Regularization of Monotone Variational Inequalities with Mosco Approximations of the Constraint Sets

YAKOV ALBER[1], DAN BUTNARIU[2] and IRINA RYAZANTSEVA[3]

[1]*Faculty of Mathematics, The Technion-Israel Institute of Technology, 32000 Haifa, Israel. [E-mail: alberya@tx.technion.ac.il]*

[2]*Department of Mathematics, University of Haifa, 31905 Haifa, Israel. [E-mail: dbutnaru@math.haifa.ac.il]*

[3]*Irina Ryazatseva, Department of Applied Mathematics, Nizhnii Novgorod State Technical University, 603155 Nizhnii Novgorod, Russia. [E-mail: nntu@adm.nntu.sci-nnov.ru]*

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Abstract

In this paper we study the convergence and stability in reflexive, smooth and strictly convex Banach spaces of a regularization method for variational inequalities with data perturbations. We prove that, when applied to perturbed variational inequalities with monotone, demiclosed, convex valued operators satisfying certain conditions of asymptotic growth, the regularization method we consider produces sequences which converge weakly to the minimal-norm solution of the original variational inequality, provided that the perturbed constraint sets converge to the constraint set of the original inequality in the sense of a modified form of Mosco convergence of order ≥ 1. If the underlying Banach space has the Kadeč-Klee property, then the sequence generated by that regularization method is strongly convergent.

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1 Introduction

Let X be a reflexive, smooth and strictly convex Banach space and let $A : X \to 2^{X^*}$ be a monotone operator. Let Ω be a nonempty, closed and convex subset of $\operatorname{Int}(\operatorname{Dom} A)$ and let $f \in X^*$. We consider the variational inequality

$$\langle Ax - f, y - x\rangle \geq 0, \ \forall y \in \Omega, \tag{1}$$

under the assumption that the operator A is convex valued and strongly-weakly demiclosed (s-w-demiclosed for short) on $\operatorname{Int}(\operatorname{Dom} A)$. A *solution* of the variational inequality (1) is a vector $x^0 \in \Omega$ such that, for some $\xi^0 \in Ax^0$, we have

$$\left\langle \xi^0 - f, y - x^0\right\rangle \geq 0, \ \forall y \in \Omega. \tag{2}$$

We denote by $\mathcal{S}(A, f, \Omega)$ the set of solutions of the variational inequality (1). Recall that an operator $A : X \to 2^{X^*}$ is called strongly-weakly demiclosed (s-w-demiclosed for short) on a subset M of $\operatorname{Dom} A$ if for any pair of sequences $\left\{u^k\right\}_{k\in\mathbb{N}} \subset M$ and $\left\{\xi^k\right\}_{k\in\mathbb{N}} \subset X^*$ such that $\xi^k \in Au^k$ for all $k \in \mathbb{N}$, $u^k \to u$ and $\xi^k \rightharpoonup \xi$ we have that $\xi \in Au$.

The problem of finding solutions for variational inequalities in the form (1) occurs in various fields and is often ill-posed in the sense that either the set $\mathcal{S}(A, f, \Omega)$ is not a singleton and/or small perturbations of the problem data A, f and Ω may lead to significant distortions of this set (see Tikhonov and Arsenin [33], [18], Kinderlehrer and Stampacchia [15], Liskovets [20], Kaplan and Tichatschke [14], Engl, Hanke and Neubauer [12], Bonnans and Shapiro [8], Lavrent'ev, Zerkal and Trofimov [17]). Ill-posedness of the problem creates difficulties in applications where the problem data are given by sequences of approximations: even if the approximative data A_k, f^k and Ω_k are close to the original data A, f and Ω, respectively, the solution set $\mathcal{S}(A_k, f^k, \Omega_k)$ of the variational inequality in which the original data are replaced by their respective approximations may happen to be empty or far from the solution set $\mathcal{S}(A, f, \Omega)$ of the variational inequality (1).

A method of overcoming some ill-posedness related difficulties in the resolution of variational inequalities is that originating in the works of Tikhonov [32] and Browder [10]. Its basic idea is to solve *the regularized variational inequality*

$$\langle (A + \alpha U)x - f, y - x\rangle \geq 0, \ \forall y \in \Omega, \tag{3}$$

where $\alpha \in (0, \infty)$ and $U : X \to X^*$ is a continuous, coercive, monotone operator with some additional regularity properties. In many instances (see

[10, Theorem 1]) the variational inequality (3) has a unique solution which, for small numbers α, is close to a solution of (1). The question is what impact the perturbation of data has on the behavior of the solution of the regularized variational inequality (3). In other words, we would like to know whether, and in which conditions, solving the regularized variational inequality (3) with approximative data A_k, f^k and Ω_k instead of the corresponding original data A, f and Ω, still produces approximations of solutions of the original variational inequality (1). The relevance of this question comes from the fact that, by contrast to (1), the regularized variational inequality is usually well-posed and, therefore, its solutions may happen to be less sensitive to data perturbations.

In this paper we assume that the approximative data preserve some basic features of the original data. Precisely, we assume that each operator A_k is monotone, each set Ω_k is nonempty, closed, convex and contained in $\operatorname{Int}(\operatorname{Dom} A_k)$ and that A_k is s-w-demiclosed and convex valued on Ω_k. The regularized variational inequality we consider is

$$\left\langle (A_k + \alpha_k J)x - f^k, y - x \right\rangle \geq 0, \ \forall y \in \Omega_k, \tag{4}$$

where α_k is a positive real number and $J : X \to X^*$ is the normalized duality mapping. We will show (see Section 2 below) that, in these circumstances, the regularized variational inequality (4) has a unique solution which we denote by x^k in that follows. We want to know what inter-correlation should exist between the convergence behavior of the sequences of perturbed data and the convergence behavior of the sequence of regularization parameters α_k in order to ensure that, by solving the (usually well-posed) problem (4), one produces good approximations of a solution for the original variational inequality (1). Our aim is to prove that, if some asymptotic growth conditions concerning the operator A (see conditions (**C1**) and (**C3**) in Section 3) or some asymptotic boundedness conditions for its approximations A_k (see condition (**C2**)) are satisfied, then the sequence $\{x^k\}_{k\in\mathbb{N}}$ is weakly convergent to the minimal-norm solution of the variational inequality (1), provided that the sequences of perturbed data converge to their corresponding original data faster than the sequence of regularization parameters α_k converges to zero (see assumption **A3** below). Moreover, we show that if the space X, in addition to being reflexive, smooth and strictly convex, also has the Kadeč-Klee property, then the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges strongly (see Theorem 3.2). In our context, the convergence of the perturbed operators A_k is in the sense of uniform on bounded sets pointwise convergence of functions from X to the family of closed subsets of X^* endowed with the

Hausdorff metric, the convergence of f^k is in the sense of the norm-topology of X^*, and the convergence of the perturbed constraint sets Ω_k is conceived in the sense of a modified form of Mosco convergence of order ≥ 1 which we call fast Mosco convergence relative to $\{\alpha_k\}_{k\in\mathbb{N}}$ – see Definition 2.1.

The convergence behavior of the solutions of regularized variational inequalities (4) with monotone operators was studied before (see [1], [2], [5], [4], [6], [9], [16], [19], [20], [21], [22], [24], [25], [26], [31]). Weak subsequential convergence of $\{x^k\}_{k\in\mathbb{N}}$ to solutions of the variational inequality (1) when the sets Ω_k are Mosco-convergent to Ω, as shown in our Theorem 3.1, was previously established under conditions more restrictive than ours concerning the operators A and A_k (see [22], [25], [26]). Convergence of $\{x^k\}_{k\in\mathbb{N}}$ in the case that only A and f are given by approximations while Ω is unperturbed was established in [1]. For monotone coercive operators A and A_k having the (S) property strong convergence of the sequence $\{x^k\}_{k\in\mathbb{N}}$ was shown in [24]. In [6] and [31] strong convergence of $\{x^k\}_{k\in\mathbb{N}}$ was proved for maximal monotone operators but under the demanding requirement that the sequence $\{\mathcal{H}(\Omega_k,\Omega)\alpha_k^{-1}\}_{k\in\mathbb{N}}$, where $\alpha_k \to 0^+$ and $\mathcal{H}$ denotes the Hausdorff distance between closed convex sets, converges to zero. A more relaxed, but still demanding, form of this requirement occurs in [5]. By contrast with these conditions, the fast Mosco-convergence relative to $\{\alpha_k\}_{k\in\mathbb{N}}$ we impose on Ω_k in our Theorem 3.2 is too weak to ensure by itself even boundedness of the sequence $\{x^k\}_{k\in\mathbb{N}}$. However, in conjunction with one of the asymptotic growth conditions mentioned above, the fast Mosco-convergence of the sets Ω_k relative to $\{\alpha_k\}_{k\in\mathbb{N}}$ becomes sufficient for guaranteeing that the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to the minimal-norm solution of the variational inequality (1). Moreover, in spaces with the Kadeč-Klee property, it ensures that $\{x^k\}_{k\in\mathbb{N}}$ converges strongly.

Our work is organized as follows. Section 2 contains some preliminary results meant to show that the regularized variational inequality (4) has unique solution when A_k is monotone and s-w-demiclosed. Among these results the most significant is Lemma 2.3 which gives a Minty type characterization of the concept of solution we deal with. In Section 3 we describe our main results concerning the behavior of the regularization method (4) and their interconnections. Section 4 consists of a sequence of lemmas which prove that the theorems stated in Section 3 are true.

A careful analysis of our arguments in Sections 2 and 4 shows that our main results can be generalized in several directions. First, our proofs can be reproduced without essential modifications in order to show that our main results still stand when one replaces the duality mapping J by a duality mapping J^μ with a gauge function $\mu : [0,\infty) \to [0,\infty)$ satisfying the condition that $\lim_{t\to\infty}\left[t^{-1}\mu(t)\right] > 0$. This is relevant because, as noted in [5], regularizing the variational inequality with a duality mapping based on a faster growing gauge function one may accelerate the convergence of the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$. Second, one can replace the s-w-demiclosedness condition we impose on the operators A and A_k by the assumption that the converse implication in Lemma 2.3 holds. Third, the proofs of Theorems 3.1 and 3.2 can be adapted for showing that these results still stand when the operators A and A_k are not monotone, but pseudo-monotone (see also [23] and [11]), but this will be the topic of another paper.

2 Preliminaries

In this section we present several technical lemmas dealing with the properties of the solutions of the variational inequality (1). Some of these results may be already known, but we were not able to find precise references for them or for their proofs in the specific forms we need in our context.

The following lemma is a variant of Theorem 5 in [29] (see also [6, Theorem 1]). It proves the claim made in the introduction that the regularized variational inequality (4) has unique solution whenever the operator involved in it is maximal monotone.

LEMMA 2.1. *If* $A : X \to 2^{X^*}$ *is a maximal monotone operator and* Ω *is a closed convex nonempty subset* Ω *of* $\mathrm{Int}(\mathrm{Dom}\, A)$, *then the variational inequality*

$$\langle (A+\alpha J)x - f, y - x\rangle \geq 0,\ \forall y \in \Omega, \tag{5}$$

has a unique solution which is exactly

$$x = (A + N_\Omega + \alpha J)^{-1} f,$$

where $N_\Omega : X \to X^*$ *is the normal cone operator associated with the set* Ω.

Proof. The operator $A + N_\Omega$ is maximal monotone because it is the sum of two maximal monotone operators and $\Omega \subseteq \mathrm{Int}(\mathrm{Dom}\, A)$ - see [29, Theorem 3]. Consequently, the operator $A + N_\Omega + \alpha J$ is maximal monotone and surjective (cf. [29, Proposition 1]). Therefore, there exists a vector

$$x \in \mathrm{Dom}\,(A + N_\Omega + \alpha J) = \Omega$$

such that $f \in (A + N_\Omega + \alpha J)\, x$. Let $\xi \in Ax$ and $\zeta \in N_\Omega x$ be such that $f = \xi + \zeta + \alpha J x$. Then we have

$$\langle \xi + \zeta + \alpha J x - f, y - x \rangle = 0, \ \forall y \in \Omega.$$

Note that, for any $y \in \Omega$, we have $\langle \zeta, y - x \rangle \leq 0$, because $\zeta \in N_\Omega x$. Hence,

$$\langle \xi + \alpha J x - f, y - x \rangle = \langle \zeta, x - y \rangle \geq 0, \ \forall y \in \Omega,$$

and this shows that x is a solution of the variational inequality (5).

In order to prove that this solution is unique, suppose by contradiction that $x' \in \Omega$ is another solution of the given variational inequality. Then, for some $\upsilon \in Ax'$, we have

$$\left\langle \upsilon + \alpha J x' - f, y - x' \right\rangle \geq 0, \ \forall y \in \Omega.$$

In particular, we have

$$\left\langle \upsilon + \alpha J x' - f, x - x' \right\rangle \geq 0.$$

Similarly, for some $\xi \in Ax$ we have

$$\left\langle \xi + \alpha J x - f, x' - x \right\rangle \geq 0.$$

Summing up the two inequalities we deduce that

$$\left\langle (\xi + \alpha J x) - \left(\upsilon + \alpha J x'\right), x' - x \right\rangle \geq 0.$$

Since the operator $A + \alpha J$ is monotone, this implies

$$\left\langle (\xi + \alpha J x) - \left(\upsilon + \alpha J x'\right), x - x' \right\rangle = 0.$$

The operator $A + \alpha J$ is strictly monotone because J is so. Consequently, the last inequality cannot hold unless $x = x'$. $\square$

The following result which we will use below, appears in [7, Lemma 5.3]. We present it here with an alternative proof because, as pointed out to us by Rainer Tichatschke, the proof given in [7] holds for single valued operators only (see also [27, Lemma 5.1.2]).

LEMMA 2.2: *Suppose that $A : X \to 2^{X^*}$ is a maximal monotone operator on its domain $\operatorname{Dom} A$, Ω is a nonempty, closed, convex subset of $\operatorname{Int}(\operatorname{Dom} A)$ and $x^0 \in \Omega$. If the vector x^0 has the property that for each $y \in \Omega$ and for any $\xi \in Ay$ we have*

$$\left\langle \xi - f, y - x^0 \right\rangle \geq 0. \tag{6}$$

then x^0 is a solution of the variational inequality (1).

Proof. Let $N_\Omega : X \to 2^{X^*}$ be the normal cone mapping of the set Ω (i.e., the subgradient of the indicator function of the set Ω). This is a maximal monotone operator and $0 \in N_\Omega(x)$ for any $x \in \Omega$. Therefore, for any $y \in \Omega$ and for any $\eta \in N_\Omega(y)$ we have

$$\left\langle \eta, y - x^0 \right\rangle = \left\langle \eta - 0, y - x^0 \right\rangle \geq 0.$$

This and (6) imply that whenever $y \in \Omega$, $\xi \in Ay$ and $\eta \in N_\Omega(y)$ we have that

$$\left\langle \eta + \xi - f, y - x^0 \right\rangle = \left\langle \xi - f, y - x^0 \right\rangle + \left\langle \eta, y - x^0 \right\rangle \geq 0. \tag{7}$$

The operator $A + N_\Omega$ is maximal monotone because A is so and

$$\operatorname{Dom} N_\Omega = \Omega \subset \operatorname{Int}\left(\operatorname{Dom} A\right).$$

By (7), we deduce that $f \in (A + N_\Omega)(x^0)$. Thus, there exist $\eta^0 \in N_\Omega(x^0)$ and $\xi^0 \in Ax^0$ such that $f = \xi^0 + \eta^0$. Clearly,

$$\left\langle \eta^0 + \xi^0 - f, y - x^0 \right\rangle = 0, \quad \forall y \in \Omega.$$

Hence, we obtain that for any $y \in \Omega$, we have that

$$\left\langle \xi^0 - f, y - x^0 \right\rangle = -\left\langle \eta^0, y - x^0 \right\rangle \geq 0,$$

where the last inequality is due to the fact that $\eta^0 \in N_\Omega(x^0)$. This implies that x^0 is a solution of (1). $\square$

Lemma 2.2 requires that the operator A should be maximal monotone. It is known (see [28]) that the implication in Lemma 2.2 still holds when A is not necessarily maximal monotone, but only monotone, s-w-demiclosed and convex valued on $\operatorname{Dom} A$ and $x^0 \in \operatorname{Int}\Omega$. In this case, if (6) is satisfied for each $y \in \Omega$ and for any $\xi \in Ay$, then $f \in Ax^0$ and, therefore, $x^0 \in \mathcal{S}(A, f, \Omega)$. The following result shows that Lemma 2.2 remains true for (not necessarily maximal) monotone operators which are s-w-demiclosed and convex valued on Ω even if x^0 is a boundary point of Ω. One should note that, in this situation, it may happen that $f \notin Ax^0$. For example, take $X = \mathbb{R}$, $\Omega = [1, 2]$, $f = 0$ and $Ax = \partial(x^2)$ and, then, $\mathcal{S}(A, f, \Omega) = \{1\}$ but $f \notin A1 = \{2\}$. Lemma 2.3 below is, in fact, a Minty type characterization of the solutions of the variational inequality (1) considered in this paper.

LEMMA 2.3. *Suppose that $A : X \to 2^{X^*}$ is a monotone operator, Ω is a closed convex subset of* $\operatorname{Dom} A$ *and $x^0 \in \Omega$.*

(*i*) *If the vector* x^0 *is a solution of the variational inequality* (1), *then for each* $y \in \Omega$ *and for any* $\xi \in Ay$ *the inequality* (6) *holds.*

(*ii*) *If* $\Omega \subseteq \operatorname{Int}(\operatorname{Dom} A)$ *and if the operator* A *is s-w-demiclosed and convex valued on* Ω, *then the converse implication is also true.*

Proof. (*i*) Suppose that x^0 is a solution of the variational inequality (1), $y \in \Omega$ and $\xi \in Ay$. Then, there exists a vector $\xi^0 \in Ax^0$ such that

$$\left\langle \xi^0 - f, x - x^0 \right\rangle \geq 0, \ \forall x \in \Omega.$$

By the monotonicity of A, we have $\left\langle \xi - \xi^0, y - x^0 \right\rangle \geq 0$ and this implies

$$\left\langle \xi, y - x^0 \right\rangle \geq \left\langle \xi^0, y - x^0 \right\rangle \geq \left\langle f, y - x^0 \right\rangle,$$

showing that condition (6) holds.

(*ii*) We first show that for any maximal monotone extension $\bar{A}$ of A we have $\bar{A}x = Ax$ for any $x \in \Omega$. To this end, let $x \in \Omega$ and observe that, since the operator A is monotone and $x \in \operatorname{Int}(\operatorname{Dom} A)$, A it is locally bounded at x – see, for instance, [28, p. 104]. Consequently, if $\{x^n\}_{k\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} x^n = x \in \Omega$, and if $\{\zeta^n\}_{k\in\mathbb{N}}$ is a sequence such that $\zeta^n \in Ax^n$ for all $n \in \mathbb{N}$, then $\{\zeta^n\}_{n\in\mathbb{N}}$ is bounded. Since X^* is reflexive, it results that the bounded sequence $\{\zeta^n\}_{n\in\mathbb{N}}$ has weak accumulation points. From the demiclosedness of A it follows that any such weak accumulation point belongs to Ax. Denote by Rx the closed convex hull of the set of weak accumulation points of all sequences $\{\zeta^n\}_{n\in\mathbb{N}}$ as described above. The set Ax is convex and, due to the demiclosedness of A on the set Ω, the set Ax is weakly closed and, therefore, closed. Hence, we have that $Rx \subseteq Ax$. Let $\bar{A}$ be a maximal monotone extension of A. Obviously, $Rx \subseteq Ax \subseteq \bar{A}x$. We claim that the inclusion $Rx \supseteq \bar{A}x$ holds too. Suppose by contradiction that this is not the case. Then, there exists $\eta \in \bar{A}x$ such that $\eta \notin Rx$. According to the strong separation theorem (see, for instance, [13, p. 64]), there exists a vector $u \in X$ such that

$$\langle \zeta - \eta, u \rangle < 0, \quad \forall \zeta \in Rx. \tag{8}$$

Let $\{t_n\}_{k\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero such that, for any $n \in \mathbb{N}$, we have that $z^n := x + t_n u \in \operatorname{Dom} A$ (such a sequence exists because $x \in \operatorname{Int}(\operatorname{Dom} A)$). For each $n \in \mathbb{N}$ let ζ^n be an arbitrary element of Az^n. The sequence $\{\zeta^n\}_{n\in\mathbb{N}}$ is bounded because $\lim_{n\to\infty} z^n = x$ and A is locally bounded at x. Thus, there exists a subsequence $\left\{\zeta^{i_n}\right\}_{n\in\mathbb{N}}$ of $\{\zeta^n\}_{n\in\mathbb{N}}$ which converges weakly to some $\bar{\zeta}$ in X^*. Clearly, $\bar{\zeta} \in Rx$. By the monotonicity of $\bar{A}$ we have that

$$\left\langle \zeta^{i_n} - \eta, z^{i_n} - x \right\rangle \geq 0, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\left\langle \zeta^{i_n} - \eta, u \right\rangle \geq 0, \quad \forall n \in \mathbb{N}.$$

Letting here $n \to \infty$ we obtain that $\left\langle \bar{\zeta} - \eta, u \right\rangle \geq 0$ and this contradicts (8). Hence, $Rx \supseteq \bar{A}x$ and this implies $Ax \subseteq \bar{A}x \subseteq Rx$ and, consequently, $Ax = \bar{A}x = Rx$.

Since $Ax = \bar{A}x$ for any $x \in \Omega$, the variational inequality (1) and the following variational inequality have the same set of solutions (in Ω):

$$\left\langle \bar{A}x - f, y - x \right\rangle \geq 0, \quad \forall y \in \Omega. \tag{9}$$

Also, the maximal monotone operator $\bar{A}$ satisfies the requirements of Lemma 2.2. Hence, x^0 is a solution of the variational inequality (9) and, therefore, it is a solution of (1) too. $\square$

Maximal monotone operators are s-w-demiclosed on their domains (cf. [9, Lemma 1.1]). Therefore, we obtain the following known result:

COROLLARY 2.1. *If A is a maximal monotone operator and $x^0 \in \Omega \subseteq \operatorname{Int}(\operatorname{Dom} A)$, then the following conditions are equivalent:*
(i) The vector x^0 is a solution of (1);
(ii) For each $y \in \Omega$ and for any $\xi \in Ay$ we have $\left\langle \xi - f, y - x^0 \right\rangle \geq 0$.

Let A be a monotone operator and let $\bar{A}$ be a maximal monotone extension of A. A solution of the variational inequality

$$\left\langle \bar{A}x - f, y - x \right\rangle \geq 0, \ \forall y \in \Omega \tag{10}$$

is called a *generalized solution* of (1). Clearly, any solution of the variational inequality (1) is a generalized solution too. We show now that, in the case of s-w-demiclosed monotone operators A, generalized solutions are necessarily solutions of (1).

LEMMA 2.4. *If the operator $A : X \to 2^{X^*}$ is monotone, if $\Omega \subseteq \operatorname{Int}(\operatorname{Dom} A)$, and if A is s-w-demiclosed and convex valued on Ω, then any generalized solution of (1) is also a solution of (1), that is, for any maximal monotone extension $\bar{A}$ of A we have that $\mathcal{S}(A, f, \Omega) = \mathcal{S}(\bar{A}, f, \Omega)$.*

Proof. Suppose that x^0 is a generalized solution of the variational inequality (1). Then, for some maximal monotone extension $\bar{A}$ of A and for some $\bar{\xi}^0 \in \bar{A}x^0$ we have

$$\left\langle \bar{\xi}^0 - f, y - x^0 \right\rangle \geq 0, \ \forall y \in \Omega.$$

According to Corollary 2.1 applied to $\bar{A}$ it results that for each $y \in \Omega$ and for any $\xi \in \bar{A}y$ we have that $\langle \xi - f, y - x^0 \rangle \geq 0$. In particular, this holds for each $y \in \Omega$ and for any $\xi \in Ay$ because $Ay \subseteq \bar{A}y$. According to Lemma 2.3, this implies that x^0 is a solution of (1). □

An immediate consequence of Lemma 2.4 is the following:

COROLLARY 2.2. *If the operator A is monotone, if $\Omega \subseteq \operatorname{Int}(\operatorname{Dom} A)$, and if A* is *s-w-demiclosed and convex valued on Ω, then, for any real number $\alpha > 0$, the variational inequality*

$$\langle (A + \alpha J)x - f, y - x \rangle \geq 0, \ \forall y \in \Omega, \tag{11}$$

has a unique solution.

Proof. The operator $A + \alpha J$ is s-w-demiclosed on Ω and monotone because A and J are so. Observe that if $\bar{A}$ is a maximal monotone extension of A, then $\bar{A} + \alpha J$ is a maximal monotone extension of $A + \alpha J$. According to Lemma 2.4 applied to $A + \alpha J$ we have that

$$\mathcal{S}(A + \alpha J, f, \Omega) = \mathcal{S}(\bar{A} + \alpha J, f, \Omega).$$

Applying Lemma 2.1 to the maximal monotone operator $\bar{A}$ we deduce that the variational inequality (11) has unique solution, that is, $\mathcal{S}(\bar{A} + \alpha J, f, \Omega)$ is a singleton and this completes the proof. □

Suppose now that the data A, f and Ω in the variational inequality (1) are given by sequences of approximations $\{A_k\}_{k\in\mathbb{N}}$, $\{f^k\}_{k\in\mathbb{N}}$ and $\{\Omega_k\}_{k\in\mathbb{N}}$, respectively. As noted in the Introduction, if the variational inequality (1) is ill-posed, then solving it with perturbed data A_k, f^k and Ω_k instead of the original data A, f and Ω, respectively, may be of no use because the variational inequality

$$\left\langle A_k x - f^k, y - x \right\rangle \geq 0, \ \forall y \in \Omega_k,$$

may be as ill-posed as the original variational inequality and the solution sets $\mathcal{S}(A, f, \Omega)$ and $\mathcal{S}(A_k, f^k, \Omega_k)$ may happen to be far from each other even if the approximative data are close to the original data. Corollary 2.2 ensures that if $\Omega_k \subseteq \operatorname{Int}(\operatorname{Dom} A)$ and the operator A_k is monotone and s-w-demiclosed on Ω_k, then, no matter how the positive real number α_k is chosen, the variational inequality (4) has a unique solution x^k. Our purpose is to show (see the next section) that, under certain conditions, by solving the well posed variational inequality (4) for small numbers α_k, one produces approximations of a solution of the original variational inequality (1). Moreover, we will show that this procedure of approximating solutions of (1) is stable in the sense that the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ converges to a solution of (1) when the sequence of perturbed constraint sets $\{\Omega_k\}_{k\in\mathbb{N}}$ converges "fast enough" to the set Ω (see Definition 2.1 below), while $f^k \to f$ and $A_k(x)$ converges to $A(x)$ uniformly on bounded sets. Recall (see [25, Definition 1.1]) that a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of subsets of X is called *Mosco convergent to the set* Ω (and we write *M-convergent* for short) if

$$\Omega = w - \overline{\lim_{k\to\infty}}\ \Omega_k = s - \underline{\lim}_{k\to\infty}\ \Omega_k,$$

where $s - \underline{\lim}_{k\to\infty}\ \Omega_k$ represents the collection of all $y \in X$ which are limits (in the strong convergence sense) of sequences with the property that $x^k \in \Omega_k$ for all $k \in \mathbb{N}$ and $w - \overline{\lim}_{k\to\infty}\ \Omega_k$ denotes the collection of all $x \in X$ such that there exists a sequence $\left\{y^k\right\}_{k\in\mathbb{N}}$ in X converging weakly to x and with the property that there exists a subsequence $\{\Omega_{i_k}\}_{k\in\mathbb{N}}$ of $\{\Omega_k\}_{k\in\mathbb{N}}$ such that $y^k \in \Omega_{i_k}$ for all $k \in \mathbb{N}$. We also use a modified form of Mosco's notion (see [25, Definition 1.3]) of "convergence of order ≥ 1".

DEFINITION 2.1: Let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero. The sequence of sets $\{\Omega_k\}_{k\in\mathbb{N}}$ is said to be *fast $M-$convergent to Ω relative to the sequence $\{\alpha_k\}_{k\in\mathbb{N}}$* if the following conditions are satisfied:

(i) If $y \in \Omega$, then $0 \in s - \underline{\lim}_{k\to\infty} \left(\alpha_{i_k}\right)^{-1} (y - \Omega_k)$;

(ii) If $\left\{z^k\right\}_{k\in\mathbb{N}}$ is a weakly convergent sequence in X such that, for some subsequence $\{\Omega_{i_k}\}_{k\in\mathbb{N}}$ of $\{\Omega_k\}_{k\in\mathbb{N}}$, we have $z^k \in \Omega_{i_k}$ for all $k \in \mathbb{N}$, then $0 \in w - \overline{\lim}_{k\to\infty} \left(\alpha_{i_k}\right)^{-1} \left(z^k - \Omega\right)$.

Clearly, the sequences of sets which are fast $M-$convergent relative to $\left\{k^{-1}\right\}_{k\in\mathbb{N}}$ are exactly those which are usually called convergent of order ≥ 1. A reasoning similar to that involved in the proof of Lemma 1.9 in [25] shows that if $\{\Omega_k\}_{k\in\mathbb{N}}$ is fast $M-$convergent to Ω relative to some sequence of positive real numbers $\{\alpha_k\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\alpha_k = 0$, then the sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ is $M-$convergent to Ω.

From a technical point of view, the key to our analysis of the behavior of this regularization method is the following result due to Alber and Notik [6, Lemma 2]:

LEMMA 2.5. *Let $F : X \to 2^{X^*}$ be a monotone operator. If $\bar{x} \in \operatorname{Int}(\operatorname{Dom} F)$, then there exists a positive real number $r := r(\bar{x}) > 0$ such that, for any pair $(x, \xi) \in \operatorname{Graph} F$, we have*

$$\langle \xi, x - \bar{x}\rangle \geq r \|\xi\|_* - (\|x - \bar{x}\| + r)\, c, \tag{12}$$

where

$$c := \sup \{ \|\eta\|_* : \|x - \bar{x}\| \le r \text{ and } \eta \in F(x) \} < \infty. \tag{13}$$

3 The Main Results

We consider the variational inequality (1) under the assumption that the operator A is monotone, Ω is a nonempty closed convex subset of $\mathrm{Int}\,(\mathrm{Dom}\, A)$ and A is s-w-demiclosed and convex valued on $\mathrm{Int}\,(\mathrm{Dom}\, A)$. All over that follows we are making the following assumption concerning the perturbed problem data:

ASSUMPTION **A1.**

For each $k \in \mathbb{N}$*, the operator* $A_k : X \to 2^{X^*}$ *is monotone,* Ω_k *is a closed, convex subset of* $\mathrm{Int}\,(\mathrm{Dom}\, A) \cap \mathrm{Int}\,(\mathrm{Dom}\, A_k)$, A_k *is s-w-demiclosed and convex valued on* Ω_k *and* $f^k \in X^*$.

In the first stage we are going to study the convergence of the sequence $\{x^k\}_{k \in \mathbb{N}}$ under the following convergence requirements concerning the approximative data A_k, Ω_k and f^k :

ASSUMPTION **A2.**

(i) *The sequence of sets* $\{\Omega_k\}_{k \in \mathbb{N}}$ *is* M*-convergent to the set* Ω*;*

(ii) *The sequence of positive real numbers* $\{\alpha_k\}_{k \in \mathbb{N}}$ *converges to zero and the following conditions are satisfied:*

$(ii-1)$ *There exists a function* $q : X \to \mathbb{R}_+$ *which is bounded on bounded sets and such that, for each* $k \in \mathbb{N}$ *and for any* $x \in \Omega_k$*, we have that*

$$\mathcal{H}(A_k x, Ax) \le q(x)\alpha_k,$$

where $\mathcal{H}$ *stands for the Hausdorff distance between sets;*

$(ii-2)$ *For some real number* $c_1 > 0$*, we have that* $\left\| f^k - f \right\|_* \le c_1 \alpha_k$*, for all* $k \in \mathbb{N}$.

As noted above, in these circumstances, Corollary 2.2 guarantees that the variational inequality (4) has a unique solution denoted x^k. Our first goal is to prove the following subsequentially weak convergence criterion for the sequence $\{x^k\}_{k \in \mathbb{N}}$:

THEOREM 3.1. *Suppose that assumptions **A1** and **A2** are satisfied. Then the following statements are true:*

(*a*) *If the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *is bounded, then it has weak accumulation points and each weak accumulation point of* $\{x^k\}_{k\in\mathbb{N}}$ *is a solution of the variational inequality* (1)*;*

(*b*) *If the variational inequality* (1) *has no solutions, then the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *is unbounded;*

(*c*) *If any of the next three conditions is satisfied, then the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *is bounded:*

(**C1**) *For any real number* $\beta > 0$, *the set*

$$L_\beta(A) := \{x \in X : \|\xi\|_* \leq \beta \|x\|, \ \forall \xi \in Ax\}$$

is bounded.

(**C2**) *There exists a bounded sequence* $\{v^k\}_{k\in\mathbb{N}}$ *in* X *such that* $v^k \in \Omega_k$ *for any* $k \in \mathbb{N}$, *and the sequence* $\{\alpha_k^{-1}\mathrm{dist}_*(A_k v^k, f^k)\}_{k\in\mathbb{N}}$ *is bounded.*

(**C3**) *There exists a vector* $\bar{x} \in \bigcap_{k\in\mathbb{N}} \Omega_k$ *at which the operator* A *is coercive, that is, for any sequence* $\{(z^k, \psi^k)\}_{k\in\mathbb{N}}$ *contained in* Graph$A$ *such that* $\lim_{k\to\infty} \|z^k\| = \infty$, *we have*

$$\lim_{k\to\infty} \frac{\langle \psi^k, z^k - \bar{x}\rangle}{\|z^k\|} = \infty.$$

The proof of this theorem consists of several lemmas presented in Section 4. A careful analysis of our proof shows that Theorem 3.1(c) is still valid when condition (**C1**) is relaxed by requiring instead boundedness of the set $L_\beta(A)$ for $\beta = a + 1$, where a is some constant given by formula (30) below.

It is interesting to observe that Theorem 3.1 leads to an existence criterion for solutions of the variational inequality (1). Note that assumptions **A1** and **A2** are satisfied when $A_k = A$, $\Omega_k = \Omega$ and $f^k = f$ for all $k \in \mathbb{N}$. Therefore, in this particular case, Theorem 3.1(c) can be re-interpreted in the following way:

COROLLARY 3.1. *If the operator* A *satisfies any of the conditions* (**C1**), (**C2**) *or* (**C3**), *then the variational inequality* (1) *has solutions.*

Theorem 3.1 gives sufficient conditions for subsequential weak convergence of the sequence $\{x^k\}_{k\in\mathbb{N}}$ to solutions of the variational inequality (1). Naturally, in practical situations we would like to have strong convergence of this sequence to a solution of the variational inequality (1). We are going to show that strong convergence of the sequence $\{x^k\}_{k\in\mathbb{N}}$ can be guaranteed when the perturbed data A_k, f^k and Ω_k have better convergence behavior than that described in assumption **A2**. Precisely, we will prove that strong convergence of $\{x^k\}_{k\in\mathbb{N}}$ to the minimal-norm solution of the variational inequality (1) happens when the perturbed data satisfy the following assumption:

ASSUMPTION **A3.**

The sequence of positive real numbers $\{\alpha_k\}_{k\in\mathbb{N}}$ *converges to zero and has the following properties:*

(i) *The sequence of sets* $\{\Omega_k\}_{k\in\mathbb{N}}$ *is fast* $M-$*convergent to* Ω *relative to the sequence* $\{\alpha_k\}_{k\in\mathbb{N}}$*;*

(ii) *There exists a sequence of positive real numbers* $\{\beta_k\}_{k\in\mathbb{N}}$ *and a function* $p : X \to \mathbb{R}$*, which is bounded on bounded subsets of* X*, such that* $\lim_{k\to\infty}(\beta_k/\alpha_k) = 0$ *and*

$$\mathcal{H}(A_k(x), A(x)) \le p(x)\beta_k, \ \forall x \in \operatorname{Dom} A,$$

for each $k \in \mathbb{N}$;

(iii) *There exists a sequence of positive real numbers* $\{\gamma_k\}_{k\in\mathbb{N}}$ *such that* $\lim_{k\to\infty}(\gamma_k/\alpha_k) = 0$ *and* $\left\|f^k - f\right\|_* \le \gamma_k$ *for all* $k \in \mathbb{N}$.

Assumptions **A2** and **A3** describe different inter-correlations between the convergence behavior of the perturbed data and that of the sequence of the regularization parameters α_k. Under assumption **A2** the regularization the parameters converge to zero as fast as the perturbed data converge to their corresponding original data. By contrast, under assumption **A3** the regularization parameters are converging to zero slower than the perturbed data approach their corresponding original data. Theorem 3.1 shows that, under assumption **A2,** we can achieve weak subsequential convergence of $\{x^k\}_{k\in\mathbb{N}}$ to solutions of the variational inequality (1). We are going to show that under assumption **A3** the sequence $\{x^k\}_{k\in\mathbb{N}}$ has better convergence behavior. Clearly, conditions (ii) and (iii) in assumption **A3**, taken together, imply condition (ii) in assumption **A2**. Since, as noted in Section 2, condition **A3**(i) implies **A2**(i), it results that assumption **A3** is stronger than **A2**. Hence, by Theorem 3.1, under assumptions **A1** and **A3** and any of the conditions (**C1**)-(**C3**), the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded, has weak accumulation points and all these points are solutions of the variational inequality (1).

For stating our strong convergence result recall that, according to Lemma 2.4, we have that $\mathcal{S}(A, f, \Omega) = \mathcal{S}(\bar{A}, f, \Omega)$, where $\bar{A}$ is a maximal monotone extension of A. This, together with Corollary 2.1, implies that the set $\mathcal{S}(A, f, \Omega)$ is convex and closed. Since the space X is reflexive and strictly convex, each closed convex nonempty subset of X has a unique minimal-norm element (the metric projection of zero on the set). Hence, if the solution set $\mathcal{S}(A, f, \Omega)$ is nonempty, then it has a minimal-norm element. According to Theorem 3.1, this happens under assumptions **A1**, **A3** and any of the conditions (**C1**)-(**C3**). That is why the following strong convergence criterion makes sense.

THEOREM 3.2. *Suppose that assumptions **A1** and **A3** are satisfied. Then the following statements are true:*

(*a*) *If the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded, then it converges weakly to the (necessarily existing) minimal-norm solution of the variational inequality* (1)*;*

(*b*) *If the space X has the Kadeč-Klee property and if the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded, then it converges strongly to the (necessarily existing) minimal-norm solution of the variational inequality* (1)*;*

(*c*) *If any of the conditions* (**C1**), (**C2**) *or* (**C3**) *of Theorem 3.1 is satisfied, then the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly, and if the space X has the Kadeč-Klee property it converges strongly, to the minimal-norm solution of the variational inequality* (1).

The proof of this theorem is given in the next section.

4 Proofs of the Main Results

From now on we presume that assumptions **A1** and **A2** are satisfied. Note that, in these circumstances, for each $k \in \mathbb{N}$, the variational inequality

$$\langle (A + \alpha_k J)y - f, z - y \rangle \geq 0, \ \forall z \in \Omega_k, \tag{14}$$

has a unique solution (because of **A1** and Corollary 2.2). We denote that solution by y^k. Since y^k is the solution of (14), there exists $\zeta^k \in Ay^k$ such that

$$\left\langle \zeta^k + \alpha_k J y^k - f, z - y^k \right\rangle \geq 0, \ \forall z \in \Omega_k. \tag{15}$$

Similarly, since x^k is the solution of the variational inequality (4), there exists $\xi^k \in A_k x^k$ such that

$$\left\langle \xi^k + \alpha_k J x^k - f^k, z - x^k \right\rangle \geq 0, \ \forall z \in \Omega_k. \tag{16}$$

With these facts in mind we start the proof of Theorem 3.1 by establishing the following technical result:

LEMMA 4.1. *If the sequence* $\left\{y^k\right\}_{k\in\mathbb{N}}$ *is bounded, then the sequence* $\left\{x^k\right\}_{k\in\mathbb{N}}$ *is bounded too.*

Proof. Denote

$$D_k := \left\langle (\xi^k + \alpha_k J x^k - f^k) - (\zeta^k + \alpha_k J y^k - f), x^k - y^k \right\rangle.$$

Note that for any $x, y \in X$ we have

$$\begin{aligned} \langle Jx - Jy, x - y\rangle &= \langle Jx, x\rangle + \langle Jy, y\rangle - \langle Jx, y\rangle - \langle Jy, x\rangle \\ &\geq \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|. \end{aligned}$$

We claim that for each $k \in \mathbb{N}$ there exists $v^k \in A_k y^k$ such that

$$\left\|\zeta^k - v^k\right\|_* \leq c_0 \alpha_k, \tag{17}$$

where $c_0 := \sup_{l\in\mathbb{N}} q(y^l)$ is finite. Indeed, according to $\mathbf{A2}(ii)$, for any pair $k, n \in \mathbb{N}$, there exists $v^{k,n} \in A_k y^k$ such that

$$\left\|\zeta^k - v^{k,n}\right\|_* \leq \left(q\left(y^k\right) + \frac{1}{n}\right)\alpha_k. \tag{18}$$

where the function $q(x)$ is bounded on bounded sets. The sequence $\left\{y^k\right\}_{k\in\mathbb{N}}$ is bounded. Hence, $c_0 = \sup_{l\in\mathbb{N}} q(y^l)$ is finite. Consequently, for each $k \in \mathbb{N}$, the sequence $\left\{v^{k,n}\right\}_{n\in\mathbb{N}}$ is bounded. Let v^k be a weak accumulation point of $\left\{v^{k,n}\right\}_{n\in\mathbb{N}}$. Since the operator A_k is s-w-demiclosed, the set $A_k y^k$ is weakly closed in X^* and, thus, $v^k \in A_k y^k$. Taking lim inf as $n \to \infty$ on both sides of (18) we obtain that v^k satisfies (17).

Observe that

$$
\begin{aligned}
D_k &= \alpha_k \left\langle Jx^k - Jy^k, x^k - y^k \right\rangle \qquad (19)\\
&+ \left\langle \xi^k - \zeta^k, x^k - y^k \right\rangle + \left\langle f - f^k, x^k - y^k \right\rangle \\
&\geq \alpha_k \left(\left\| x^k \right\| - \left\| y^k \right\| \right)^2 + \left\langle \xi^k - \upsilon^k + \upsilon^k - \zeta^k, x^k - y^k \right\rangle \\
&+ \left\langle f - f^k, x^k - y^k \right\rangle \\
&= \alpha_k \left(\left\| x^k \right\| - \left\| y^k \right\| \right)^2 + \left\langle \xi^k - \upsilon^k, x^k - y^k \right\rangle \\
&+ \left\langle \upsilon^k - \zeta^k, x^k - y^k \right\rangle + \left\langle f - f^k, x^k - y^k \right\rangle \\
&\geq \alpha_k \left(\left\| x^k \right\| - \left\| y^k \right\| \right)^2 + \left\langle \upsilon^k - \zeta^k, x^k - y^k \right\rangle + \left\langle f - f^k, x^k - y^k \right\rangle,
\end{aligned}
$$

where the last inequality is due to the monotonicity of A_k. By (15) and (16), we deduce that

$$\left\langle \zeta^k + \alpha_k Jy^k - f, x^k - y^k \right\rangle \geq 0,$$

and

$$\left\langle \xi^k + \alpha_k Jx^k - f^k, y^k - x^k \right\rangle \geq 0.$$

Consequently,

$$\left\langle \zeta^k + \alpha_k Jy^k - f, x^k - y^k \right\rangle \geq 0 \geq \left\langle \xi^k + \alpha_k Jx^k - f^k, x^k - y^k \right\rangle,$$

that is,

$$0 \geq \left\langle \xi^k + \alpha_k Jx^k - f^k, x^k - y^k \right\rangle + \left\langle \zeta^k + \alpha_k Jy^k - f, y^k - x^k \right\rangle = D_k.$$

This and (19) imply that

$$\left\langle \zeta^k - \upsilon^k, x^k - y^k \right\rangle + \left\langle f^k - f, x^k - y^k \right\rangle \geq \alpha_k \left(\left\| x^k \right\| - \left\| y^k \right\| \right)^2.$$

Thus, we have that

$$
\begin{aligned}
\alpha_k \left(\left\| x^k \right\| - \left\| y^k \right\| \right)^2 &\leq \left\| x^k - y^k \right\| \left(\left\| \zeta^k - \upsilon^k \right\|_* + \left\| f^k - f \right\|_* \right) \qquad (20)\\
&\leq \left(\left\| x^k \right\| + \left\| y^k \right\| \right) \left(\left\| \zeta^k - \upsilon^k \right\|_* + \left\| f^k - f \right\|_* \right).
\end{aligned}
$$

According to condition $\mathbf{A2}(ii)$ and (20), we obtain that

$$\left(\left\|x^k\right\| - \left\|y^k\right\|\right)^2 \leq \alpha_k^{-1} \left(\left\|\zeta^k - \upsilon^k\right\|_* + \left\|f^k - f\right\|_*\right) \left(\left\|x^k\right\| + \left\|y^k\right\|\right)$$
$$\leq (c_0 + c_1) \left(\left\|x^k\right\| + \left\|y^k\right\|\right). \tag{21}$$

Now, assume by contradiction that the sequence $\{x^k\}_{k\in\mathbb{N}}$ is unbounded. Then there exists a subsequence $\{x^{i_k}\}_{k\in\mathbb{N}}$ of it such that $\lim_{k\to\infty} \left\|x^{i_k}\right\| = \infty$. Taking into account (21), we deduce that there exists a positive integer k_0 such that for any integer $k \geq k_0$ we have

$$\left\|x^{i_k}\right\| \left(1 - \frac{\left\|y^{i_k}\right\|}{\left\|x^{i_k}\right\|}\right)^2 \left(1 + \frac{\left\|y^{i_k}\right\|}{\left\|x^{i_k}\right\|}\right)^{-1} \leq c_0 + c_1 < \infty.$$

Consequently, since the sequence $\{y^k\}_{k\in\mathbb{N}}$ is bounded, letting $k \to \infty$ in this inequality we get a contradiction. □

Recall that, for each $k \in \mathbb{N}$, $\xi^k \in A_k x^k$ is a vector for which (16) holds.

LEMMA 4.2. *If the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *is bounded, then the sequence* $\{\xi^k\}_{k\in\mathbb{N}}$ *is bounded too.*

Proof. According to $\mathbf{A2}(ii)$, one can reproduce without essential modifications the reasoning which proves the claim (17) in order to show that for each $k \in \mathbb{N}$, there exists $\psi^k \in Ax^k$ such that

$$\left\|\psi^k - \xi^k\right\|_* \leq \bar{c}\alpha_k, \tag{22}$$

where $\bar{c} := \sup_{k\in\mathbb{N}} q(x^k)$ is finite because the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded and the function $q(x)$ is bounded on bounded sets. Let $\bar{x}$ be a vector in Ω and denote

$$E_k = \left\langle \psi^k - f, x^k - \bar{x} \right\rangle.$$

Let $\{z^k\}_{k\in\mathbb{N}}$ be a sequence on X which converges strongly to $\bar{x}$ and such that $z^k \in \Omega_k$ for all $k \in \mathbb{N}$. According to Lemma 2.5 applied to the operator $Fx = Ax - f$ at the point $\bar{x}$, we deduce that there exist two real numbers $r > 0$ and $c \geq 0$ such that

$$E_k \geq r\left\|\psi^k - f\right\|_* - c\left(r + \left\|x^k - \bar{x}\right\|\right), \ \forall k \in \mathbb{N}. \tag{23}$$

Clearly,

$$\begin{aligned}
E_k &= \left\langle \psi^k - \xi^k, x^k - \bar{x} \right\rangle + \left\langle \xi^k + \alpha_k J x^k - f^k, x^k - \bar{x} \right\rangle \\
&+ \left\langle f^k - f, x^k - \bar{x} \right\rangle - \alpha_k \left\langle J x^k, x^k - \bar{x} \right\rangle \\
&= \left\langle \psi^k - \xi^k, x^k - \bar{x} \right\rangle + \left\langle f^k - f, x^k - \bar{x} \right\rangle - \alpha_k \left\langle J x^k, x^k - \bar{x} \right\rangle \\
&+ \left\langle \xi^k + \alpha_k J x^k - f^k, x^k - z^k \right\rangle + \left\langle \xi^k + \alpha_k J x^k - f^k, z^k - \bar{x} \right\rangle,
\end{aligned}$$

where, by (16), $\langle \xi^k + \alpha_k J x^k - f^k, x^k - z^k \rangle \le 0$. Thus, we deduce that

$$\begin{aligned}
E_k &\le \left\langle \psi^k - \xi^k, x^k - \bar{x} \right\rangle + \left\langle f^k - f, x^k - \bar{x} \right\rangle - \alpha_k \left\langle J x^k, x^k - \bar{x} \right\rangle \\
&+ \left\langle \xi^k + \alpha_k J x^k - f^k, z^k - \bar{x} \right\rangle \\
&\le \left(\left\| \psi^k - \xi^k \right\|_* + \left\| f^k - f \right\|_* + \alpha_k \left\| x^k \right\| \right) \left\| x^k - \bar{x} \right\| \\
&+ \left\langle \xi^k + \alpha_k J x^k - f^k, z^k - \bar{x} \right\rangle \\
&\le \alpha_k \left(\bar{c} + c_1 + \left\| x^k \right\| \right) \left\| x^k - \bar{x} \right\| + \left\langle \xi^k + \alpha_k J x^k - f^k, z^k - \bar{x} \right\rangle,
\end{aligned}$$

where the last inequality is due to **A2**. This inequality, (23), **A2**(ii) and (22) imply that

$$\begin{aligned}
r \left\| \psi^k - f \right\|_* &\le \alpha_k \left(\bar{c} + c_1 + \left\| x^k \right\| \right) \left\| x^k - \bar{x} \right\| + c \left(r + \left\| x^k - \bar{x} \right\| \right) \\
&+ \left\langle \xi^k - \psi^k, z^k - \bar{x} \right\rangle + \left\langle \psi^k + \alpha_k J x^k - f^k, z^k - \bar{x} \right\rangle \\
&\le \alpha_k \left(\bar{c} + c_1 + \left\| x^k \right\| \right) \left\| x^k - \bar{x} \right\| + c \left(r + \left\| x^k - \bar{x} \right\| \right) \\
&+ \left(\alpha_k \bar{c} + \left\| \psi^k - f^k \right\|_* + \alpha_k \left\| x^k \right\| \right) \left\| z^k - \bar{x} \right\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left(r - \left\| z^k - \bar{x} \right\| \right) \left\| \psi^k - f \right\|_* \\
&\le \alpha_k \left(\bar{c} + c_1 + \left\| x^k \right\| \right) \left\| x^k - \bar{x} \right\| + c \left(r + \left\| x^k - \bar{x} \right\| \right) \\
&+ \left(\alpha_k \bar{c} + \left\| f - f^k \right\|_* + \alpha_k \left\| x^k \right\| \right) \left\| z^k - \bar{x} \right\|,
\end{aligned}$$

for all $k \in \mathbb{N}$. Since the sequence $\{z^k\}_{k \in \mathbb{N}}$ converges to $\bar{x}$, there exists a positive integer k_0 such that, for any $k \ge k_0$, we have $r - \left\| z^k - \bar{x} \right\| > r/2$.

Hence, the last inequality shows that, if $k \geq k_0$, then

$$\begin{aligned}\left\|\psi^k - f\right\|_* &\leq 2r^{-1}\alpha_k\left(\bar{c} + c_1 + \left\|x^k\right\|\right)\left(\left\|x^k - \bar{x}\right\| + \left\|z^k - \bar{x}\right\|\right)\\ &\quad + 2r^{-1}c\left(r + \left\|x^k - \bar{x}\right\|\right),\end{aligned}$$

where the right hand side is bounded. By consequence, the sequence $\left\{\psi^k\right\}_{k\in\mathbb{N}}$ is bounded. Now, taking into account (22), we deduce that the sequence $\left\{\left\|\psi^k - \xi^k\right\|\right\}_{k\in\mathbb{N}}$ is also bounded and, therefore, $\left\{\xi^k\right\}_{k\in\mathbb{N}}$ is bounded too.□

We are now in position to prove Theorem 3.1(a). Clearly, Theorem 3.1(b) directly follows from Theorem 3.1(a).

LEMMA 4.3. *If the sequence* $\left\{x^k\right\}_{k\in\mathbb{N}}$ *is bounded, then it has weak accumulation points and any weak accumulation point of it is a solution of the variational inequality* (1).

Proof. Existence of weak accumulation points for the bounded sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is due to the reflexivity of X. Let x^* be a weak accumulation point of $\left\{x^k\right\}_{k\in\mathbb{N}}$. Let $\left\{x^{i_k}\right\}_{k\in\mathbb{N}}$ be a subsequence of $\left\{x^k\right\}_{k\in\mathbb{N}}$ which converges weakly to x^*. By **A2**(i), we have that $x^* \in \Omega$. Let x be an arbitrary element of Ω. By **A2**(i) again, there exists a sequence $\left\{z^k\right\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} z^k = x$ and $z^k \in \Omega_k$ for all $k \in \mathbb{N}$. Denote

$$F_k := F_k(x) = \left\langle \xi^k + \alpha_k J x^k - f^k, x - x^k \right\rangle. \tag{24}$$

Note that

$$\begin{aligned}F_k &= \left\langle \xi^k + \alpha_k J x^k - f^k, x - z^k \right\rangle + \left\langle \xi^k + \alpha_k J x^k - f^k, z^k - x^k \right\rangle \qquad (25)\\ &\geq \left\langle \xi^k + \alpha_k J x^k - f^k, x - z^k \right\rangle\\ &\geq -\left\|\xi^k + \alpha_k J x^k - f^k\right\|_* \left\|x - z^k\right\|,\end{aligned}$$

where the first inequality is due to (16). For any $\eta^k \in Ax^k$ and for any $\eta \in Ax$, we have

$$\begin{aligned}F_k &= \left\langle \xi^k - \eta, x - x^k \right\rangle + \left\langle \eta - f^k, x - x^k \right\rangle + \alpha_k \left\langle Jx^k, x - x^k \right\rangle \qquad (26)\\ &= \left\langle \xi^k - \eta^k, x - x^k \right\rangle + \left\langle \eta^k - \eta, x - x^k \right\rangle + \left\langle \eta - f, x - x^k \right\rangle\\ &\quad + \left\langle f - f^k, x - x^k \right\rangle + \alpha_k \left\langle Jx^k, x - x^k \right\rangle\\ &\leq \left\langle \xi^k - \eta^k, x - x^k \right\rangle + \left\langle \eta - f, x - x^k \right\rangle\\ &\quad + \left\langle f - f^k, x - x^k \right\rangle + \alpha_k \left\langle Jx^k, x - x^k \right\rangle,\end{aligned}$$

where the inequality holds because $\langle \eta^k - \eta, x - x^k \rangle \leq 0$ as A is monotone. This implies

$$F_k \leq \left\langle \eta - f, x - x^k \right\rangle + \left\| f - f^k \right\|_* \left\| x - x^k \right\| + \alpha_k \left\| x^k \right\| \left\| x - x^k \right\| + \left\langle \xi^k - \eta^k, x - x^k \right\rangle.$$

Combining this inequality with (25), we deduce that

$$\begin{aligned} &\left\langle \eta - f, x - x^k \right\rangle + \left\| f - f^k \right\|_* \left\| x - x^k \right\| \\ &+ \alpha_k \left\| x^k \right\| \left\| x - x^k \right\| + \left\langle \xi^k - \eta^k, x - x^k \right\rangle \\ &\geq - \left\| \xi^k + \alpha_k J x^k - f^k \right\|_* \left\| x - z^k \right\|. \end{aligned}$$

According to $\mathbf{A2}(ii)$, the sequence $\left\{\eta^k\right\}_{k\in\mathbb{N}}$ with $\eta^k \in Ax^k$ for all $k \in \mathbb{N}$, can be chosen such that

$$\lim_{k\to\infty} \left\| \xi^k - \eta^k \right\|_* = 0.$$

Hence, for any $k \in \mathbb{N}$, we have that

$$\begin{aligned} &\langle \eta - f, x - x^{i_k} \rangle + \left[\left\| f - f^{i_k} \right\|_* + \alpha_{i_k} \left\| x^{i_k} \right\| + \left\| \xi^{i_k} - \eta^{i_k} \right\|_* \right] \left\| x - x^{i_k} \right\| \\ &\geq - \left\| \xi^{i_k} + \alpha_{i_k} J x^{i_k} - f^{i_k} \right\|_* \left\| x - z^{i_k} \right\|, \end{aligned}$$

where the quantity between the square brackets, as well as the quantity on the right hand side, converge to zero as $k \to \infty$ (cf. Lemma 4.2). Therefore, taking on both sides of the last inequality the limit for $k \to \infty$, we obtain that $\langle \eta - f, x - x^* \rangle \geq 0$. Since x was arbitrarily chosen in Ω and η was arbitrarily chosen in Ax, Lemma 2.3 implies that x^* is a solution of the variational inequality (1). □

In conjunction with Lemma 4.1, the following result shows that Theorem 3.1(c) holds when condition (**C1**) is satisfied.

LEMMA 4.4. *If condition* **(C1)** *holds, then the sequence* $\left\{y^k\right\}_{k\in\mathbb{N}}$ *is bounded.*

Proof. Let $\bar{x}$ be any element of the set Ω. Since $\Omega \subseteq \operatorname{Int}(\operatorname{Dom} A)$ and the operator $Fx := Ax - f$ is monotone, we can apply Lemma 2.5 in order to deduce that there exist two real numbers $r = r(\bar{x}) > 0$ and $c \geq 0$ such that, for any $y \in \operatorname{Dom} A$ and for any $\zeta \in Ay$, we have

$$\langle \zeta - f, y - \bar{x} \rangle \geq r \left\| \zeta - f \right\|_* - (\left\| y - \bar{x} \right\| + r)\, c. \tag{27}$$

According to **A1**, $\Omega_k \subseteq \operatorname{Dom} A$ and, therefore, the inequality (27) still holds for $y = y^k \in \Omega_k$ and $\zeta = \zeta^k \in Ay^k$. In other words, we have that

$$\left\langle \zeta^k - f, y^k - \bar{x} \right\rangle \geq r \left\| \zeta^k - f \right\|_* - \left(\left\| y^k - \bar{x} \right\| + r \right) c, \quad \forall k \in \mathbb{N}. \tag{28}$$

By **A2**, there exists a sequence $\left\{ z^k \right\}_{k \in \mathbb{N}}$ in X such that $\lim_{k \to \infty} z^k = \bar{x}$ and $z^k \in \Omega_k$ for all $k \in \mathbb{N}$. Observe that, for each $k \in \mathbb{N}$, we have that

$$\begin{aligned}
\left\langle \zeta^k - f, y^k - \bar{x} \right\rangle &= \left\langle \zeta^k + \alpha_k J y^k - f, y^k - \bar{x} \right\rangle - \alpha_k \left\langle J y^k, y^k - \bar{x} \right\rangle \\
&\leq \left\langle \zeta^k + \alpha_k J y^k - f, y^k - z^k \right\rangle \\
&+ \left\langle \zeta^k + \alpha_k J y^k - f, z^k - \bar{x} \right\rangle \\
&- \alpha_k \left\| y^k \right\|^2 + \alpha_k \left\| y^k \right\| \left\| \bar{x} \right\|,
\end{aligned}$$

where $\left\langle \zeta^k + \alpha_k J y^k - f, y^k - z^k \right\rangle \leq 0$ because of (15). Thus, we obtain that

$$\left\langle \zeta^k - f, y^k - \bar{x} \right\rangle \leq \left\langle \zeta^k + \alpha_k J y^k - f, z^k - \bar{x} \right\rangle + \alpha_k \left\| y^k \right\| \left\| \bar{x} \right\|, \quad \forall k \in \mathbb{N}.$$

Combining this and (28) yields

$$\begin{aligned}
& r \left\| \zeta^k - f \right\|_* - \left(\left\| y^k - \bar{x} \right\| + r \right) c \\
& \leq \left\langle \zeta^k + \alpha_k J y^k - f, z^k - \bar{x} \right\rangle + \alpha_k \left\| y^k \right\| \left\| \bar{x} \right\| \\
& \leq \left\| z^k - \bar{x} \right\| \left\| \zeta^k - f \right\|_* + \alpha_k \left\| y^k \right\| \left(\left\| z^k - \bar{x} \right\| + \left\| \bar{x} \right\| \right),
\end{aligned}$$

for any $k \in \mathbb{N}$. This implies that

$$\begin{aligned}
& \left(r - \left\| z^k - \bar{x} \right\| \right) \left\| \zeta^k - f \right\|_* \\
& \leq \left\| y^k \right\| \left[\alpha_k \left(\left\| z^k - \bar{x} \right\| + \left\| \bar{x} \right\| \right) + c \right] + c \left(\left\| \bar{x} \right\| + r \right).
\end{aligned}$$

Since the sequence $\left\{ z^k \right\}_{k \in \mathbb{N}}$ converges to $\bar{x}$, there exists a positive integer k_0 such that for any integer $k \geq k_0$ we have that $r - \left\| z^k - \bar{x} \right\| > r/2$. Thus, for any $k \geq k_0$, we have

$$\begin{aligned}
& \frac{r}{2} \left(\left\| \zeta^k \right\|_* - \left\| f \right\|_* \right) \\
& \leq \left\| y^k \right\| \left[\alpha_k \left(\left\| z^k - \bar{x} \right\| + \left\| \bar{x} \right\| \right) + c \right] + c \left(\left\| \bar{x} \right\| + r \right),
\end{aligned}$$

where the quantity between the square brackets is bounded by some positive real number ρ. Hence, for any $k \geq k_0$, we have

$$\left\|\zeta^k\right\|_* \leq a\left\|y^k\right\| + b, \tag{29}$$

where

$$a := 2r^{-1}\rho \text{ and } b := 2cr^{-1}\left(\|\bar{x}\| + r\right) + \|f\|_* . \tag{30}$$

Suppose by contradiction that the sequence $\{y^k\}_{k\in\mathbb{N}}$ is unbounded. Then there exists a subsequence $\{y^{i_k}\}_{k\in\mathbb{N}}$ of $\{y^k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\left\|y^{i_k}\right\| = \infty$. Let $k_1 \geq k_0$ be an integer such that, for any $k \geq k_0$, we have $b \leq \left\|y^{i_k}\right\|$. By (29), we have that

$$\frac{\|\zeta^{i_k}\|_*}{\|y^{i_k}\|} \leq a + \frac{b}{\|y^{i_k}\|} \leq a + 1,$$

whenever $k \geq k_0$. In other words, taking $\beta := a+1$, we have that $y^{i_k} \in L_\beta(A)$ for any integer $k \geq k_0$. According to condition (**C1**), the set $L_\beta(A)$ is bounded and this contradicts the unboundedness of the sequence $\{y^{i_k}\}_{k\in\mathbb{N}}$.□

The next lemma, combined with Lemma 4.1, shows that Theorem 3.1(c) is true when condition (**C2**) is satisfied.

LEMMA 4.5. *Suppose that the operators* A_k, $(k \in \mathbb{N})$, *satisfy condition* (**C2**). *Then, the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *is bounded.*

Proof. From Lemma 2.4 we know that

$$\mathcal{S}(A_k + \alpha_k J, \Omega_k, f^k) = \mathcal{S}(\bar{A}_k + \alpha_k J, \Omega_k, f^k),$$

where $\bar{A}_k$ is a maximal monotone extension of A_k. According to Lemma 2.1, the only element in $\mathcal{S}(\bar{A}_k + \alpha_k J, \Omega_k, f^k)$ is the vector

$$x^k = \left(\bar{A}_k + N_{\Omega_k} + \alpha_k J\right)^{-1} f^k. \tag{31}$$

According to (31), for each $k \in \mathbb{N}$, there exist $\chi^k \in \bar{A}_k x^k$ and $\eta^k \in N_{\Omega_k} x^k$ such that

$$f^k = \chi^k + \eta^k + \alpha_k J x^k. \tag{32}$$

Let $\{v^k\}_{k\in\mathbb{N}}$ be the sequence in X whose existence follows from condition (**C2**). By (**C2**) it results that the sets $A_k v^k$ are nonempty. Also, for each

$k \in \mathbb{N}$, the set $\bar{A}_k v^k$ is convex and closed and contains $A_k v^k$. Hence, there exists $\gamma^k \in \bar{A}_k v^k$ such that

$$\left\| \gamma^k - f^k \right\|_* = \mathrm{dist}_*(f^k, \bar{A}_k v^k). \tag{33}$$

Taking into account that $\bar{A}_k$ is monotone, $\chi^k \in \bar{A}_k x^k$ and $\gamma^k \in \bar{A}_k v^k$, we deduce that $\langle \chi^k - \gamma^k, x^k - v^k \rangle \geq 0$. Hence,

$$\begin{aligned} \left\langle \gamma^k, x^k - v^k \right\rangle &\leq \left\langle \chi^k, x^k - v^k \right\rangle = \left\langle f^k - \alpha_k J x^k - \eta_k, x^k - v^k \right\rangle \\ &= \left\langle f^k - \alpha_k J x^k, x^k - v^k \right\rangle + \left\langle \eta_k, v^k - x^k \right\rangle \\ &\leq \left\langle f^k, x^k - v^k \right\rangle - \alpha_k \left\langle J x^k, x^k \right\rangle + \alpha_k \left\langle J x^k, v^k \right\rangle \\ &= \left\langle f^k, x^k - v^k \right\rangle - \alpha_k \left\| x^k \right\|^2 + \alpha_k \left\langle J x^k, v^k \right\rangle \\ &\leq \left\langle f^k, x^k - v^k \right\rangle - \alpha_k \left\| x^k \right\|^2 + \alpha_k \left\| v^k \right\| \left\| x^k \right\|, \end{aligned}$$

where the first equality follows from (32), the second inequality results from the fact that $\langle \eta_k, v^k - x^k \rangle \leq 0$ (which is true because $v^k \in \Omega_k$ and $\eta_k \in N_{\Omega_k}(x^k)$) and the second equality, as well as the last inequality, follow from the properties of J. By consequence, we have

$$\begin{aligned} \alpha_k \left\| x^k \right\| \left(\left\| x^k \right\| - \left\| v^k \right\| \right) &\leq \left\langle f^k - \gamma^k, x^k - v^k \right\rangle \\ &\leq \left\| f^k - \gamma^k \right\|_* \left\| x^k \right\| + \left\| f^k - \gamma^k \right\|_* \left\| v^k \right\| \end{aligned} \tag{34}$$

for all $k \in \mathbb{N}$. Suppose, by contradiction, that $\{x^k\}_{k\in\mathbb{N}}$ is unbounded. Then, for some subsequence $\{x^{i_k}\}_{k\in\mathbb{N}}$ of it we have that $\lim_{k\to\infty} \left\| x^{i_k} \right\| = +\infty$. From (34) we deduce that, for sufficiently large k,

$$\left\| x^{i_k} \right\| - \left\| v^{i_k} \right\| \leq \frac{1}{\alpha_{i_k}} \left\| f^{i_k} - \gamma^{i_k} \right\|_* \left(1 + \frac{\left\| v^{i_k} \right\|}{\left\| x^{i_k} \right\|} \right). \tag{35}$$

According to (33) and the hypothesis, the sequence $\{\alpha_k^{-1} \left\| f^k - \gamma^k \right\|_*\}_{k\in\mathbb{N}}$ is bounded because

$$\alpha_k^{-1} \left\| f^k - \gamma^k \right\|_* \leq \alpha_k^{-1} \mathrm{dist}_*(A_k v^k, f^k).$$

Taking on both sides of (35) the upper limit as $k \to \infty$ we obtain that the limit on the left hand side is $+\infty$ while that on the right hand side is finite, that is, a contradiction. This shows that $\{x^k\}_{k\in\mathbb{N}}$ is bounded. □

The next lemma completes the proof of Theorem 3.1.

LEMMA 4.6. *Suppose that the operator A satisfies condition* (**C3**). *Then, the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded.*

Proof. According to Lemma 4.1, it is sufficient to show that the sequence $\{y^k\}_{k\in\mathbb{N}}$ is bounded. Let $\bar{x}$ be any vector in $\bigcap_{k\in\mathbb{N}}\Omega_k$ at which the operator A is coercive. Such a vector exists by (**C3**). It follows from (15) that, for each $k\in\mathbb{N}$, we have

$$\left\langle \zeta^k, y^k - \bar{x}\right\rangle + \alpha_k \left\langle Jy^k, y^k - \bar{x}\right\rangle \le \left\langle f, y^k - \bar{x}\right\rangle .$$

Therefore, we have

$$\begin{aligned} &\left\langle \zeta^k, y^k - \bar{x}\right\rangle - \alpha_k \left\langle Jy^k, \bar{x}\right\rangle \qquad (36)\\ &\le \left\langle \zeta^k, y^k - \bar{x}\right\rangle + \alpha_k \left\|y^k\right\|^2 - \alpha_k \left\langle Jy^k, \bar{x}\right\rangle\\ &= \left\langle \zeta^k, y^k - \bar{x}\right\rangle + \alpha_k \left\langle Jy^k, y^k - \bar{x}\right\rangle\\ &\le \|f\|_* \left(\left\|y^k\right\| + \|\bar{x}\|\right) . \end{aligned}$$

Suppose, by contradiction, that the sequence $\{y^k\}_{k\in\mathbb{N}}$ is unbounded. Then there exists a subsequence $\{y^{i_k}\}_{k\in\mathbb{N}}$ of $\{y^k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\left\|y^{i_k}\right\| = \infty$. Hence, according to (36), there exists a positive integer k_0 such that, for any integer $k \ge k_0$, we have

$$\frac{\left\langle \zeta^{i_k}, y^{i_k} - \bar{x}\right\rangle}{\|y^{i_k}\|} \le \|f\|_* + \frac{\|f\|_* \|\bar{x}\|}{\|y^{i_k}\|} + \alpha \|\bar{x}\| ,$$

where $\alpha := \sup_{k\in\mathbb{N}}\alpha_k$. Letting here $k\to\infty$ and taking into account (**C3**) we get a contradiction. □

We have already shown that, under the hypothesis of Theorem 3.1, if the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded, then all its weak accumulation points are solutions of the variational inequality (1). Suppose, from now on, that we are under the stronger hypothesis of Theorem 3.2. We are going to show that, in these circumstances, if the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded, then it has a unique weak accumulation point which is exactly the minimal-norm solution of the variational inequality (1). This proves Theorem 3.2(a).

LEMMA 4.7. *Suppose that assumptions **A1** and **A3** are satisfied. If the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded, then it converges weakly to the minimal-norm solution of the variational inequality* (1).

Proof. It was noted in Section 3 that, under the hypothesis of Theorem 3.2, the set of solutions of the variational inequality (1) is convex, closed and nonempty and, therefore, it contains a minimal-norm element $\bar{x}$. We are going to show that the only weak accumulation point of the sequence $\{x^k\}_{k\in\mathbb{N}}$ is $\bar{x}$. To this end, let x^* be an arbitrary weak accumulation point of the sequence $\{x^k\}_{k\in\mathbb{N}}$ and let $\{x^{i_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{x^k\}_{k\in\mathbb{N}}$ whose weak limit is x^*. According to $\mathbf{A3}(i)$, there exists a sequence $\{z^k\}_{k\in\mathbb{N}}$ in X such that

$$\lim_{k\to\infty} \alpha_k^{-1} \left\| \bar{x} - z^k \right\| = 0 \text{ and } z^k \in \Omega_k \text{ for all } k \in \mathbb{N}. \tag{37}$$

Let $G_k := F_k(\bar{x})$, where $F_k(x)$ is given by (24). Reproducing the reasonings made in (25) with $\bar{x}$ instead of x we deduce that

$$G_k \geq - \left\| A_k x^k - f^k + \alpha_k J x^k \right\|_* \left\| \bar{x} - z^k \right\|. \tag{38}$$

Take $\eta \in A\bar{x}$ such that

$$\langle \eta - f, x - \bar{x} \rangle \geq 0, \quad \forall x \in \Omega. \tag{39}$$

Such a vector exists because $\bar{x}$ is a solution of the variational inequality (1). Taking into account $\mathbf{A3}(ii)$, one can follow the pattern of the arguments involved in the proof of (17) in order to show that for each $k \in \mathbb{N}$, there exists $\eta^k \in Ax^k$ such that

$$\left\| \xi^k - \eta^k \right\|_* \leq c_2 \beta_k, \tag{40}$$

where $c_2 := \sup_{k\in\mathbb{N}} p(x^k) < \infty$ and $\lim_{k\to\infty} \beta_k = 0$. One can reproduce the reasonings in (26) in order to deduce that

$$\begin{aligned} G_k &\leq \left\langle \xi^k - \eta^k, \bar{x} - x^k \right\rangle + \left\langle \eta - f, \bar{x} - x^k \right\rangle \\ &\quad + \left\langle f - f^k, \bar{x} - x^k \right\rangle + \alpha_k \left\langle J x^k, \bar{x} - x^k \right\rangle. \end{aligned}$$

This and (38) imply that

$$\begin{aligned} &- \left\| A_k x^k - f^k + \alpha_k J x^k \right\|_* \left\| \bar{x} - z^k \right\| \\ &\leq \left\langle \xi^k - \eta^k, \bar{x} - x^k \right\rangle + \left\langle \eta - f, \bar{x} - x^k \right\rangle \\ &+ \left\langle f - f^k, \bar{x} - x^k \right\rangle + \alpha_k \left\langle J x^k, \bar{x} - x^k \right\rangle \\ &\leq \left(\left\| \xi^k - \eta^k \right\|_* + \left\| f - f^k \right\|_* \right) \left\| \bar{x} - x^k \right\| \\ &+ \left\langle \eta - f, \bar{x} - x^k \right\rangle + \alpha_k \left\langle J x^k, \bar{x} - x^k \right\rangle. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\alpha_k \left\langle Jx^k, x^k - \bar{x} \right\rangle \\
&\leq \left\| A_k x^k - f^k + \alpha_k J x^k \right\|_* \left\| \bar{x} - z^k \right\| \\
&+ \left(\left\| \xi^k - \eta^k \right\|_* + \left\| f - f^k \right\|_* \right) \left\| \bar{x} - x^k \right\| + \left\langle \eta - f, \bar{x} - x^k \right\rangle,
\end{aligned}$$

for all $k \in \mathbb{N}$. Since $\langle Jx^k, x^k - \bar{x} \rangle \geq \|x^k\|^2 - \|x^k\| \|\bar{x}\|$, we obtain

$$\begin{aligned}
&\left\| x^k \right\|^2 - \left\| x^k \right\| \|\bar{x}\| \\
&\leq \alpha_k^{-1} \left\| \bar{x} - z^k \right\| \left\| A_k x^k - f^k + \alpha_k J x^k \right\|_* \\
&+ \alpha_k^{-1} \left(\left\| \xi^k - \eta^k \right\|_* + \left\| f - f^k \right\|_* \right) \left\| \bar{x} - x^k \right\| \\
&+ \alpha_k^{-1} \left\langle \eta - f, \bar{x} - x^k \right\rangle.
\end{aligned} \tag{41}$$

The sequence $\{x^{i_k}\}_{k \in \mathbb{N}}$ converges weakly and $x^{i_k} \in \Omega_{i_k}$ for all $k \in \mathbb{N}$. Thus, by $\mathbf{A3}(i)$, we deduce that there exists a sequence $\{u^k\}_{k \in \mathbb{N}} \subseteq \Omega$ having the property that the sequence $\left\{ \alpha_{i_k}^{-1} (x^{i_k} - u^k) \right\}_{k \in \mathbb{N}}$ converges weakly to zero. Therefore, we have that

$$\lim_{k \to \infty} \alpha_{i_k}^{-1} \left\langle \eta - f, u^k - x^{i_k} \right\rangle = 0. \tag{42}$$

Using the inequality (41), we obtain that

$$\begin{aligned}
&\|x^{i_k}\|^2 - \|x^{i_k}\| \|\bar{x}\| \\
&\leq \alpha_{i_k}^{-1} \|\bar{x} - z^{i_k}\| \|A_{i_k} x^{i_k} - f^{i_k} + \alpha_{i_k} J x^{i_k}\|_* \\
&+ \left(\alpha_{i_k}^{-1} \|\xi^{i_k} - \eta^{i_k}\|_* + \alpha_{i_k}^{-1} \|f - f^{i_k}\|_* \right) \|\bar{x} - x^{i_k}\| \\
&+ \alpha_{i_k}^{-1} \left\langle \eta - f, \bar{x} - u^k \right\rangle + \alpha_{i_k}^{-1} \left\langle \eta - f, u^k - x^{i_k} \right\rangle,
\end{aligned}$$

where $\alpha_k^{-1} \langle \eta - f, \bar{x} - u^k \rangle \leq 0$ because of (39). This yields

$$\begin{aligned}
&\|x^{i_k}\|^2 - \|x^{i_k}\| \|\bar{x}\| \\
&\leq \alpha_{i_k}^{-1} \|\bar{x} - z^{i_k}\| \|A_{i_k} x^{i_k} - f^{i_k} + \alpha_{i_k} J x^{i_k}\|_* \\
&+ \left(\alpha_{i_k}^{-1} \|\xi^{i_k} - \eta^{i_k}\|_* + \alpha_{i_k}^{-1} \|f - f^{i_k}\|_* \right) \|\bar{x} - x^{i_k}\| \\
&+ \alpha_{i_k}^{-1} \left\langle \eta - f, u^k - x^{i_k} \right\rangle.
\end{aligned} \tag{43}$$

By $\mathbf{A3}(ii)$, $\mathbf{A3}(iii)$ and (40) we also have that

$$\lim_{k\to\infty} \alpha_k^{-1} \left\| f - f^k \right\|_* = 0 = \lim_{k\to\infty} \alpha_k^{-1} \left\| \xi^k - \eta^k \right\|_* .$$

This and (42) show that by taking on both sides of the inequality (43) the upper limit as $k \to \infty$ we obtain

$$\limsup_{k\to\infty} \left(\left\| x^{i_k} \right\|^2 - \|\bar{x}\| \left\| x^{i_k} \right\| \right) \le 0.$$

By consequence, we have

$$\begin{aligned} \left(\liminf_{k\to\infty} \left\| x^{i_k} \right\| \right)^2 &\le \liminf_{k\to\infty} \left\| x^{i_k} \right\|^2 \\ &\le \limsup_{k\to\infty} \left\| x^{i_k} \right\|^2 \le \|\bar{x}\| \liminf_{k\to\infty} \left\| x^{i_k} \right\| . \end{aligned} \tag{44}$$

If $\liminf_{k\to\infty} \left\| x^{i_k} \right\| = 0$, then $\|x^*\| = 0$ (by the weak lower semicontinuity of $\|\cdot\|$) and this implies that $\|x^*\| = 0 = \|\bar{x}\|$ because $\bar{x}$ is the minimal-norm solution of (1). Thus, in this case, $x^* = 0 = \bar{x}$. If $\liminf_{k\to\infty} \left\| x^{i_k} \right\| > 0$, then (44) implies that

$$\|x^*\| \le \liminf_{k\to\infty} \left\| x^{i_k} \right\| \le \|\bar{x}\| ,$$

and this can not hold unless $x^* = \bar{x}$ because $\bar{x}$ is the unique minimal-norm solution of (1). Hence, in all possible situations, $x^* = \bar{x}$. Since x^* is an arbitrary accumulation point of $\left\{x^k\right\}_{k\in\mathbb{N}}$ it results that the unique weak accumulation point of the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is $\bar{x}$. □

Up to this point in our proofs we did not need the assumption that the space X has the Kadeč-Klee property. We are going to use this feature of the space for establishing strong convergence of $\left\{x^k\right\}_{k\in\mathbb{N}}$. In this way we prove Theorem 3.2(b). Theorem 3.2(c) follows directly from Theorem 3.1(c) combined with Theorem 3.2(b).

LEMMA 4.8. *Suppose that the space X has the Kadeč-Klee property and that assumptions **A1** and **A3** are satisfied. If the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ is bounded, then it converges strongly to the minimal-norm solution of the variational inequality* (1).

Proof. According to Lemma 4.7, the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ converges weakly to the minimal-norm solution $\bar{x}$ of the variational inequality (1). Hence, since the space X has the Kadeč-Klee property, it is sufficient to prove that

$\lim_{k\to\infty} \left\| x^k \right\| = \|\bar{x}\|$. Observe that the sequence $\left\{\left\| x^k \right\|\right\}_{k\in\mathbb{N}}$ is bounded and, therefore, for proving that $\lim_{k\to\infty} \left\| x^k \right\| = \|\bar{x}\|$, it is sufficient to show that the only accumulation point of the sequence of numbers $\left\{\left\| x^k \right\|\right\}_{k\in\mathbb{N}}$ is $\|\bar{x}\|$. To this end, let $\left\{x^{i_k}\right\}_{k\in\mathbb{N}}$ be a subsequence of $\left\{x^k\right\}_{k\in\mathbb{N}}$ such that $\left\{\left\| x^{i_k} \right\|\right\}_{k\in\mathbb{N}}$ converges. The inequality (43) still holds and it yields

$$\limsup_{k\to\infty} \left(\left\| x^{i_k} \right\|^2 - \|\bar{x}\| \left\| x^{i_k} \right\| \right) \leq 0.$$

Taking into account that $\left\{x^{i_k}\right\}_{k\in\mathbb{N}}$ converges weakly to $\bar{x}$ (cf. Lemma 4.7) and that $\left\{\left\| x^{i_k} \right\|\right\}_{k\in\mathbb{N}}$ converges, we deduce that

$$\begin{aligned} \|\bar{x}\|^2 &\leq \left(\liminf_{k\to\infty} \left\| x^{i_k} \right\| \right)^2 = \left(\lim_{k\to\infty} \left\| x^{i_k} \right\| \right)^2 \\ &= \lim_{k\to\infty} \left\| x^{i_k} \right\|^2 = \limsup_{k\to\infty} \left\| x^{i_k} \right\|^2 \\ &\leq \|\bar{x}\| \liminf_{k\to\infty} \left\| x^{i_k} \right\| = \|\bar{x}\| \lim_{k\to\infty} \left\| x^{i_k} \right\| . \end{aligned} \tag{45}$$

If $\lim_{k\to\infty} \left\| x^{i_k} \right\| = 0$, then (45) implies that $\|\bar{x}\| = 0$ and, then, we have $\lim_{k\to\infty} \left\| x^{i_k} \right\| = 0 = \|\bar{x}\|$. If $\lim_{k\to\infty} \left\| x^{i_k} \right\| > 0$, then

$$\|\bar{x}\| \leq \liminf_{k\to\infty} \left\| x^{i_k} \right\| \leq \|\bar{x}\| ,$$

where the last inequality is due to (45). Thus, $\lim_{k\to\infty} \left\| x^{i_k} \right\| = \|\bar{x}\|$ in this case too and that completes the proof. □

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