducible T-measure $\mathbf{m}^*$ (that is a T-measure which is either identically zero, or such

that there is no T-measure of the form $(*)$ differing monotonically from $\mathbf{m}^*$).

The relevance of our results may be seen under several aspects. First, we describe analytically a large class of T-tribes, which are in fact abstractions of the concept of a Boolean ring (see Schmidt [32]), and we characterize fundamental triangular norm-based measures defined on general T-tribes. These are among the generalizations of ordinary probability measures naturally involved in problems of *Pattern Recognition* and *Plausibility Theory* (cf. Sugeno [38], Höhle and Klement [18]), *Automata Theory* (Eilenberg [14]), *Capacity Theory* (Frank [15]), *Mathematical Economics* (Aczel and Alsina [2]), and *Game Theory* (Butnariu [10]). On the other hand, one may look at our results from a probabilistic point of view. In such a context, Theorems 3.5 and 4.1 say that fundamental triangular norm-based T-measures, which are defined on generated T-tribes and T_∞-measures on arbitrary T_∞-tribes, are "totally disintegrable" (i.e., they can be written as integrals of Markov kernels). Theorem 5.3 implies that, in general, fundamental triangular norm-based measures are disintegrable up to a *hard core* which is essentially irreducible. These facts open a way to a proof that on a significant space of coalitional games (known in the literature as pM) a maximally monotone *multivalued value operator* exists. On the other hand, Theorem 4.1 allows formulation of an alternative interpretation of the concept of Lebesgue integral; i.e., it shows that a Lebesgue integral on the set X is precisely a T_∞-measure on a T_∞-tribe in the unit cube $[0, 1]^X$.

Finally, we must point out that our representation theorems for triangular norm-based measures are valid for monotone T-measures only. The question whether they are true for nonmonotone T-measures, too, is equivalent to whether for triangular norm-based measures there exist Jordan decompositions (by monotone T-measures). It follows from a result of Schmidt [31] that T-measures on T-tribes can be written as differences of monotone T-countable additive functions, but this does not mean automatically that for T-measures Jordan decompositions exist (except in the case of T_∞-measures, where Schmidt's results apply according to Example 3.2 (iii) and Remark 4.2 (iii)).

1. Triangular Norms, T-Clans and T-Tribes

A *triangular norm* (*t-norm* for short) is a two-place function $T: [0, 1] \times [0, 1] \to [0, 1]$ which is commutative, associative, monotone in each component and satisfies the boundary condition $T(x, 1) = x$. A t-norm T is called *strict* if it is continuous und satisfies $T(x, y) < T(x, z)$ whenever $y < z$. It is called *Archimedian* if it satisfies $T(x, x) < x$ for all $x \in]0, 1[$. The corresponding *t-conorm* of T is the function $S: [0, 1] \times [0, 1] \to [0, 1]$ defined by $S(x, y) = 1 - T(1-x, 1-y)$.

As an example, a most important family of t-norms $\{T_s\}_{s \in [0, \infty]}$ (cf. [15]), which we call *fundamental t-norms*, is given by

$$\begin{aligned} T_s(x, y) &= \min(x, y) && \text{if } s = 0, \\ &= x \cdot y && \text{if } s = 1, \\ &= \max(0, x + y - 1) && \text{if } s = \infty, \\ &= \log_s \left[1 + \frac{(s^x - 1) \cdot (s^y - 1)}{s - 1} \right] && \text{if } s \in]0, \infty[\setminus \{1\}. \end{aligned}$$

Their corresponding t-conorms are

$$\begin{aligned} S_s(x, y) &= \max(x, y) && \text{if } s = 0, \\ &= x + y - x \cdot y && \text{if } s = 1, \\ &= \min(1, x + y) && \text{if } s = \infty, \\ &= 1 - \log_s \left[1 + \frac{(s^{1-x} - 1) \cdot (s^{1-y} - 1)}{s - 1} \right] && \text{if } s \in]0, \infty[\setminus \{1\}. \end{aligned}$$

This is a "continuous" family of t-norms in the sense that $\lim_{s \to t} T_s = T_t$. Moreover, each pair (T_s, S_s) satisfies the functional equation

$$T(x, y) + S(x, y) = x + y. \qquad (1)$$

T_0 is not Archimedean (and hence not strict), T_∞ is Archimedean but not strict, and each T_s with $s \in]0, \infty[$ is strict (and hence Archimedean).

Consider a countable set J, a family $\{]a_j, b_j[\}_{j \in J}$ of mutually disjoint open subintervals of $[0, 1]$, and a family of t-norms $\{T_j\}_{j \in J}$. Then the function $T: [0, 1] \times [0, 1] \to [0, 1]$ defined by

$$\begin{aligned} T(x, y) &= a_j + (b_j - a_j) \cdot T_j\left(\frac{x - a_j}{b_j - a_j}, \frac{y - a_j}{b_j - a_j}\right) && \text{if } x, y \in]a_j, b_j[\text{ for some } j \text{ in } J, \\ &= \min(x, y) && \text{otherwise,} \end{aligned}$$

is a t-norm called *ordinal sum of the t-norms* $\{T_j\}_{j \in J}$ *over the intervals* $\{]a_j, b_j[\}_{j \in J}$ (see [37]). Ordinal sums of t-conorms can be defined dually. Frank [15] proved that the only pairs (T, S) of continuous t-norms and corresponding t-conorms solving Eq. (1) are the fundamental t-norms and the ordinal sums of fundamental t-norms T_s $(s > 0)$ together with their corresponding t-conorms S_s $(s > 0)$.

The function

$$\begin{aligned} W(x, y) &= \min(x, y) && \text{if } \max(x, y) = 1, \\ &= 0 && \text{otherwise,} \end{aligned}$$

is a t-norm, its corresponding t-conorm is

$$\begin{aligned} V(x, y) &= \max(x, y) && \text{if } \min(x, y) = 0, \\ &= 1 && \text{otherwise.} \end{aligned}$$

W is not Archimedean (and hence not strict) and not continuous. It is the "smallest" t-norm, and the fundamental t-norm T_0 is the "largest" t-norm, i.e., for any t-norm T we have

$$W \leqslant T \leqslant T_0.$$

Given a t-norm T and its corresponding t-conorm S their associativity allows to extend them to n-ary operations $\mathbf{T}_{i=1}^n: [0, 1]^n \to [0, 1]$ and $\mathbf{S}_{i=1}^n: [0, 1]^n \to [0, 1]$. In what follows we write $\mathbf{T}_{i=1}^n x_i$ and $\mathbf{S}_{i=1}^n x_i$ instead of $\mathbf{T}_{i=1}^n (x_1, ..., x_n)$ and $\mathbf{S}_{i=1}^n (x_1, ..., x_n)$, respectively. For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, 1]$ the sequence $\{\mathbf{T}_{i=1}^n x_i\}_{n \in \mathbb{N}}$ is nonincreasing; therefore its limit $\mathbf{T}_{n=1}^\infty x_n = \lim_{n \to \infty} \mathbf{T}_{i=1}^n x_i$ always exists. By duality, the sequence $\{\mathbf{S}_{i=1}^n x_i\}_{n \in \mathbb{N}}$ is nondecreasing, its limit, denoted $\mathbf{S}_{n=1}^\infty x_n$, exists, and we have $\mathbf{S}_{n=1}^\infty x_n = 1 - \mathbf{T}_{n=1}^\infty (1 - x_n)$.

1.1 Proposition. *Let T be a continuous Archimedean t-norm and let $\{x_n\}_{n \in \mathbb{N}}$ be a constant sequence in* $[0, 1[$. *Then we have*

$$\mathbf{T}_{n=1}^\infty x_n = 0.$$

Proof. Assume that $x_n = a \neq 0$ for each $n \in \mathbb{N}$. Let us consider the continuous function h from X to $[0, 1]$ defined by $h(x) = T(x, x)$. Putting $h^1 = h$ and $h^{n+1} = h \circ h^n$, we have, for each $x \in]0, 1[$, $h(x) < x$ and $h^{n+1}(x) \leqslant h^n(x)$. Then for $b = \lim_{n \to \infty} h^n(a)$ we obtain the equality $h(b) = h(\lim_{n \to \infty} h^n(a)) = \lim_{n \to \infty} h^{n+1}(a) = b$ which implies $b = 0$. Since the sequence $\{h^n(a)\}_{n \in \mathbb{N}}$ is a subsequence of the convergent sequence $\{\mathbf{T}_{i=1}^n x_i\}_{n \in \mathbb{N}}$, our result follows. ∎

1.2 Proposition. (i) *If T is a t-norm which is either fundamental or the ordinal sum of a family of fundamental t-norms, then $T_\infty \leqslant T \leqslant T_0$.*

(ii) *If $0 \leqslant s < 1 < t \leqslant \infty$, and T_s and T_t are the corresponding fundamental t-norms, then $T_t \leqslant T_1 \leqslant T_s$.*

Proof. (i) Let T be a t-norm which is either fundamental or the ordinal sum of a family of fundamental t-norms. If T is fundamental, then $T_\infty \leqslant T$ because T and its corresponding t-conorm S satisfy (1). If T itself is not fundamental but an ordinal sum of a family of fundamental t-norms $\{T_j\}_{j\in J}$ over a family of subintervals $\{]a_j, b_j[\}_{j\in J}$ of $[0, 1]$, then for any $j \in J$ and for any $x, y \in]a_j, b_j[$ we have

$$\begin{aligned} T(x, y) &\geqslant a_j + (b_j - a_j)\, T_\infty\left(\frac{x-a_j}{b_j-a_j}, \frac{y-a_j}{b_j-a_j}\right) \\ &= \max[x + y - b_j, a_j] \\ &\geqslant T_\infty(x, y), \end{aligned}$$

and this, together with $T \leqslant T_0$ completes the proof of (i).

(ii) Consider two fundamental t-norms T_s and T_t with $0 < s < 1 < t < \infty$. If $x \in \{0, 1\}$ or $y \in \{0, 1\}$, then $T_t(x, y) = T_1(x, y) = T_s(x, y)$. It remains to show that if $x, y \in]0, 1[$ then $T_t(x, y) \leqslant T_1(x, y) \leqslant T_s(x, y)$. The first inequality is equivalent to

$$\log_t \left[1 + \frac{(t^x - 1)\cdot(t^y - 1)}{t-1}\right] \leqslant x \cdot y.$$

This inequality is equivalent to

$$\frac{t^x - 1}{t^{xy} - 1} \leqslant \frac{t-1}{t^y - 1} \qquad (x, y \in]0, 1[, t > 1)$$

and, substituting $v = t^x$, to

$$\frac{v-1}{v^y - 1} \leqslant \frac{t-1}{t^y - 1} \qquad (1 < v < t \text{ and } y \in]0, 1[).$$

Thus, it is sufficient to show that the function $f_y(v) = (v-1)/(v^y - 1)$ is nondecreasing in the interval $]1, t[$ for any fixed y in $]0, 1[$. Computing the derivative we get

$$f_y'(v) = y \cdot \frac{\dfrac{v^y - 1}{y - 0} - \dfrac{v^y - v^{y-1}}{y - (y-1)}}{(v^y - 1)^2} \qquad (v > 1, y \in]0, 1[).$$

The function $y \to v^y$ is convex on $\mathbb{R}$ for any fixed $v > 0$. Therefore, the denominator of f_y' is nonnegative in $]1, \infty[$, and f_y is nondecreasing. For the second inequality fix $s \in]0, 1[$. In fact, it is sufficient to prove that the function f_y attains its minimal value in the interval $[s, 1[$ at the point $v = s$ for any $y \in]0, 1[$. Since, for $v \in]s, 1[$, f_y' is as above and the denominator of the derivative is still nonnegative for $v \in [s, 1[$, it follows that f_y is nondecreasing on $[s, 1[$ for any y fixed in $]0, 1[$. By consequence, the minimal value of f_y on $[s, 1[$ is attained at $v = s$. ∎

A function $A : X \to [0, 1]$ has been called a *fuzzy subset* of the ordinary set X (Zadeh [41]). This generalizes the concept of a (Cantorian) subset A of X which can be identified with its characteristic function $A : X \to \{0, 1\}$ defined by $A(x) = 1$ if $x \in A$, and $A(x) = 0$ if $x \notin A$. If A is a fuzzy subset of X, then the value $A(x)$ is interpreted as the degree of membership of the point x in A. The collection of all fuzzy subsets of X is denoted $[0, 1]^X$, as usual.

Let T be a t-norm and S be its corresponding t-conorm. We extend T and S to $[0, 1]^X$ pointwise, i.e., $(A\,\mathbf{T}\,B)(x) = T(A(x), B(x))$ and $(A\,\mathbf{S}\,B)(x) = S(A(x), B(x))$. These operations can be considered as "intersection" and "union" of fuzzy subsets, respectively. Also, finite (countable) "intersections" $\mathbf{T}_{i=1}^n A_i$ ($\mathbf{T}_{n=1}^\infty A_i$) and "unions" $\mathbf{S}_{i=1}^n A_i$ ($\mathbf{S}_{n=1}^\infty A_i$) of fuzzy subsets are defined in the straightforward way. They satisfy the De Morgan laws

$$\left(\mathop{\mathbf{T}}_{n=1}^{\infty} A_n\right)' = \mathop{\mathbf{S}}_{n=1}^{\infty} A_n' \qquad \text{and} \qquad \left(\mathop{\mathbf{S}}_{n=1}^{\infty} A_n\right)' = \mathop{\mathbf{T}}_{n=1}^{\infty} A_n',$$

where the "complement" A' is defined by $A'(x) = 1 - A(x)$. Restricted to ordinary sets (i.e., characteristic functions), these operations coincide with intersection, union, and complement, respectively, regardless which t-norm and t-conorm is considered. The class $[0, 1]^X$ of the fuzzy subsets of X together with the operations $\mathbf{T}$ and $\mathbf{S}$ form a partially ordered commutative semigroup having $\varnothing$ as smallest (and as null) element and X as largest (and as unit) element. However, $[0, 1]^X$ provided with the operations $\mathbf{T}$, $\mathbf{S}$, and the complement "$'$" is not a Boolean algebra. It is not even a lattice, except in the case $T = T_0$ and $S = S_0$. In general, $\mathbf{T}$ and $\mathbf{S}$ are not distributive with respect to each other, $A\,\mathbf{T}\,A'$ may be different from $\varnothing$ and $A\,\mathbf{S}\,A'$ may be different from X.

Let T be a t-norm. A subfamily $\mathscr{C}$ of $[0, 1]^X$ containing $\varnothing$ and being closed under the operation $\mathbf{T}$ and under complementation will be called a *T-clan*. Obviously, by the duality of $\mathbf{T}$ and $\mathbf{S}$, the closedness with respect to $\mathbf{T}$ can be replaced by the closedness with respect to $\mathbf{S}$ in the definition above.

1.3 Example. (i) Since we identify ordinary subsets of X with their characteristic functions, any algebra of subsets of X is a T-clan with respect to any t-norm T.

(ii) For any $n \in \mathbb{N}$ the family $\mathscr{C}_n(X) = \{0, 1/n, ..., (n-1)/n, 1\}^X$ is a T-clan for $T = T_0$, and also for $T = T_\infty$, but not with respect to any other fundamental t-norm.

(iii) If the t-norm T is continuous (measurable), and if X is a topological (measurable) space, then the family of all continuous (measurable) fuzzy subsets of X is a T-clan.

A T-clan $\mathscr{T}$ which is also closed under countable "intersections," i.e., which satisfies

$$\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T} \Rightarrow \mathop{\mathbf{T}}_{n=1}^{\infty} A_n \in \mathscr{T},$$

is called a *T-tribe*. A pair $(X, \mathscr{T})$, where X is a set and $\mathscr{T}$ is a T-tribe, is called a *T-measurable space*.

1.4 EXAMPLE. (i) Obviously, not any T-clan is a T-tribe. For instance, the family of all constant functions on X with values in $\mathbb{Q} \cap [0, 1]$ is a T_∞-clan but not a T_∞-tribe.

(ii) Any σ-algebra of subsets of X is a T-tribe with respect to any t-norm T.

(iii) Given a σ-algebra $\mathscr{A}$ of subsets of X, the family $\mathscr{A}^\vee$ of all $\mathscr{A}$-measurable fuzzy subsets of X is a T-tribe with respect to any Borel-measurable t-norm T.

(iv) Given a T-tribe $\mathscr{T}$, the family $\mathscr{T}^\vee$ of all characteristic functions contained in $\mathscr{T}$ is a σ-algebra, and hence a T-tribe.

(v) (Klement [20]) The family $\mathscr{T}$ consisting of all fuzzy subsets of $X = [0, 1]$ which are either constant or have all their values in the interval $[\frac{1}{3}, \frac{2}{3}]$ is a T-tribe in the case $T = W$ and for $T = T_0$, but it is not a T-tribe for $T = T_t$ with $t \in]0, \infty]$. It is interesting to note that there is no σ-algebra of sets $\mathscr{A}$ such that $\mathscr{T} = \mathscr{A}^\wedge$.

(vi) (Klement [22]) If in Example (v) we additionally require all elements of $\mathscr{T}$ to be continuous, then $\mathscr{T}$ is a W-tribe but not a T_0-tribe.

(vii) Consider a nonempty subset Y of X such that $Y \neq X$. The family $\mathscr{T}$ of fuzzy subsets of X which are constant on Y and assume only values 0 and 1 outside of Y, is a T-tribe with respect to any t-norm T, but it does not contain any constant fuzzy subset (except $\varnothing$ and X).

1.5 THEOREM. *If $s \in]0, \infty[$ and T_s is the corresponding fundamental t-norm, then any T_s-tribe $\mathscr{T}$ is a T_∞-tribe. Moreover, any T_∞-tribe is a T_0-tribe.*

Proof. We fix an arbitrary $s \in]0, \infty[$. The proof is carried out in several steps.

(a) First we prove that if $A, B \in \mathscr{T}$ then there exists a $C \in \mathscr{T}$ such that $A\,\mathbf{S}_\infty\, B = A\,\mathbf{S}_s\, C$. Let A and B be any two fuzzy subsets in the T_s-tribe

$\mathscr{T}$. Define a double sequence as follows: $A_1 = A$, $B_1 = B$, $A_{n+1} = A_n\,\mathbf{S}_s\, B_n$, and $B_{n+1} = A_n\,\mathbf{T}_s\, B_n$. The sequence $\{A_n\}_{n\in\mathbb{N}}$ is nondecreasing, the sequence $\{B_n\}_{n\in\mathbb{N}}$ is nonincreasing, and both sequences are contained in $\mathscr{T}$. Since the pair (T_s, S_s) satisfies (1), by induction we get for all $n \in \mathbb{N}$

$$A_n + B_n = A + B. \tag{2}$$

Claim 1. For each $a \in [0, 1[$ there exists a number $c \in [0, 1[$ such that for all b in $[0, a]$ we have

$$T_s(a, T_s(a, b)) \leqslant c \cdot T_s(a, b). \tag{3}$$

Indeed, from Proposition 1.2 we have that $T_s \leqslant T_1$ for $s \geqslant 1$; and this implies that if $s \geqslant 1$ we can choose $c = a$. If $s < 1$, consider $c = (s^a - 1)/(s - 1)$. It is clear that $c < 1$. Then for each $b \in [0, a[$ the inequality (3) is equivalent to

$$\ln[1 + c^2(s^b - 1)] \geqslant c \cdot \ln[1 + c(s^b - 1)].$$

The expansion of the logarithms in power series leads to

$$\sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{c^{2i}(s^b - 1)^i}{i} \geqslant \sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{c^{i+1}(s^b - 1)^i}{i}$$

which is equivalent to $c^{i-1} \leqslant 1$. Since this inequality holds for all $i \in \mathbb{N}$, it follows that (3) is valid for all $s \in]0, \infty[$.

Claim 2. We have

$$A\,\mathbf{S}_\infty\, B = \mathop{\mathbf{S}_s}_{n=1}^{\infty} C_n, \tag{4}$$

where $C_1 = A_1$ and $C_{n+1} = B_n$, $(n \in \mathbb{N})$. In order to prove that, we fix an $x \in X$ and put $\alpha = (A\,\mathbf{S}_\infty\, B)(x)$. If $\alpha < 1$, then $A_n(x) \leqslant \alpha < 1$ for all $n \in \mathbb{N}$ because of (2). Let c be a number in $[0, \alpha[$ such that (3) is satisfied. Then, by the monotonicity of the t-norm T_s we get

$$\begin{aligned} B_n(x) &\leqslant c^{n-2} \cdot \alpha && \text{if} \quad s < 1 \\ &\leqslant \alpha^n && \text{if} \quad s \geqslant 1 \end{aligned}$$

for all $n \geqslant 2$. Since α and c are both in $[0, 1[$, it follows that

$$\lim_{n\to\infty} B_n(x) = 0$$

and, because of (2),

$$\lim_{n\to\infty} A_n(x) = (A\,\mathbf{S}_\infty\, B)(x), \tag{5}$$

which is exactly (4) by the definition of the double sequence. Now, assume that $\alpha = 1$. In this case (5) also holds, since assuming the contrary we get

$$\lim_{n \to \infty} A_n(x) < 1,$$

and this means that there exists a number d in $[0, 1[$ such that $A_n(x) \leqslant d < 1$ for all $n \in \mathbb{N}$. But using analogous arguments as above, with d instead of α, we deduce that

$$\lim_{n \to \infty} A_n(x) \geqslant 1$$

contradicting our assumption. Hence (4) is always true. Putting $C = \mathbf{S}_{n=2}^{\infty}\, C_n$, the proof of part (a) is complete. This also shows that the T_∞-tribe $\mathscr{T}$ is a T_∞-clan.

(b) Let $\{D_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathscr{T}$. There exists a sequence $\{E_n\}_{n \in \mathbb{N}}$ in $\mathscr{T}$ such that for each $n \in \mathbb{N}$ we have

$$\mathbf{S}_{\infty\, j=1}^{n} D_j = \mathbf{S}_{s\, k=1}^{n} E_k. \tag{6}$$

For $n = 2$ this follows from part (a). Suppose we have proved (6) for $n \in \mathbb{N}$. Then we get, again using part (a),

$$\mathbf{S}_{\infty\, j=1}^{n+1} D_j = \Big(\mathbf{S}_{\infty\, j=1}^{n} D_j\Big) \mathbf{S}_\infty D_{n+1} = \Big(\mathbf{S}_{s\, j=1}^{n} E_j\Big) \mathbf{S}_\infty D_{n+1}$$
$$= \Big(\mathbf{S}_{s\, j=1}^{n} E_j\Big) \mathbf{S}_s E_{n+1} = \mathbf{S}_{s\, j=1}^{n+1} E_j.$$

Now, because of (6), we obtain

$$\mathbf{S}_{\infty\, n=1}^{\infty} D_n = \lim_{n \to \infty} \mathbf{S}_{\infty\, j=1}^{n} D_j = \lim_{n \to \infty} \mathbf{S}_{s\, j=1}^{n} E_j = \mathbf{S}_{s\, n=1}^{\infty} E_n,$$

the latter fuzzy subset being an element of $\mathscr{T}$. This shows that $\mathscr{T}$ is a T_∞-tribe.

(c) In order to show that a T_∞-tribe $\mathscr{T}$ is a T_0-clan it suffices to observe that for any two fuzzy subsets A and B one has $A\, \mathbf{T}_0\, B = A\, \mathbf{T}_\infty\, (B\, \mathbf{S}_\infty\, A')$. Actually, $\mathscr{T}$ is even a T_0-tribe: If $\{A_n\}_{n \in \mathbb{N}}$ is an increasing sequence in $\mathscr{T}$ put $B_n = A_n\, \mathbf{T}_\infty\, A'_{n-1}$ for each $n \in \mathbb{N}$ with $A_0 = \varnothing$ and observe that

$$A_n = \mathbf{S}_{\infty\, i=1}^{n} B_i \qquad (n \in \mathbb{N}).$$

Hence,

$$\mathbf{S}_{0\, n=1}^{\infty} A_i = \lim_{n \to \infty} A_n = \mathbf{S}_{\infty\, n=1}^{\infty} B_n \in \mathscr{T}. \qquad \blacksquare$$

Theorem 1.5 shows that T-tribes based on fundamental t-norms T are implicitly σ-complete lattices with respect to the pointwise order and that they are closed sets with respect to the weak topology on the unit cube $[0, 1]^X$. Moreover, viewed as T_∞-tribes, the fundamental norm based T_s-tribes with $s \in]0, \infty]$ are implicitly "clans" in the sense of Wyler [40], where the subtraction is defined by $A \ominus B = A\, \mathbf{T}_\infty\, B'$ (see also Schmidt [22]).

2. Representation of T-Tribes, Disjointness

We already observed that any T-tribe $\mathscr{T}$ on X includes a σ-algebra $\mathscr{T}^\vee$ of subsets of X, and that the family $(\mathscr{T}^\vee)^\wedge$ of all $\mathscr{T}^\vee$-measurable functions $X \to [0, 1]$ is a T-tribe (if T is Borel-measurable (see Example 1.4)). Now we study the precise relationship between $\mathscr{T}$ and $(\mathscr{T}^\vee)^\wedge$. In particular, we are interested to know under which conditions they coincide. In this case, the T-tribe $\mathscr{T}$ is called *generated*. For a nongenerated T-tribe see Example 1.4 (v).

2.1 Theorem. *For any fundamental t-norm T_s with $s > 0$, and for each T_s-tribe $\mathscr{T}$ we have $\mathscr{T} \subseteq (\mathscr{T}^\vee)^\wedge$.*

Proof. For any $A \in \mathscr{T}$ and for any $a \in [0, 1]$ we denote $A_a = \{x \in X;\ A(x) \geqslant a\}$, and we must show that $A_a \in \mathscr{T}^\vee$. For $a = 0$ this is trivial. Assume $a = 1$. Then, because of Proposition 1.1, for any fuzzy subset B we have

$$B(x) > 0 \Leftrightarrow \mathbf{S}_{s\, n=1}^{\infty} B_n(x) = 1 \qquad (\text{where } B_n = B \text{ for each } n \in \mathbb{N}).$$

Putting $B = A'$ this yields $A_1 \in \mathscr{T}^\vee$. Now choose $a \in]0, 1[$. Then there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of positive rational numbers increasing to a, and we have

$$A_a = \bigcap_{n=1}^{\infty} A_{a_n}.$$

Thus, it suffices to show that $A_a \in \mathscr{T}^\vee$ for any $a \in]0, 1[$ which can be represented in the form

$$a = \sum_{i=1}^{k} \frac{a(i)}{2^i} \qquad \text{with} \quad a(i) \in \{0, 1\} \qquad \text{for} \quad 1 \leqslant i \leqslant k, \quad \text{and} \quad k \in \mathbb{N}. \tag{7}$$

We proceed by induction upon the positive integer k involved in (7). If $k=1$ than $a(1)=1$ and $a=\frac{1}{2}$. Thus we have $A_a=\{x\in X;\ (A\,\mathbf{S}_\infty\,A)(x)=1\}=(A\,\mathbf{S}_\infty\,A)_1$. But $A\,\mathbf{S}_\infty\,A\in\mathscr{T}$ because of Theorem 1.5. Therefore, $A_a=(A\,\mathbf{S}_\infty\,A)_1$ belongs to $\mathscr{T}^\vee$. Let us assume that for every A in $\mathscr{T}$ and for every $a\in\,]0,1[$ of the form (7) with $k\leqslant n$ we have $A_a\in\mathscr{T}^\vee$. Suppose that

$$a=\sum_{i=1}^{n+1}\frac{a(i)}{2^i}\qquad\text{with}\quad a(i)\in\{0,1\}$$

$$\text{for}\quad 1\leqslant i\leqslant n+1\quad\text{and}\quad a(n+1)\neq 0. \tag{8}$$

Then $a=b/2$, where $b=\sum_{i=1}^{n+1}(a(i)/2^{i-1})$. If $b\in[0,1]$, then $A_a=(A\,\mathbf{T}_\infty\,A)_b\in\mathscr{T}^\vee$ by the inductive assumption. If $b\notin[0,1]$, then $a>\frac{1}{2}$, and it can be written as

$$a=(c+1)/2\qquad\text{with}\quad c=\sum_{i=1}^{n}\frac{a(i+1)}{2^i}\in\,]0,1].$$

Thus we get $A_a=(A'\,\mathbf{T}_\infty\,A')'_c\in\mathscr{T}^\vee$ by Theorem 1.5 and the inductive assumption. ∎

In general a T_s-tribe $\mathscr{T}$ is not generated (i.e., $\mathscr{T}$ may be different from $(\mathscr{T}^\vee)^\wedge$) even if T_s is a fundamental t-norm with $s>0$ (see Example 1.4 (v)). However, we have the following result:

2.2 THEOREM. *For any fundamental t-norm T_s with $s>0$, a T_s-tribe on X is generated if and only if it contains all the constant fuzzy subsets of X.*

Proof. Necessity is obvious. Conversely, assume that the T_s-tribe $\mathscr{T}$ contains all the constant fuzzy subsets of X. We must show that each A in $(\mathscr{T}^\vee)^\wedge$ is contained in $\mathscr{T}$ (cf. Theorem 2.1). Define the sequence of fuzzy subsets

$$U_n=\mathop{\mathbf{S}_\infty}_{k=1}^{2^n-1}(A_k^{[n]}\,\mathbf{T}_\infty\,V_k^{[n]}),$$

where

$$\begin{aligned}V_k^{[n]}&=\{x\in X;\ (2k-1)/2^n\leqslant A(x)<2k/2^n\} &&\text{if}\quad k<2^{n-1},\\ &=\{x\in X;\ (2^n-1)/2^n\leqslant A(x)\} &&\text{otherwise}\end{aligned}$$

and $A_k^{[n]}$ is the constant fuzzy subset $A_k^{[n]}(x)=a_k^{[n]}$, with the number $a_k^{[n]}$ chosen such that $S_s((2k-2)/2^n, a_k^{[n]})=(2k-1)/2^n$. Note that this choice of $a_k^{[n]}$ is possible since the function $S_s((2k-2)/2^n,\cdot)$: $[0,1]\to[(2k-2)/2^n,1]$ is a surjection. Because of the $\mathscr{T}^\vee$-measurability of A each $V_k^{[n]}$ is contained in $\mathscr{T}^\vee$, and hence in $\mathscr{T}$. Since $\mathscr{T}$ contains the constant fuzzy subsets $A_k^{[n]}$, and since it is a T_∞-clan (cf. Theorem 1.5), it follows that $U_n\in\mathscr{T}$. Now it is a matter of computation to check that

$$\mathop{\mathbf{S}_s}_{i=1}^{n}U_i=\sum_{k=1}^{2^n}\frac{k-1}{2^n}\cdot W_k^{[n]}, \tag{9}$$

where

$$\begin{aligned}W_k^{[n]}&=\{x\in X;\ (k-1)/2^n\leqslant A(x)<k/2^n\} &&\text{if}\quad k<2^n,\\ &=\{x\in X;\ (2^n-1)/2^n\leqslant A(x)\} &&\text{otherwise.}\end{aligned}$$

Since the functions on the right-hand side of (9) are convergent to A, we get $A=\mathop{\mathbf{S}_\infty}_{n=1}^{\infty}U_n$, showing that $A\in\mathscr{T}$. ∎

2.3 *Remark.* (i) A T_s-tribe $\mathscr{T}$ may be not generated even if T_s is a fundamental t-norm with $s>0$. See, for instance, Example 1.4 (vii).

(ii) By virtue of Theorem 1.5, in Theorem 2.2 the condition "$\mathscr{T}$ contains all the constant fuzzy subsets of X" can be replaced by the condition "$\mathscr{T}$ contains a sequence $\{A_n\}_{n\in\mathbb{N}}$ of constant fuzzy subsets of X with $A_n(x)=1/z^n$ $(n\in\mathbb{N})$, where $z\geqslant 2$ is an integer."

In order to introduce the important concept of disjointness of fuzzy subsets with respect to t-norms, let X be a nonempty set, T a t-norm and S its corresponding t-conorm. A *finite family* of fuzzy subsets $A_1, A_2, \ldots, A_n$ of X is said to be *T-disjoint* if

$$\Big(\mathop{\mathbf{S}}_{j\neq k}A_j\Big)\,\mathbf{T}\,A_k=\varnothing\qquad(1\leqslant k\leqslant n). \tag{10}$$

An *infinite sequence* $\{A_j\}_{j\in\mathbb{N}}$ of fuzzy subsets of X is called *T-disjoint* if for any $n\in\mathbb{N}$, $n\geqslant 2$, the finite family $A_1, \ldots, A_n$ is T-disjoint.

2.4 *Remark.* (i) Any subfamily $\{A_i\}_{i\in I}$ of a countable T-disjoint family $\{A_i\}_{i\in J}$ is also T-disjoint. Obviously, it suffices to prove this for a finite set $J=\{1,2,\ldots,n\}$ and a subset $I=\{i_1,\ldots,i_k\}\subseteq J$. Indeed, for any $i_r\in I$ we have

$$\Big(\mathop{\mathbf{S}}_{\substack{h=1\\h\neq r}}^{k}A_{i_h}\Big)\,\mathbf{T}\,A_{i_r}\leqslant\Big(\mathop{\mathbf{S}}_{\substack{h=1\\h\neq r}}^{n}A_{i_h}\Big)\,\mathbf{T}\,A_{i_r}=\varnothing.$$

(ii) The definition of T-disjointness does not depend on the order in which the fuzzy subsets $\{A_j\}_{j\in\mathbb{N}}$ are numbered, i.e., if π is a permutation of $\mathbb{N}$ and $\{A_j\}_{j\in\mathbb{N}}$ is T-disjoint so is $\{A_{\pi(j)}\}_{j\in\mathbb{N}}$.

Different t-norms may lead to different concepts of "disjointness." However, for some classes of t-norms the corresponding "disjointness" concepts do not depend on the choice of the t-norm in that class.

2.5 EXAMPLE. Let $\{A_j\}_{j\in\mathbb{N}}$ be a countable family of fuzzy subsets.

(i) If all A_j are (characteristic functions of) ordinary sets then T-disjointness is equivalent with pairwise disjointness with respect to any t-norm.

(ii) T-disjointness implies pairwise T-disjointness according to 2.4 (i), but the converse is not generally true: if we take $A_i = \frac{1}{2}$ for $i = 1, 2, 3$, then A_1, A_2, A_3 are pairwise T_∞-disjoint, but they are not T_∞-disjoint.

(iii) For $s \in [0, \infty[$ we get: $\{A_j\}_{j\in\mathbb{N}}$ is T_s-disjoint if and only if each x is "contained" in at most one A_k (that is if and only if $A_k(x) > 0$ for at most one k).

(iv) W-disjointness of $\{A_j\}_{j\in\mathbb{N}}$ means that for each $x \in X$ exactly one of the following conditions holds:

(1) There is at most one $k \in \mathbb{N}$ such that $A_k(x) = 1$.

(2) There are at most two indices $k, l \in \mathbb{N}$ such that $0 < A_k(x)$, $A_l(x) < 1$.

2.6 PROPOSITION. *Let T be a t-norm and S be its corresponding t-conorm such that* (1) *holds. Then for any $n \geq 2$ the following conditions are equivalent*:

(i) *$A_1, ..., A_n$ are T-disjoint.*

(ii) *For any $k = 2, ..., n$: $(\mathbf{S}_{i=1}^{k-1} A_i)\,\mathbf{T}\,A_k = \varnothing$.*

(iii) *For any $k = 2, ..., n$: $\mathbf{S}_{i=1}^{k} A_i = \sum_{i=1}^{k} A_i$.*

(iv) *For each set $I \subseteq \{1, 2, ..., n\}$ containing at least $n-1$ elements*: $\mathbf{S}_{i\in I} A_i = \sum_{i\in I} A_i$.

Proof. (i) ⇒ (ii) is an immediate consequence of Remark 2.4 (i).

(ii) ⇒ (iii). Using (1) we have

$$\left(\mathop{\mathbf{S}}_{i=1}^{k-1} A_i\right)\mathbf{T}\,A_k + \left(\mathop{\mathbf{S}}_{i=1}^{k-1} A_i\right)\mathbf{S}\,A_k = \mathop{\mathbf{S}}_{i=1}^{k-1} A_i + A_k, \tag{11}$$

which implies

$$\mathop{\mathbf{S}}_{i=1}^{k} A_i = \mathop{\mathbf{S}}_{i=1}^{k-1} A_i + A_k.$$

Repeating this $(k-1)$ times gives the desired result.

(iii) ⇒ (iv). If $n = 2$ or $I = \{1, 2, ..., n\}$ or $I = \{1, 2, ..., n-1\}$ nothing is to prove. Otherwise, observe that for $2 \leq k \leq n$ we have (11), which implies

$$\left(\mathop{\mathbf{S}}_{i=1}^{k-1} A_i\right)\mathbf{T}\,A_k = \varnothing.$$

Now, for $k = 2, ..., n+1$ and $j \leq k-1$ define $I_{k,j} = \{1, ..., k-1\}\setminus\{j\}$. Then (11) together with the monotonicity of S and T implies

$$\left(\mathop{\mathbf{S}}_{i\in I_{k,j}} A_i\right)\mathbf{T}\,A_k = \varnothing. \tag{12}$$

Since from (1) we have

$$\left(\mathop{\mathbf{S}}_{i\in I_{k,j}} A_i\right)\mathbf{T}\,A_k + \left(\mathop{\mathbf{S}}_{i\in I_{k,j}} A_i\right)\mathbf{S}\,A_k = \mathop{\mathbf{S}}_{i\in I_{k,j}} A_i + A_k$$

and because of (12) we get

$$\mathop{\mathbf{S}}_{i\in I_{k+1,j}} A_i = \mathop{\mathbf{S}}_{i\in I_{k,j}} A_i + A_k. \tag{13}$$

Now, put $k = n$ and $j \leq n-1$. If $j = n-1$ we obtain the desired result from (13). If $j < n-1$, compute $\mathbf{S}_{i\in I_{k,j}} A_i$ using (13), and insert it in (13) again. Continue until $j = k-1$, and this gives again the desired result.

(iv) ⇒ (i). For $1 \leq k \leq n$ put $I_k = \{1, 2, ..., n\}\setminus\{k\}$. Then because of (1) we have

$$\left(\mathop{\mathbf{S}}_{i\in I_k} A_i\right)\mathbf{T}\,A_k + \left(\mathop{\mathbf{S}}_{i\in I_k} A_i\right)\mathbf{S}\,A_k = \mathop{\mathbf{S}}_{i\in I_k} A_i + A_k,$$

which immediately implies T-disjointness. ∎

2.7 COROLLARY. *Let T be a t-norm and S its corresponding t-conorm such that* (1) *holds. Then the following assertions are equivalent*:

(i) *The family $\{A_n\}_{n\in\mathbb{N}}$ is T-disjoint.*

(ii) *For any $k \geq 2$ we have*: $(\mathbf{S}_{i=1}^{k-1} A_i)\,\mathbf{T}\,A_k = \varnothing$.

(iii) *For any $k \geq 2$ we have*: $\mathbf{S}_{i=1}^{k} A_i = \sum_{i=1}^{k} A_i$.

(iv) *For each finite subset I of $\mathbb{N}$ we have*: $\mathbf{S}_{i\in I} A_i = \sum_{i\in I} A_i$.

2.8 *Remark.* Let $\{A_j\}_{j\in J}$ be a countable family of fuzzy subsets.

(i) $\{A_j\}_{j\in J}$ is T_∞-disjoint if and only if $\sum_{i\in J} A_i \leq 1$.

(ii) From Proposition 2.6 and Corollary 2.7 we know that if T and its corresponding t-conorm S satisfy (1) and if $\{A_j\}_{j\in J}$ is T-disjoint, then

$\sum_{i\in J} A_i \leq 1$. However, the converse is not generally true (see Example 2.5(iii)).

(iii) The requirement that T and S satisfy (1) cannot be dropped in Proposition 2.6 and in Corollary 2.7. If, for instance, we take $S = W'$ and $T = W$, then the conditions (i), (ii), and (iii) are no longer equivalent.

3. T-Measures and a First Representation Theorem

Throughout this paragraph let X be a nonempty set, T a t-norm, and S its corresponding t-conorm. For a T-clan $\mathscr{T} \subseteq [0, 1]^X$ we consider functions $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ which assume at most one of the values $-\infty$ and $+\infty$. A function $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ is called a *T-valuation* (*on* $\mathscr{T}$) if it satisfies the following conditions:

$$\mathbf{m}(\emptyset) = 0 \tag{14}$$

$$A, B \in \mathscr{T} \Rightarrow \mathbf{m}(A\,\mathbf{T}\,B) + \mathbf{m}(A\,\mathbf{S}\,B) = \mathbf{m}(A) + \mathbf{m}(B). \tag{15}$$

A function $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ is said to be T-additive if it satisfies (14) and

$$(A, B \in \mathscr{T} \text{ and } A\,\mathbf{T}\,B = \emptyset) \Rightarrow \mathbf{m}(A\,\mathbf{S}\,B) = \mathbf{m}(A) + \mathbf{m}(B). \tag{16}$$

3.1 *Remark.* (i) If $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ is a T-valuation on the T-clan $\mathscr{T}$ then $\mathbf{m}$ is also T-additive, the converse not being generally true since, for instance, if $\mathscr{T}$ consists of all the constant functions in $[0, 1]^X$ and if $s \in [0, +\infty[$, then, because of the absence of any nontrivial T_s-disjoint elements in the T_s-clan $\mathscr{T}$, each function $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ which satisfies (14) is T_s-additive without necessarily being a T_s-valuation. This shows that our T-valuations are particular *additive functions* in the sense of Schmidt [31, p. 558] and that, consequently, if they are finite, they can be represented as differences of monotone T-additive functions (cf. Schmidt [31, Theorem 2.2]). However, this is not sufficient to conclude directly that T-valuations always have Jordan decompositions.

(ii) If $\mathscr{T}$ is a T_∞-clan and if $\mathbf{m}$ is a finite T_∞-additive function on $\mathscr{T}$, then $\mathbf{m}$ is also a T_∞-valuation.

(iii) If $\mathscr{T}$ is a T-clan consisting of characteristic functions only, then the finite T-additive functions are Q-valuations for any t-norm Q.

A function $\mathbf{m}$ from a T-clan $\mathscr{T}$ to $[-\infty, +\infty]$ is called a *T-measure* if it is a T-valuation and if the following *left-continuity* is satisfied

$$(\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T},\ A_n \uparrow A \text{ and } A \in \mathscr{T}) \Rightarrow \lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(A). \tag{17}$$

A function $\mathbf{m}$ from a T-tribe $\mathscr{T}$ to $[-\infty, +\infty]$ is said to be *T-countably additive* if it satisfies (14) and if

$$\mathbf{m}\left(\mathop{\mathbf{S}}_{i=1}^{\infty} A_n\right) = \sum_{i=1}^{\infty} \mathbf{m}(A_n) \tag{18}$$

for any T-disjoint sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T}$.

3.2 Example. (i) T-countably additive functions are T-additive.

(ii) T-measures on T-tribes are T-countably additive: Take a T-disjoint sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathscr{T}$, and define $B_n = \mathbf{S}_{i=1}^{n} A_i$; then $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T}$ and $B_n \uparrow (\mathbf{S}_{n=1}^{\infty} A_n)$ and

$$\mathbf{m}\left(\mathop{\mathbf{S}}_{n=1}^{\infty} A_n\right) = \mathbf{m}(\lim_{n\to\infty} B_n) = \lim_{n\to\infty} \mathbf{m}(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{m}(A_i) = \sum_{n=1}^{\infty} \mathbf{m}(A_n).$$

(iii) T_∞-countably additive functions on T_∞-tribes are necessarily T_∞-measures (cf. Butnariu [5]), but, in general, for arbitrary t-norms T and T-tribes $\mathscr{T}$ the T-countably additivity does not imply the left-continuity (17). For example, if $T = T_s$ with $s \in [0, \infty[$ and $\mathscr{T} = [0, 1]^X$, then for any fixed $x_0 \in X$ the function $\mathbf{m}$ from $\mathscr{T}$ to $[-\infty, +\infty]$ defined by $\mathbf{m}(A) = 1$ if $A(x_0) = 1$, and $\mathbf{m}(A) = 0$ if $A(x_0) < 1$ is T_s-countably additive, but it is not a T_s-measure since it does not satisfy (17).

(iv) If $\mathscr{T}$ is a T-tribe which consists of characteristic functions only, then the family of T-countable additivity functions coincides with the family of T-measures for any t-norm T, since in this case all t-norms on $\mathscr{T}$ coincide with T_∞.

(v) One can strengthen in some way the left-continuity (17) replacing it by

$$(\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T},\ A_n \uparrow A \text{ and } A \in \mathscr{T}) \Rightarrow \lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}\left(\mathop{\mathbf{S}}_{n=1}^{\infty} A_n\right). \tag{19}$$

(vi) For T-tribes consisting of characteristic functions only conditions (17) and (19) are equivalent.

(vii) If $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ is *monotone* in the sense that

$$(A, B \in \mathscr{T} \text{ and } A \leq B) \Rightarrow \mathbf{m}(A) \leq \mathbf{m}(B), \tag{20}$$

then (19) implies (17) for any t-norm T and for any T-tribe $\mathscr{T}$, since for $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T}$, $A_n \uparrow A$ and $A \in \mathscr{T}$ we have

$$A_n \leqslant A = \mathbf{S}_0{}_{n=1}^{\infty} A_n \leqslant \mathbf{S}_{n=1}^{\infty} A_n,$$

which implies $\lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(A) \leqslant \mathbf{m}(\mathbf{S}_{n=1}^{\infty} A_n)$.

3.3 Proposition. *If $\mathscr{T}$ is both a T-clan and a T_0-clan, then each T-valuation is a T_0-valuation.*

Proof. For any A and B in $\mathscr{T}$ we have

$$\begin{aligned}\mathbf{m}(A\,\mathbf{T}_0\,B) &+ \mathbf{m}(A\,\mathbf{S}_0\,B)\\ &= \mathbf{m}((A\,\mathbf{T}_0\,B)\,\mathbf{T}\,(A\,\mathbf{S}_0\,B)) + \mathbf{m}((A\,\mathbf{T}_0\,B)\,\mathbf{S}\,(A\,\mathbf{S}_0\,B))\\ &= \mathbf{m}(A\,\mathbf{T}\,B) + \mathbf{m}(A\,\mathbf{S}\,B) = \mathbf{m}(A) + \mathbf{m}(B). \quad ▮\end{aligned}$$

3.4 *Remark.* (i) Proposition 3.3 shows that if $\mathbf{m}$ is a T-measure on $\mathscr{T}$ ($\mathscr{T}$ being both a T- and a T_0-clan), then it is also a T_0-measure.

(ii) If $\mathbf{m}$ is a T_s-measure on a T_s-tribe $\mathscr{T}$ with $s \in]0, \infty]$, then it is also a T_0-measure (cf. Theorem 1.5).

(iii) The converse of Proposition 3.3 does not generally hold: Let $\mathscr{T}$ be the family of all Borel-measurable fuzzy subsets on $X = [0, 1]$. Then the function $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ defined by

$$\mathbf{m}(A) = \int_{\{A>0\}} (1 + A(x))\, dx$$

is a T_0-valuation (even a T_0-measure) but not a T_∞-valuation.

The T_0-measures play a fundamental role in the following. In order to give an integral representation for them let $(X, \mathscr{A})$ be a measurable space, $\mathscr{B}_0$ be the family of all Borel subsets of $[0, 1]$ and $\mathscr{B}_1 = \mathscr{B}_0 \cap [0, 1[$. A function $K: X \times \mathscr{B}_1 \to \mathbb{R}$ is called an *$\mathscr{A}$-Markov kernel* if it satisfies the following conditions:

(a) For each $x \in X$, the function $K(x, \cdot): \mathscr{B}_1 \to \mathbb{R}$ is a probability measure on $\mathscr{B}_1$;

(b) For each $B \in \mathscr{B}_1$, the function $K(\cdot, B): X \to \mathbb{R}$ is measurable.

It was observed above (see Example 1.4 (iii)) that, if $(X, \mathscr{A})$ is a measurable space then the family $\mathscr{A}^\wedge$ of all $\mathscr{A}$-measurable functions from X to $[0, 1]$ is a T_0-tribe. The following result shows that T_0-measures on $\mathscr{A}^\wedge$ can be represented as integrals of Markov kernels.

3.5 Theorem (First Representation Theorem, Klement [21]). *If T_s is a fundamental t-norm with $s \in [0, \infty]$, if $\mathscr{T}$ is a generated T_s-tribe and if $\mathbf{m}$ is a finite monotone T_s-measure on $\mathscr{T}$, then there exists a unique measure $\check{\mathbf{m}}$ on $\mathscr{T}^\vee$ and an $\check{\mathbf{m}}$-a.e. uniquely determined $\mathscr{T}^\vee$-Markov kernel $K: X \times \mathscr{B}_1 \to \mathbb{R}$ such that*

$$\mathbf{m}(A) = \int_X K(x, [0, A(x)[)\, d\check{\mathbf{m}}(x) \qquad (A \in \mathscr{A}). \tag{21}$$

Proof. Immediate if one combines Proposition 3.3, Remark 3.4 (ii), and the representation theorem in Klement [21, Section 6]. ▮

4. Integral Representation of $\mathbf{T}_\infty$-Measures

The First Representation Theorem shows that monotone finite measures based on fundamental t-norms T and defined on generated T-tribes can be represented as integrals of Markov kernels. It is clear that this holds for $T = T_\infty$, too. However, in this particular case the condition that $\mathscr{T}$ must be generated can be dropped. This is a consequence of the results of [8] showing that for finite T_∞-measures on T_∞-tribes nonnegativity implies *continuity* in the sense of

$$(\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T} \text{ and } \lim_{n\to\infty} A_n = A) \Rightarrow \lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(A), \tag{22}$$

and that nonnegativity is equivalent to monotonicity. The following Representation Theorem of T_∞-measures is essentially Theorem 2.6 (c) of Butnariu [10]. We present it here with an alternative proof.

4.1 Theorem. *If $\mathscr{T}$ is a T_∞-tribe and if $\mathbf{m}$ is a finite nonnegative T_∞-measure on $\mathscr{T}$ then there exists a unique measure $\check{\mathbf{m}}$ on $\mathscr{T}^\vee$, namely the restriction of $\mathbf{m}$ to $\mathscr{T}^\vee$, such that for any A in $\mathscr{T}$*

$$\mathbf{m}(A) = \int_X A(x)\, d\check{\mathbf{m}}(x). \tag{23}$$

Proof. It is clear that if (23) holds then $\check{\mathbf{m}}$ must be the restriction of $\mathbf{m}$ to $\mathscr{T}^\vee$.

Claim 1. If $A \in \mathscr{T}$, $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$ then the set

$$A_{\alpha,\beta} = \{x \in X: \alpha < A(x) \leqslant \beta\}$$

belongs to $\mathscr{T}^\vee$, $A \cdot A_{\alpha,\beta}$ belongs to $\mathscr{T}$, and

$$\alpha \cdot \mathbf{m}(A_{\alpha,\beta}) \leqslant \mathbf{m}(A \cdot A_{\alpha,\beta})$$

The first assertion follows from Theorem 2.1. The second results from the fact that for any $M \in \mathcal{T}^{\vee}$ we have $A \cdot M = A \mathbf{T}_{\infty} M \in \mathcal{T}$. Now, in order to prove (24) it is sufficient to show that it holds for any A in $\mathcal{T}$ and for any α, β in $[0, 1]$, where $\alpha < \beta$ and $\alpha = a$ with a of the form (8). Indeed, if (24) is true in this case, then for any $0 < \alpha < \beta \leqslant 1$ we can find a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ which is nonnegative, nondecreasing, and convergent to α, and such that each α_n is of the form (8). Using the continuity of $\mathbf{m}$ we get

$$\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) = \lim_{n \to \infty} \alpha_n \cdot \check{\mathbf{m}}(A_{\alpha_n,\beta}) \leqslant \lim_{n \to \infty} \mathbf{m}(A \cdot A_{\alpha_n,\beta}) = \mathbf{m}(A \cdot A_{\alpha,\beta}),$$

since in our setting we have $A_{\alpha_n,\beta} \downarrow A_{\alpha,\beta}$. Let us assume $\alpha = a$, where a is of the form (8). If $a = 0$ there is nothing to prove. Suppose $a > 0$. In this situation we proceed by induction upon the number k involved in (8). If $k = 1$, then $\alpha = 0$ or $\alpha = \frac{1}{2}$. In the first case (24) clearly holds. In the second case we have $A_{\alpha,\beta} = (A \cdot A_{\alpha,\beta}) \mathbf{S}_{\infty} (A \cdot A_{\alpha,\beta})$, and this implies

$$\begin{aligned}\check{\mathbf{m}}(A_{\alpha,\beta}) &= \mathbf{m}(A_{\alpha,\beta}) \\ &= \mathbf{m}(A \cdot A_{\alpha,\beta}) + \mathbf{m}(A \cdot A_{\alpha,\beta}) - \mathbf{m}[(A \cdot A_{\alpha,\beta}) \mathbf{T}_{\infty} (A \cdot A_{\alpha,\beta})] \\ &\leqslant 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta})\end{aligned}$$

which is exactly (24) with $\alpha = \frac{1}{2}$. Suppose that (24) holds for all $A \in \mathcal{T}$ and for all α, β in $[0, 1]$ with $\alpha < \beta$ and $\alpha = a$, where a is of the form (8) with $k \leqslant m$. Consider

$$\alpha = \sum_{i=1}^{m+1} \frac{a(i)}{2^i} \quad \text{with} \quad a(i) \in \{0, 1\}, \quad (1 \leqslant i \leqslant m+1), \quad \text{and} \quad a(m+1) \neq 0.$$

Then we get $\alpha = \theta/2$ with

$$\theta = \sum_{i=1}^{m+1} \frac{a(i)}{2^{i-1}}. \tag{25}$$

Case 1. Assume $\theta < 1$ and $\beta < \frac{1}{2}$. Then $A_{\alpha,\beta} = (A \mathbf{S}_{\infty} A)_{\theta,2\beta}$. Using the inductive assumption for the set $(A \mathbf{S}_{\infty} A)_{\theta,2\beta}$ (this is possible since the sum in (25) has at most m nonzero terms in our case) we get $\theta \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) \leqslant \mathbf{m}[(A \mathbf{S}_{\infty} A) \cdot A_{\alpha,\beta}]$ and, observing that $(A \mathbf{S}_{\infty} A) \cdot A_{\alpha,\beta} \leqslant (A \cdot A_{\alpha,\beta}) \mathbf{S}_{\infty} (A \cdot A_{\alpha,\beta})$, we obtain

$$\begin{aligned}\theta \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) &\leqslant \mathbf{m}[(A \mathbf{S}_{\infty} A) \cdot A_{\alpha,\beta}] \leqslant \mathbf{m}[(A \cdot A_{\alpha,\beta}) \mathbf{S}_{\infty} (A \cdot A_{\alpha,\beta})] \\ &= 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta}) - \mathbf{m}[(A \cdot A_{\alpha,\beta}) \mathbf{T}_{\infty} (A \cdot A_{\alpha,\beta})] \leqslant 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta})\end{aligned}$$

by the monotonicity and additivity of $\mathbf{m}$ (see [8]). This implies (24) in this specific case.

Case 2. Assume $\theta < 1$ and $\beta = \frac{1}{2}$. Let $\{\gamma_n\}_{n \in \mathbb{N}}$ be an increasing sequence in $]\alpha, \beta[$ which converges to β. By Case 1 we have $\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\gamma_n}) \leqslant \mathbf{m}(A \cdot A_{\alpha,\gamma_n})$ for each $n \in \mathbb{N}$. Since $\mathbf{m}$ is continuous (cf. [8]) and because of

$$A_{\alpha,\gamma_n} \uparrow \mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}$$

we get

$$\alpha \cdot \check{\mathbf{m}}\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right) \leqslant \mathbf{m}\left[A \cdot \left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\right],$$

where

$$\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right) \cup \{x \in X;\, A(x) = \beta\} = A_{\alpha,\beta}.$$

Since the sets forming the union are disjoint one may write

$$\begin{aligned}\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) &= \alpha \cdot \check{\mathbf{m}}\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right) + \alpha \cdot \check{\mathbf{m}}(\{A = \beta\}) \\ &\leqslant \mathbf{m}\left[A\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\right] + \alpha \cdot \check{\mathbf{m}}(\{A = \beta\}).\end{aligned} \tag{26}$$

Since for $\beta = \frac{1}{2}$ the fuzzy subsets B_1 and B_2 defined by

$$\begin{aligned}B_1(x) = B_2(x) &= 0 \quad \text{if} \quad A(x) \neq \beta \\ &= \beta \quad \text{if} \quad A(x) = \beta\end{aligned}$$

are T_{∞}-disjoint and elements of $\mathcal{T}$, we have $\mathbf{m}(B_1 \mathbf{S}_{\infty} B_2) = 2 \cdot \mathbf{m}(\beta \cdot \{A = \beta\})$ (cf. Remark 3.1 (i)) and $B_1 \mathbf{S}_{\infty} B_2 = \{A = \beta\}$. Hence,

$$\mathbf{m}(\beta \cdot \{A = \beta\}) = \beta \cdot \check{\mathbf{m}}(\{A = \beta\}) \geqslant \alpha \cdot \mathbf{m}(\{A = \beta\}).$$

Combining this with (26) and using the additivity of $\mathbf{m}$, we deduce

$$\begin{aligned}\alpha \cdot \mathbf{m}(A_{\alpha,\beta}) &\leqslant \mathbf{m}\left[A \cdot \left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\right] + \mathbf{m}(\beta \cdot \{A = \beta\}) \\ &= \mathbf{m}\left[A \cdot \left(\{A = \beta\} \cup \left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\right)\right] = \mathbf{m}(A \cdot A_{\alpha,\beta}).\end{aligned}$$

This proves (24) in this case.

Case 3. Assume $\theta < 1$ and $\beta > \frac{1}{2}$. Then $A \cdot A_{\alpha,\beta} = (A \cdot A_{\alpha,1/2})\, \mathbf{S}_\alpha\, (A \cdot A_{1/2,\beta})$, where the fuzzy subsets on the right-hand side are T_α-disjoint. Hence, according to Remark 3.1 (i) we have $\mathbf{m}(A \cdot A_{\alpha,\beta}) = \mathbf{m}(A \cdot A_{\alpha,1/2}) + \mathbf{m}(A \cdot A_{1/2,\beta})$. The first term on the right-hand side falls under the circumstances of Case 2, and the second one falls under the circumstances of the inductive assumption. Hence

$$\check{\mathbf{m}}(A_{\alpha,\beta}) \geqslant \alpha \cdot \check{\mathbf{m}}(A_{1/2,\beta}) + \tfrac{1}{2} \cdot \check{\mathbf{m}}(A_{1/2,\beta})$$
$$\geqslant \alpha \cdot [\check{\mathbf{m}}(A_{\alpha,1/2}) + \check{\mathbf{m}}(A_{1/2,\beta})] = \alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}),$$

showing that (24) holds in this case too.

Case 4. Assume finally $\theta \geqslant 1$. In this case α can be written as

$$\alpha = (\varepsilon + 1)/2, \qquad \text{where} \quad \varepsilon = \sum_{i=1}^{m} \frac{a(i)}{2^i} \in [0, 1].$$

We also have $A_{\alpha,\beta} = [(A' \mathbf{S}_\alpha A')']_{\varepsilon,2\delta}$ with $\delta = \beta - \frac{1}{2}$. Using the inductive assumption for the set $[(A' \mathbf{S}_\alpha A')']_{\varepsilon,2\delta}$, we obtain

$$\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) = \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \frac{\varepsilon}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta})$$
$$= \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \frac{\varepsilon}{2} \cdot \check{\mathbf{m}}([(A' \mathbf{S}_\alpha A')']_{\varepsilon,2\delta})$$
$$\leqslant \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \frac{1}{2} \cdot \check{\mathbf{m}}([(A' \mathbf{S}_\alpha A')']_{\varepsilon,2\delta}). \tag{27}$$

Observe that $A_{\alpha,\beta}[(A' \mathbf{S}_\alpha A')'] = D\, \mathbf{T}_\alpha\, C'$ with $D = A_{\alpha,\beta}$ and $C = A_{\alpha,\beta}(A' \mathbf{S}_\alpha A')$. According to Remark 3.1 (i) we have that

$$(E, F \in \mathscr{T} \text{ and } E \geqslant F) \Rightarrow \mathbf{m}(E\, \mathbf{T}_\alpha\, F') = \mathbf{m}(E) - \mathbf{m}(F). \tag{28}$$

Since we clearly have $D \geqslant C$, (27) combined with (28) gives

$$\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) \leqslant \tfrac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \tfrac{1}{2} \cdot [\mathbf{m}(D) - \mathbf{m}(C)] = \check{\mathbf{m}}(A_{\alpha,\beta}) - \tfrac{1}{2} \cdot \mathbf{m}(C). \tag{29}$$

Now, taking into account that $\theta \geqslant 1$ we deduce $C = (A' \cdot A_{\alpha,\beta})\, \mathbf{S}_\alpha\, (A' \cdot A_{\alpha,\beta})$ and $(A' \cdot A_{\alpha,\beta})\, \mathbf{T}_\alpha\, (A' \cdot A_{\alpha,\beta}) = \varnothing$. Thus $\mathbf{m}(C) = 2 \cdot \mathbf{m}(A' \cdot A_{\alpha,\beta})$ by the additivity of $\mathbf{m}$. Since $A' \cdot A_{\alpha,\beta} = A_{\alpha,\beta}\, \mathbf{T}_\alpha\, (A \cdot A_{\alpha,\beta})'$, we get $\mathbf{m}(C) = 2 \cdot [\mathbf{m}(A_{\alpha,\beta}) - \mathbf{m}(A \cdot A_{\alpha,\beta})]$ because of (28). Substituting this in (29) we obtain (24), and Claim 1 is completely proved.

Claim 2. If $A \in \mathscr{T}$, if $0 \leqslant \alpha < \beta \leqslant 1$, and if we put $\bar{A}_{\alpha,\beta} = \{x \in X; \alpha \leqslant A(x) < \beta\}$, then $\bar{A}_{\alpha,\beta} \in \mathscr{T}^{\vee}$, $A \cdot \bar{A}_{\alpha,\beta} \in \mathscr{T}$, and

$$\mathbf{m}(A \cdot \bar{A}_{\alpha,\beta}) \leqslant \beta \cdot \check{\mathbf{m}}(\bar{A}_{\alpha,\beta}). \tag{30}$$

The first two assertions are obvious. To prove (30) observe that

$$A_{\alpha,\beta} = \bar{A}'_{\beta',\alpha'}, \qquad \text{where} \quad \alpha' = 1 - \alpha \quad \text{and} \quad \beta' = 1 - \beta. \tag{31}$$

Then

$$\mathbf{m}[(A \cdot A_{\alpha,\beta})'] = \check{\mathbf{m}}(X) - \mathbf{m}(A \cdot \bar{A}_{\alpha,\beta}) = \check{\mathbf{m}}(\bar{A}_{\alpha,\beta}) + \check{\mathbf{m}}([\bar{A}_{\alpha,\beta}]') - \mathbf{m}(A \cdot \bar{A}_{\alpha,\beta})$$
$$= \mathbf{m}(A' \cdot \bar{A}_{\alpha,\beta}) + \check{\mathbf{m}}([\bar{A}_{\alpha,\beta}]') = \mathbf{m}(A' \cdot A'_{\beta',\alpha'}) + \check{\mathbf{m}}([\bar{A}_{\alpha,\beta}]')$$
$$\geqslant \beta' \cdot \check{\mathbf{m}}(X) + \check{\mathbf{m}}([\bar{A}_{\alpha,\beta}]') = \check{\mathbf{m}}(X) - \beta \cdot \check{\mathbf{m}}(\bar{A}_{\alpha,\beta}),$$

where the inequality and the last equality are consequences of (31) and Claim 1, respectively. This implies (30), and Claim 2 is proved.

In order to complete the proof of our theorem let A be in $\mathscr{T}$. Denote

$$G_{n,i} = \{x \in X; A(x) = 0\} \quad \text{if} \quad i = 0,$$
$$= A_{(i-1)/2^n,\, i/2^n} \quad \text{if} \quad 1 \leqslant i \leqslant 2^n,$$

and

$$H_{n,i} = \bar{A}_{i/2^n,\, (i+1)/2^n} \quad \text{if} \quad 1 \leqslant i < 2^n,$$
$$= \{x \in X; A(x) = 1\} \quad \text{if} \quad i = 2^n.$$

From Claims 1 and 2 it follows that the step functions

$$s_n = \sum_{i=1}^{m} \frac{i-1}{m} \cdot G_{n,i} \qquad \text{and} \qquad t_n = \sum_{i=0}^{m-1} \frac{i+1}{m} \cdot H_{n,i} + H_{n,m},$$

where $m = 2^n$ are $\mathscr{T}^{\vee}$-measurable. It is clear that $s_n \uparrow A$ and $t_n \downarrow A$. Taking into account (24) and (30), we deduce

$$\int_X s_n \, d\check{\mathbf{m}} = \sum_{i=1}^{m} \frac{i-1}{m} \cdot \check{\mathbf{m}}(G_{n,i}) \leqslant \sum_{i=0}^{m-1} \mathbf{m}(A \cdot G_{n,i}) = \mathbf{m}(A)$$

and

$$\int_X t_n \, d\check{\mathbf{m}} \geqslant \sum_{i=1}^{m-1} \frac{i+1}{m} \cdot \check{\mathbf{m}}(H_{n,i}) \geqslant \sum_{i=0}^{m} \mathbf{m}(A \cdot H_{n,i}) = \mathbf{m}(A).$$

Taking the limit $n \to \infty$ in these relations we obtain (23), therefore completing the proof of the theorem. ∎

4.2 *Remark*. (i) Comparing the results of [8] with the First Representation Theorem, one can easily see that on a generated T-tribe, with T being a fundamental t-norm, the T_α-measures are exactly those

T-measures for which the corresponding Markov kernel K in the representation (21) is given by

$$K(x, [\alpha, \beta[) = \beta - \alpha \qquad (x \in X).$$

(ii) For fundamental t-norms T_s with $s \in]0, \infty[$ and generated T_s-tribes, one can also specify the form of the Markov kernel involved in (21). To be precise, it was shown in [24] that if T_s is a fundamental t-norm with $s \in]0, \infty[$, and if the T_s-tribe $\mathcal{T}$ is generated, then for any monotone finite T_s-measure $\mathbf{m}$ on $\mathcal{T}$ there exists a unique measure $\check{\mathbf{m}}$ on $\mathcal{T}^\vee$, namely the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$, and an $\check{\mathbf{m}}$–a.e. uniquely determined $\mathcal{T}^\vee$-measurable function $f: X \to [0, 1]$ such that for all $A \in \mathcal{T}$

$$\mathbf{m}(A) = \int_{\{A>0\}} [f + (1-f) \cdot A] \, d\check{\mathbf{m}}. \tag{32}$$

(iii) Theorems 4.1 and 1.5 imply that T_∞-measures on T_s-tribes with $s \in]0, \infty[$ are also T_s-measures. However, not each T_s-measure is necessarily a T_∞-measure, even on generated T_s-tribes. It was shown in [21] that it is necessary and sufficient for a monotone finite T_s-measure $\mathbf{m}$ with $s \in]0, \infty[$ to be a T_∞-measure, that the following condition be satisfied:

$$(\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{T} \text{ and } A_n \downarrow \varnothing) \Rightarrow \lim_{n\to\infty} \mathbf{m}(A_n) = 0. \tag{33}$$

5. Decompositions of T-Measures

According to (32), finite monotone measures $\mathbf{m}$, based on fundamental t-norms T_s with $s \in]0, \infty]$, on generated tribes differ from T_∞-measures (i.e., from integrals, according to Theorem 4.2) by functions of the form $A \to \int_{\{A>0\}} f \, d\check{\mathbf{m}}$, which are also monotone finite T_s-measures. The question is now how much a T_s-measure, defined on a nongenerated T_s-tribe, differs from a T_∞-measure (i.e., an integral).

5.1 Proposition. *Let T_s be a fundamental t-norm with $s \in [0, \infty]$. If $\mathcal{T}$ is a T_s-tribe and if $\mathbf{m}$ is a finite monotone T_s-measure on $\mathcal{T}$, then there exists a unique pair $(\mathbf{m}_\infty, \mathbf{m}_s)$ of functions from $\mathcal{T}$ to $\mathbb{R}_+$ such that:*

(a) $\mathbf{m}_\infty$ *is a T_∞-measure on $\mathcal{T}$;*

(b) $\mathbf{m}_s$ *is a T_s-measure on $\mathcal{T}$;*

(c) $\mathbf{m} = \mathbf{m}_\infty + \mathbf{m}_s$;

(d) $\mathbf{m}_\infty$ *is "maximal" in the sense that if $\mathbf{m}': \mathcal{T} \to \mathbb{R}_+$ is another T_∞-measure such that $\mathbf{m} - \mathbf{m}'$ is monotone, then $\mathbf{m}' \leqslant \mathbf{m}_\infty$.*

Moreover, the functions $\mathbf{m}_\infty$ and $\mathbf{m}_s$ have the property that there exists a unique measure $\check{\mathbf{m}}$ on $\mathcal{T}^\vee$, namely the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$, and an $\check{\mathbf{m}}$-a.e. unique $\mathcal{T}^\vee$-measurable function $f: X \to [0, 1]$ such that for all $A \in \mathcal{T}$

$$\mathbf{m}_\infty(A) = \int_X (1-f) \cdot A \, d\check{\mathbf{m}}, \tag{34}$$

and for all $M \in \mathcal{T}^\vee$

$$\mathbf{m}_s(M) = \int_M f \, d\check{\mathbf{m}}. \tag{35}$$

Proof. Denote by $\mathcal{M}_\infty$ the family of all T_∞-measures $\mathbf{p}: \mathcal{T} \to \mathbb{R}_+$ such that $\mathbf{m} - \mathbf{p}$ is monotone. The family $\mathcal{M}_\infty$ is nonempty since it contains the zero T_∞-measure on $\mathcal{T}$. The family $\mathcal{M}_\infty$ is provided with the partial order

$$\mathbf{p} \leqslant \mathbf{p}' \Leftrightarrow (\forall A \in \mathcal{T}: \mathbf{p}(A) \leqslant \mathbf{p}'(A)). \tag{36}$$

If $\{\mathbf{p}_\alpha\}_{\alpha\in J}$ is a chain in $\mathcal{M}_\infty$, then the function $\mathbf{p}: \mathcal{T} \to [0, \infty]$ defined by

$$\mathbf{p}(A) = \sup_{\alpha\in J} \mathbf{p}_\alpha(A) \qquad (A \in \mathcal{T}) \tag{37}$$

is a T_∞-valuation. Indeed, $\mathbf{p}(\varnothing) = 0$ clearly holds, and for any A and B in $\mathcal{T}$ we have

$$\begin{aligned}\mathbf{p}(A \, \mathbf{S}_\infty \, B) + \mathbf{p}(A \, \mathbf{T}_\infty \, B) &= \sup_{\alpha\in J} \mathbf{p}_\alpha(A \, \mathbf{S}_\infty \, B) + \sup_{\alpha\in J} \mathbf{p}_\alpha(A \, \mathbf{T}_\infty \, B) \\ &= \sup_{\alpha\in J} [\mathbf{p}_\alpha(A \, \mathbf{S}_\infty \, B) + \mathbf{p}_\alpha(A \, \mathbf{T}_\infty \, B)] \\ &= \sup_{\alpha\in J} [\mathbf{p}_\alpha(A) + \mathbf{p}_\alpha(B)] = \mathbf{p}(A) + \mathbf{p}(B),\end{aligned}$$

where the second and the last equality hold because of the monotonicity of $\{\mathbf{p}_\alpha(C)\}_{\alpha\in J}$ for every C in $\mathcal{T}$. It is clear that $\mathbf{p}$ is monotone. Hence, $0 \leqslant \mathbf{p}(A) \leqslant \mathbf{m}(A) \leqslant \mathbf{m}(X)$ for all A in $\mathcal{T}$, implying that $\mathbf{p}$ is also finite. If $\{A_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence in $\mathcal{T}$, then

$$\begin{aligned}\lim_{n\to\infty} \mathbf{p}(A_n) &= \sup_{n\in\mathbb{N}} \mathbf{p}(A_n) = \sup_{n\in\mathbb{N}} (\sup_{\alpha\in J} \mathbf{p}_\alpha(A_n)) \\ &= \sup_{\alpha\in J} (\sup_{n\in\mathbb{N}} \mathbf{p}_\alpha(A_n)) = \sup_{\alpha\in J} (\lim_{n\to\infty} \mathbf{p}_\alpha(A_n)) = \mathbf{p}(\lim_{n\to\infty} A_n),\end{aligned}$$

showing that $\mathbf{p}$ is left-continuous. Hence, $\mathbf{p}$ is a finite nonnegative T_∞-measure on $\mathcal{T}$. It is easy to see that $\mathbf{m} - \mathbf{p}$ is also monotone, since

$\mathbf{m}-\mathbf{p}_\alpha$ is monotone for each $\alpha \in J$. Thus we have $\mathbf{p} \in \mathscr{M}_\alpha$. In other words, each nondecreasing chain in $\mathscr{M}_\alpha$ has an upper bound in $\mathscr{M}_\alpha$ and, by Zorn's lemma, $\mathscr{M}_\alpha$ has a maximal element denoted $\mathbf{m}_\alpha$. Since $\mathbf{m}_\alpha$ is a T_α-measure it is a T_s-measure, too, and so is the difference $\mathbf{m}_s = \mathbf{m} - \mathbf{m}_\alpha$. By the definition of $\mathbf{m}_\alpha$, $\mathbf{m}_s$ is monotone and finite, and for the pair $(\mathbf{m}_\alpha, \mathbf{m}_s)$ the conditions (a), (b), and (c) are satisfied. By the maximality of $\mathbf{m}_\alpha$ in $\mathscr{M}_\alpha$ the condition (d) is also satisfied; (a) and (d) imply uniqueness. It remains to show that there exists an f such that (34) and (35) hold. To this end observe that $\mathscr{T}$ is a T_α-tribe (cf. Theorem 1.5) and that $\mathbf{m}_\alpha$ is a T_α-measure on $\mathscr{T}$. Hence, according to Theorem 4.1, $\mathbf{m}_\alpha$ can be written as

$$\mathbf{m}_\alpha = \int_X A \, d\check{\mathbf{m}}_\alpha \qquad (A \in \mathscr{T}), \tag{38}$$

where $\check{\mathbf{m}}_\alpha$ is the restriction of $\mathbf{m}_\alpha$ to $\mathscr{T}^\vee$. Since $\mathbf{m}_\alpha \leqslant \mathbf{m}$, it follows that $\check{\mathbf{m}}_\alpha$ is absolutely continuous with respect to the restriction $\check{\mathbf{m}}$ of $\mathbf{m}$ to $\mathscr{T}^\vee$, and the Radon–Nikodym derivative $d\check{\mathbf{m}}_\alpha/d\check{\mathbf{m}}$ is $\check{\mathbf{m}}$-a.e. equal to a function g mapping X into $[0, 1]$. Putting $f = 1-g$, then f is a $\mathscr{T}^\vee$-measurable function with values in $[0, 1]$, and (34) is satisfied due to (38). Now, taking into account the definition of $\mathbf{m}_s$, we can write

$$\mathbf{m}_s(M) = \check{\mathbf{m}}(M) - \check{\mathbf{m}}_\alpha(M) = \check{\mathbf{m}}(M) - \int_M (1-f)\, d\check{\mathbf{m}} = \int_M f \, d\check{\mathbf{m}}$$

for all $M \in \mathscr{T}^\vee$, and therefore (35) also holds. It also follows that if (34) and (35) hold then $\check{\mathbf{m}}$ must be equal to the restriction of $\mathbf{m}$ to $\mathscr{T}^\vee$. ∎

5.2 *Remark.* (i) The component $\mathbf{m}_s$ of a T_s-measure in Proposition 8.2 is a "pure" T_s-measure in the sense that it has a zero T_α-component: $(\mathbf{m}_s)_\alpha = 0$. In fact, assuming the contrary would contradict the maximality of $\mathbf{m}_\alpha$.

(ii) If in Proposition 5.1 one assumes the T_s-tribe $\mathscr{T}$ to be generated, then (34) holds for any M in $\mathscr{T}$ (and not only for M in the σ-algebra $\mathscr{T}^\vee$). Indeed, this follows comparing (32) with (34) and (35), and keeping in mind that the function f in (32) must be $\check{\mathbf{m}}$-a.e. unique.

Consider a T_s-tribe $\mathscr{T}$, where T_s is a fundamental t-norm with $s \in]0, \infty[$. If $\check{\mathbf{p}}$ is any measure on $\mathscr{T}^\vee$, and if g and h are any $\mathscr{T}^\vee$-measurable functions from X to $[0, \infty]$, then the function $\mathbf{m}: \mathscr{T} \to [0, +\infty]$ defined by

$$\mathbf{m}(A) = \int_{\{A>0\}} (g + h \cdot A)\, d\check{\mathbf{p}} \tag{39}$$

is a monotone T_s-measure on $\mathscr{T}$. Monotonicity and $\mathbf{m}(\varnothing) = 0$ are obvious; the left continuity of $\mathbf{m}$ follows from the Lebesgue monotone convergence theorem, taking into account that for a nondecreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathscr{T}$ whose pointwise limit is A, one has $\bigcup_{n=1}^\infty \{A_n > 0\} = \{A > 0\}$; the T_s-additivity of $\mathbf{m}$ is shown as

$$\begin{aligned}
\mathbf{m}(A\, \mathbf{T}_s\, B) + \mathbf{m}(A\, \mathbf{S}_s\, B) &= \int_{\{A>0\}\cap\{B>0\}} [g + h\cdot(A\, \mathbf{T}_s\, B)]\, d\check{\mathbf{p}} \\
&\quad + \int_{\{A>0\}\cup\{B>0\}} [g + h\cdot(A\, \mathbf{S}_s\, B)]\, d\check{\mathbf{p}} \\
&= \int_{\{A>0\}\cap\{B>0\}} [2\cdot g + h\cdot(A\, \mathbf{T}_s\, B + A\, \mathbf{S}_s\, B)]\, d\check{\mathbf{p}} \\
&\quad + \int_{\{A>0\}\cap\{B=0\}} [g + h(A\, \mathbf{S}_s\, B)]\, d\check{\mathbf{p}} \\
&\quad + \int_{\{A=0\}\cap\{B>0\}} [g + h(A\, \mathbf{S}_s\, B)]\, d\check{\mathbf{p}} \\
&= \int_{\{A>0\}\cap\{B>0\}} [2\cdot g + h\cdot(A + B)]\, d\check{\mathbf{p}} \\
&\quad + \int_{\{A>0\}\cap\{B=0\}} (g + h\cdot A)\, d\check{\mathbf{p}} \\
&\quad + \int_{\{A=0\}\cap\{B>0\}} (g + h\cdot B)\, d\check{\mathbf{p}} \\
&= \int_{\{A>0\}} (g + h\cdot A)\, d\check{\mathbf{p}} + \int_{\{B>0\}} (g + h\cdot B)\, d\check{\mathbf{p}} \\
&= \mathbf{m}(A) + \mathbf{m}(B).
\end{aligned}$$

A T_s-measure $\mathbf{m}$ on $\mathscr{T}$, which can be represented in the form (39) by some nonnegative measure $\check{\mathbf{p}}$ on $\mathscr{T}^\vee$ and by some pair (g, h) of nonnegative $\mathscr{T}^\vee$-measurable functions on X, is said to be *generated (by* $\check{\mathbf{p}}$, g *and* h*)*.

It follows from [24] (see Remark 4.2) that if $\mathscr{T}$ is generated then all finite monotone T_s-measures with $s \in]0, \infty[$ on $\mathscr{T}$ are generated. From Theorem 4.2 we already know that the T_α-measures on $\mathscr{T}$ are generated, even when $\mathscr{T}$ is not generated. Thus, it is natural to ask whether, in general, T_s-measures on T_s-tribes are always generated. In order to answer this question, we define a T_s-measure $\mathbf{m}$ on the T_s-tribe $\mathscr{T}$ to be *monotonically irreducible*, if it is monotone and if there is no nonidentically zero generated T_s-measure $\mathbf{q}$ on $\mathscr{T}$ such that $\mathbf{m} - \mathbf{q}$ is monotone on $\mathscr{T}$.

Now, it is obvious that a T_s-measure $\mathbf{m}$ on $\mathscr{T}$ is generated if and only if it can be extended to a T_s-measure on the generated T_s-tribe $(\mathscr{T}^{\vee})^{\wedge}$ (since, if $\mathbf{m}$ is generated then (39) defines $\mathbf{m}$ on $(\mathscr{T}^{\vee})^{\wedge}$, the converse following from Theorem 5.1). By contrast, monotonically irreducible T_s-measures, except for the trivial one, are not generated and, hence, they cannot be extended to $(\mathscr{T}^{\vee})^{\wedge}$.

5.3 THEOREM. *If T_s is a fundamental t-norm with $s \in]0, \infty]$, if $\mathscr{T}$ is a T_s-tribe, and if $\mathbf{m}$ is a finite monotone T_s-measure on $\mathscr{T}$, then $\mathbf{m}$ can be uniquely decomposed in a monotonically irreducible and a generated T_s-measure; that is there exist a unique monotonically irreducible T_s-measure $\mathbf{m}^*$ on $\mathscr{T}$, a measure $\check{\mathbf{m}}$ on $\mathscr{T}^{\vee}$ (which is exactly the restriction of $\mathbf{m}$ to $\mathscr{T}^{\vee}$), and two $\check{\mathbf{m}}$-a.e. uniquely determined $\mathscr{T}^{\vee}$-measurable functions $g, h: X \to [0, 1]$, such that for all $A \in \mathscr{T}^{\vee}$ one has*

$$\mathbf{m}(A) - \mathbf{m}^*(A) = \int_{\{A>0\}} (g + h \cdot A)\, d\check{\mathbf{m}}. \tag{40}$$

Proof. If $s = +\infty$, then the result follows from Theorem 4.1 putting $g(x) = 0$, $h(x) = 1$ for $x \in X$ and $\mathbf{m}^* = 0$. Assume $s \in]0, \infty[$. In this case the theorem is proved in several steps.

Claim 1. If $\mathbf{p}$ is a finite monotone T_s-measure on $\mathscr{T}$, then there exists a unique finite monotone T_s-measure $|\mathbf{p}|$ on the generated T_s-tribe $(\mathscr{T}^{\vee})^{\wedge}$ which is *monotonically maximal* in the sense that for any T_s-measure $\mathbf{p}'$ on $(\mathscr{T}^{\vee})^{\wedge}$, for which $\mathbf{p} - \mathbf{p}'$ is monotone on $\mathscr{T}$, the difference $|\mathbf{p}| - \mathbf{p}'$ is also monotone on $(\mathscr{T}^{\vee})^{\wedge}$.

In order to prove that, denote by $\mathscr{M}(\mathbf{p})$ the family of all T_s-measures $\mathbf{q}$ on $(\mathscr{T}^{\vee})^{\wedge}$ such that $\mathbf{p} - \mathbf{q}$ is monotone on $\mathscr{T}$. This family is partially ordered by the *dominance relation* defined by

$$\mathbf{q} \gg \mathbf{q}' \Leftrightarrow \mathbf{q} - \mathbf{q}' \text{ is monotone on } (\mathscr{T}^{\vee})^{\wedge}. \tag{41}$$

Let $\{\mathbf{q}_\alpha\}_{\alpha \in J}$ be a chain in $\mathscr{M}(\mathbf{p})$ with respect to the partial order (41) and define

$$\mathbf{q}(A) = \sup_{\alpha \in J} \mathbf{q}_\alpha(A) \qquad (A \in (\mathscr{T}^{\vee})^{\wedge}).$$

Similarly as in the proof of Proposition 5.1, one can prove that $\mathbf{q}$ is a monotone finite T_s-measure on $(\mathscr{T}^{\vee})^{\wedge}$. For $A, B \in \mathscr{T}$ with $A \leqslant B$ we have that

$$\mathbf{p}(A) - \mathbf{q}(A) = \inf_{\alpha \in J} [\mathbf{p}(A) - \mathbf{q}_\alpha(A)] \leqslant \inf_{\alpha \in J} [\mathbf{p}(B) - \mathbf{q}_\alpha(B)] = \mathbf{p}(B) - \mathbf{q}(B),$$

i.e., $\mathbf{p} - \mathbf{q}$ is monotone on $\mathscr{T}$. Clearly, $\mathbf{q} \gg \mathbf{q}_\alpha$ $(\alpha \in J)$. Hence, each chain in $\mathscr{M}(\mathbf{p})$ has an upper bound in $\mathscr{M}(\mathbf{p})$ and, according to Zorn's lemma, $\mathscr{M}(\mathbf{p})$ has a maximal element denoted $|\mathbf{p}|$. It is obvious that $|\mathbf{p}|$ is the only T_s-measure with this property. Hence, Claim 1 is proved.

Let $|\mathbf{m}|$ be the T_s-measure existing by Claim 1 for $\mathbf{p} = \mathbf{m}$. Since $|\mathbf{m}|$ is defined on the generated tribe $(\mathscr{T}^{\vee})^{\wedge}$, it can be represented according to Theorem 5.1 by

$$|\mathbf{m}|(A) = \int_{\{A>0\}} [w + (1 - w) \cdot A]\, d|\mathbf{m}|^{\vee},$$

where $|\mathbf{m}|^{\vee}$ is the restriction of $|\mathbf{m}|$ to $((\mathscr{T}^{\vee})^{\wedge})^{\vee} = \mathscr{T}^{\vee}$, and w is a $\mathscr{T}^{\vee}$-measurable function from X to $[0, 1]$. As observed above (Remark 5.2 (ii)), the unique decomposition $(|\mathbf{m}|_\infty, |\mathbf{m}|_s)$ of $|\mathbf{m}|$ according to Proposition 5.1 is given by

$$|\mathbf{m}|_\infty (A) = \int_X (1 - w) \cdot A\, d|\mathbf{m}|^{\vee}$$

and

$$|\mathbf{m}|_s (A) = \int_{\{A>0\}} w\, d|\mathbf{m}|^{\vee}. \tag{42}$$

Furthermore, let $(\mathbf{m}_\infty, \mathbf{m}_s)$ be the unique decomposition pair provided by Proposition 5.1 for $\mathbf{m}$, and let $f: X \to [0, 1]$ be the function satisfying (34) and (35).

Claim 2. For all $A \in (\mathscr{T}^{\vee})^{\wedge}$ we have

$$|\mathbf{m}|_\infty (A) = \int_X (1 - f) \cdot A\, d\check{\mathbf{m}}. \tag{43}$$

To prove this consider $\mathbf{q}_\infty : (\mathscr{T}^{\vee})^{\wedge} \to \mathbb{R}$ defined by

$$\mathbf{q}_\infty(A) = \int_X (1 - f) \cdot A\, d\check{\mathbf{m}}.$$

Obviously, this is a finite nonnegative T_∞-measure on $(\mathscr{T}^{\vee})^{\wedge}$ whose restriction to $\mathscr{T}$ coincides with $\mathbf{m}_\infty$. According to the definition of $|\mathbf{m}|$, the function $\mathbf{p}: \mathscr{T} \to \mathbb{R}$ given by $\mathbf{p}(A) = \mathbf{m}(A) - |\mathbf{m}|(A)$ is a monotone T_s-measure. Thus $\mathbf{m} - |\mathbf{m}|_\infty$ $(= \mathbf{p} + |\mathbf{m}|_s)$ is also a monotone T_s-measure on $\mathscr{T}$. Because of the maximality of $\mathbf{m}_\infty$ we have

$$\mathbf{m}_\infty(A) \geqslant |\mathbf{m}|_\infty (A) \qquad (A \in \mathscr{T}). \tag{44}$$

Taking into account Theorem 4.1 we can write for each $A \in (\mathscr{T}^\vee)^\wedge$

$$\mathbf{q}_\infty(A) = \int_X A \, d\check{\mathbf{m}}_\infty$$

and

$$|\mathbf{m}|_\infty(A) = \int_X A \, d|\mathbf{m}|_\infty^\vee,$$

where $\check{\mathbf{m}}_\infty$ and $|\mathbf{m}|_\infty^\vee$ are the restrictions of $\mathbf{m}_\infty$ and $|\mathbf{m}|_\infty$ to $\mathscr{T}^\vee$, respectively. Since by (44) we have also $\check{\mathbf{m}}_\infty \geq |\mathbf{m}|_\infty^\vee$, it follows that

$$\mathbf{q}_\infty(A) \geq |\mathbf{m}|_\infty(A) \qquad (A \in (\mathscr{T}^\vee)^\wedge) \tag{45}$$

On the other hand, we know that $\mathbf{m} - \mathbf{m}_\infty$ and $\mathbf{m} - \mathbf{q}_\infty$ coincide on $\mathscr{T}$, and that the first one is monotone. Since $|\mathbf{m}|_\infty$ is a maximal T_∞-measure dominated (in the sense of (41)) by $|\mathbf{m}|$, it follows that $|\mathbf{m}|_\infty - \mathbf{q}_\infty$ is also monotone on $(\mathscr{T}^\vee)^\wedge$, and this implies $|\mathbf{m}|_\infty \geq \mathbf{q}_\infty$ on $(\mathscr{T}^\vee)^\wedge$, which combined with (45) proves Claim 2.

According to the definition of $|\mathbf{m}|$ (see Claim 1) we have that $\check{\mathbf{m}} \geq |\mathbf{m}|^\vee$ on $\mathscr{T}^\vee$. Therefore there exists a $[0, 1]$-valued Radon–Nikodym derivative u of $|\mathbf{m}|^\vee$ with respect to $\check{\mathbf{m}}$. Putting $g = w \cdot u$ and using (42) one gets

$$|\mathbf{m}|_s(A) = \int_{\{A>0\}} g \, d\check{\mathbf{m}} \qquad (A \in (\mathscr{T}^\vee)^\wedge), \tag{46}$$

where g is a function with values in $[0, 1]$. From Claim 2 and (46) we deduce

$$\begin{aligned}\mathbf{m}(A) &= |\mathbf{m}|(A) + (\mathbf{m}_s(A) - |\mathbf{m}|_s(A)) \\ &= \mathbf{m}^*(A) + \int_{\{A>0\}} [g + (1-f) \cdot A] \, d\check{\mathbf{m}}\end{aligned} \tag{47}$$

with

$$\mathbf{m}^*(A) := \mathbf{m}_s(A) - |\mathbf{m}|_s(A) \qquad (A \in \mathscr{T}).$$

Claim 3. $\mathbf{m}^*$ is a finite monotonically irreducible T_s-measure on $\mathscr{T}$. It is clear that $\mathbf{m}^*$ is a finite monotone T_s-measure on $\mathscr{T}$. In order to show that it is also irreducible, it is sufficient to show that $|\mathbf{m}_s| = |\mathbf{m}|_s$ on $(\mathscr{T}^\vee)^\wedge$, where $|\mathbf{m}_s|$ is the T_s-measure existing for $\mathbf{m}_s$ by Claim 1. First observe that $|\mathbf{m}_s|_\infty = \mathbf{0}$ on $(\mathscr{T}^\vee)^\wedge$ because of Proposition 5.1 and of the fact, that $\mathbf{m} = \mathbf{m}_\infty + |\mathbf{m}_s|_\infty + |\mathbf{m}_s|_s$ (on $\mathscr{T}$) implies $|\mathbf{m}_s|_\infty = \mathbf{0}$ on $\mathscr{T}$ (by the maximality of $\mathbf{m}_\infty$), which in turn implies that the restriction of $|\mathbf{m}_s|_\infty$ to $\mathscr{T}^\vee$ is also identically zero. Now, observe that on $\mathscr{T}$ we have $\mathbf{m} - |\mathbf{m}_s| = \mathbf{m}_\infty + (\mathbf{m}_s - |\mathbf{m}_s|)$, where the right-hand side is a monotone T_s-measure on $\mathscr{T}$. Thus, $|\mathbf{m}| - |\mathbf{m}_s|$ must be monotone on $(\mathscr{T}^\vee)^\wedge$ (cf. Claim 1). But by Claim 2 we have $\mathbf{m}_s - |\mathbf{m}|_s = \mathbf{m} - |\mathbf{m}|$ on $\mathscr{T}$ implying that $\mathbf{m}_s - |\mathbf{m}|_s$ is monotone on $\mathscr{T}$. Hence, because of the maximality of $|\mathbf{m}_s|$ we know that $|\mathbf{m}_s| - |\mathbf{m}|_s$ is monotone on $(\mathscr{T}^\vee)^\wedge$. Suppose that $|\mathbf{m}_s| - |\mathbf{m}|_s \neq 0$. Then the function $\bar{\mathbf{m}}$, which is defined by $\bar{\mathbf{m}}(A) = |\mathbf{m}|_\infty(A) + |\mathbf{m}_s|(A)$ $(A \in (\mathscr{T}^\vee)^\wedge)$, is a monotone T_s-measure on $(\mathscr{T}^\vee)^\wedge$ which dominates $|\mathbf{m}|$ (in the sense of (41)) and satisfies $\mathbf{m} - \bar{\mathbf{m}} = \mathbf{m}_s - |\mathbf{m}_s|$ on $\mathscr{T}$ (cf. Claim 2), where the right-hand side is monotone on $\mathscr{T}$. This contradicts the maximality of the T_s-measure $|\mathbf{m}|$. Claim 3 is completely proved.

Now, putting $h := 1 - f$ in (47), and taking into account Claim 3, we obtain a representation of the form (40) for $\mathbf{m}$. Suppose that $\mathbf{m} = \mathbf{m}' + \mathbf{p}'$ is another decomposition of $\mathbf{m}$ by a monotonically irreducible T_s-measure $\mathbf{m}'$ and a generated T_s-measure $\mathbf{p}'$. The generated measure $\mathbf{p}'$ can be extended in the canonical way to $(\mathscr{T}^\vee)^\wedge$. Since $\mathbf{m} - \mathbf{p}' = \mathbf{m}'$ is monotone on $\mathscr{T}$ it follows that $|\mathbf{m}| \geq \mathbf{p}'$ (cf. Claim 1). Hence on $\mathscr{T}$ we have $\mathbf{m}' = (\mathbf{m} - |\mathbf{m}|) + (|\mathbf{m}| - \mathbf{p}')$, which shows that there exists a generated T_s-measure, namely the difference $|\mathbf{m}| - \mathbf{p}'$, which differs from $\mathbf{m}'$ by a monotone T_s-measure, namely the difference $|\mathbf{m}| - \mathbf{p}'$, which differs from $\mathbf{m}'$ by a monotone T_s-measure on $\mathscr{T}$. Thus, $\mathbf{m}'$ cannot be monotonically irreducible. Consequently the representation (40) of $\mathbf{m}$ as a sum of a monotonically irreducible and of a generated T_s-measure is unique. This also shows that the generated component of the decomposition has to be the restriction of $|\mathbf{m}|$ to $\mathscr{T}$, whose unique decomposition provided by Proposition 5.1 is given by (43) and (46). Therefore the functions g and h involved in the representation (40) are $\check{\mathbf{m}}$-a.e. uniquely determined, completing the proof of the theorem. ∎

Combining Theorem 3.5 with Theorem 5.3 we deduce the following result:

5.4 COROLLARY. *If T_s is a fundamental t-norm with $s \in]0, \infty]$, if $\mathscr{T}$ is a T_s-tribe and if $\mathbf{m}$ is a finite monotone T_s-measure on $\mathscr{T}$, then there exists a unique finite nonnegative measure $\mathbf{p}$ on $\mathscr{T}^\vee$, a $\mathbf{p}$-a.e. uniquely determined $\mathscr{T}^\vee$-Markov kernel K from $X \times \mathscr{B}_1$ to $\mathbb{R}$ and a unique monotonically irreducible T_s-measure $\mathbf{m}^*$ on $\mathscr{T}$ such that for every $A \in \mathscr{T}$*

$$\mathbf{m}(A) = \mathbf{m}^*(A) + \int_X K(x, [0, A(x)[) \, d\mathbf{p}(x).$$

ACKNOWLEDGMENTS

The authors thank Alain Chateauneuf, Ulrich Höhle, Moshe Roitman, and Klaus Schmidt for many stimulating discussions and their helpful suggestions which led to improvements of earlier versions of this work.

REFERENCES

1. J. ACZEL, "Lectures on Functional Equations and Their Applications," Academic Press, New York, 1969.
2. J. ACZEL AND C. ALSINA, Characterizations of some classes quasilinear functions with applications to triangular norms and synthesis judgments, *in* "Contributions to Production Theory" (R. Henn and D. Pallaschke, Eds.), Vol. 48, pp. 3–21, Methods of Oper. Res., Nor. Verlagsgruppe, Athenäum, Hanstein, 1984.
3. J. P. AUBIN, "Mathematical Methods of Games and Economic Theory," North-Holland, Amsterdam, 1979.
4. R. J. AUMANN AND L. S. SHAPLEY, "Values of Non-Atomic Games," Princeton University Press, Princeton, 1974.
5. D. BUTNARIU, Additive fuzzy measures and integrals. I, *J. Math. Anal. Appl.* **93** (1983), 436–452.
6. D. BUTNARIU, Decompositions and range for additive fuzzy measures, *Fuzzy Sets and Systems* **10** (1983), 135–155.
7. D. BUTNARIU, Non-atomic fuzzy measures and games, *Fuzzy Sets and Systems* **17** (1985), 39–52.
8. D. BUTNARIU, Fuzzy measurability and integrability, *J. Math. Anal. Appl.* **117** (1986), 385–410.
9. D. BUTNARIU, Additive measures and integrals. III, *J. Math. Anal. Appl.* **125** (1987), 288–303.
10. D. BUTNARIU, Values and cores for fuzzy games with infinitely many players, *Internat. J. Game Theory* **16** (1987), 43–68.
11. A. C. CLIMESCU, Sur l'équation fonctionelle de l'associativité, *Bull. École Polytech. Iassy* **1** (1946), 1–16.
12. M. DE GLASS, Fuzzy σ-fields and fuzzy measures, *J. Math. Anal. Appl.* **124** (1987), 281–289.
13. A. DVORETZKY, A. WALD, AND J. WOLFOWITZ, Relations among certain ranges of vector measures, *Pacific J. Math.* **1** (1951), 59–74.
14. S. EILENBERG, "Automata, Languages and Machines," Vol. A, Academic Press, New York, 1974.
15. M. D. FRANK, On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$, *Aequationes Math.* **19** (1979), 194–226.
16. P. R. HALMOS, "Measure Theory," Van Nostrand-Reinhold Company, New York Cincinnati/Toronto/London/Melbourne, 1950.
17. H. HAMACHER, "Über logische Aggregationen nicht-binär explizierter Entscheidungskriterien," Rita G. Fischer Verlag, Frankfurt, 1978.
18. U. HÖHLE AND E. P. KLEMENT, Plausibility measures –A general framework for possibility and fuzzy probability measures, *in* "Aspects of Vagueness" (H. J. Skala, S. Termini, and E. Trillas, Eds.), pp. 31–50, Reidel, Dordrecht, 1984.
19. C. KIMBERLING, On a class of associative functions, *Publ. Math. Debrecen* **20** (1973), 21–39.
20. E. P. KLEMENT, Fuzzy σ-algebras and fuzzy measurable functions, *Fuzzy Sets and Systems* **4** (1980), 83–93.
21. E. P. KLEMENT, Characterization of finite fuzzy measures using Markoff-kernels, *J. Math. Anal. Appl.* **75** (1980), 330–339.
22. E. P. KLEMENT, Construction of fuzzy σ-algebras using triangular norms, *J. Math. Anal. Appl.* **85** (1982), 543–565.
23. E. P. KLEMENT, A theory of fuzzy measures: A survey, *in* "Fuzzy Information and Decision Processes" (M. M. Gupta and E. Sanchez, Eds.), pp. 59–65, North-Holland, Amsterdam/New York/Oxford, 1982.

24. E. P. KLEMENT, Characterization of fuzzy measures constructed by means of triangular norms, *J. Math. Anal. Appl.* **86** (1982), 345–358.
25. E. P. KLEMENT, R. LOWEN, AND W. SCHWYHLA, Fuzzy probability measures, *Fuzzy Sets and Systems* **5** (1981), 21–30.
26. E. P. KLEMENT AND W. SCHWYHLA, Correspondence between fuzzy measures and classical measures, *Fuzzy Sets and Systems* **7** (1982), 57–70.
27. C. H. LING, Representation of the associative functions, *Publ. Math. Debrecen* **12** (1965), 189–212.
28. K. MENGER, Statistical metrics, *Proc. Nat. Acad. Sci. USA* **28** (1942), 353–537.
29. A. B. PAALMAN-DE MIRANDA, "Topological Semigroups," Mathematisch Centrum Amsterdam, Amsterdam, 1964.
30. A. ROSE AND J. B. ROSSER, Fragments of many valued statement calculus, *Trans. Amer. Math. Soc.* **87** (1958), 1–53.
31. K. D. SCHMIDT, A general Jordan decomposition, *Arch. Math.* **38** (1982), 556–564.
32. K. D. SCHMIDT, A common abstraction of Boolean rings and lattice ordered groups, *Comput. Math.* **54** (1985), 51–62.
33. K. D. SCHMIDT, Decompositions of vector measures in Riesz spaces and Banach lattices, *Proc. Edinburgh Math. Soc.* **29** (1986), 23–39.
34. B. SCHWEIZER AND A. SKLAR, Statistical metric spaces, *Pacific J. Math.* **10** (1960), 313–334.
35. B. SCHWEIZER AND A. SKLAR, Associative functions and statistical triangle inequalities, *Publ. Math. Debrecen* **8** (1961), 169–186.
36. B. SCHWEIZER AND A. SKLAR, Associative functions and abstract semigroups, *Publ. Math. Debrecen* **10** (1963), 69–81.
37. B. SCHWEIZER AND A. SKLAR, "Probabilistic Metric Spaces," North-Holland, New York/Amsterdam/Oxford, 1983.
38. M. SUGENO, "Theory of Fuzzy Integrals and Applications," Thesis, Tokyo Inst. of Technology, Tokyo, 1974.
39. A. WALD, On statistical generalizations of metric systems, *Proc. Nat. Acad. Sci. USA* **29** (1943), 196–197.
40. O. WYLER, Clans, *Comput. Math.* **17** (1966), 172–189.
41. L. A. ZADEH, Fuzzy Sets, *Inform. Control* **8** (1965), 338–353.
42. L. A. ZADEH, Probability measures of fuzzy events, *J. Math. Anal. Appl.* **23** (1968), 421–427.