Existence and Approximation of Solutions for Fredholm Equations of the First Kind with Applications to a Linear Moment Problem

Dan Butnariu and Ben Zion Schlyar

Abstract. The Cimmino algorithm is an interative projection method for finding almost common points of measurable families of closed convex sets in a Hilbert space. When applied to Fredholm equations of the first kind the Cimmino algorithm produces weak approximations of solutions provided that solutions exist. We show that for consistent Fredholm equations of the first kind whose data satisfy some spectral conditions the sequences produced by the Cimmino algorithm converge not only weakly but also in norm. Using this fact we obtain an existence criterion for solutions to a class of moment problems and show that if problems in this class have solutions, then the Cimmino algorithm generate norm approximations of such solutions.

1. Introduction

In this paper we study the convergence behavior of the Cimmino algorithm applied to a class of Fredholm equations of the first kind in a separable, real or complex, Hilbert space X. Our aim is to use the properties of this algorithm in order to deduce existence and strong approximation criteria for solutions of a class of Fredholm equations and, in particular, for a class of moment problems. In its general form the Cimmino algorithm (see (2.1) and (2.7)) is an iterative method of approximating solutions to stochastic convex feasibility problems, i.e., problems of finding almost common points of measurable families of closed convex subsets in X. The basic convergence properties of the Cimmino algorithm are summarized in Theorem 2.1. Theorem 2.1 shows that the Cimmino algorithm produces sequences in X which are weakly convergent to solutions of the stochastic convex feasibility problem provided that such solutions exist. It is known (see [**4**]) that, even if solutions of the stochastic convex feasibility problem exist, the sequences generated by the Cimmino algorithm may converge weakly without being strongly convergent. Theorem 2.1(D) also shows that, if the solution set of the convex feasibility problem is solid, then the sequences produced by the Cimmino algorithm converge strongly. The Fredholm equations of the first kind we consider here (see (2.11)) are among

2000 *Mathematics Subject Classification.* Primary: 45B05, 44A60; Secondary: 45A05, 49M99.

Key words and phrases. almost common point, asymptotic center of a sequence, Cimmino type algorithm, discrete linear moment problem, eigenvalue of a linear operator, Fredholm equation of the first kind, Gram matrix, projection method, spectrum of a linear operator.

the most typical stochastic convex feasibility problems for which the Cimmino algorithm is applicable and relatively easy implementable (see Corollary 2.1 which summarizes the special features of the Cimmino algorithm applied to such equations). However, their solution sets are never solid and, therefore, Theorem 2.1(D) is no guarantee that the Cimmino algorithm will produce norm approximations (as opposed to weak approximations) of solutions for Fredholm equations of the first kind. Theorem 2.2 presents a sufficient condition for strong convergence of the Cimmino algorithm when the Fredholm equation to which it is applied is consistent. The condition which ensures strong convergence in Theorem 2.2 requires positivity of the minimal spectral value of the linear, bounded and self-adjoint mapping M defined by (2.17). It is interesting to note that Theorem 2.2 also provides a tool for deciding whether the Fredholm equation has solutions. It essentially says that, in the given circumstances, by computing a sequence $\{x^k\}_{k\in\mathbb{N}}$ generated according to the Cimmino procedure and evaluating the corresponding nonnegative numerical sequence $\{\mathbf{g}(x^k)\}_{k\in\mathbb{N}}$, where $\mathbf{g}$ is the function given by (2.15), one can decide whether the Fredholm equation has, or has not, solutions by observing whether or not $\{\mathbf{g}(x^k)\}_{k\in\mathbb{N}}$ converges to zero. When $\{\mathbf{g}(x^k)\}_{k\in\mathbb{N}}$ converges to zero, solutions of the Fredholm equation exist and the convergence of $\{x^k\}_{k\in\mathbb{N}}$ to such a solution is strong.

Our interest in the behavior of the Cimmino algorithm applied to Fredholm equations of the first kind is motivated by the fact that many discrete linear moment problems occurring in mathematical statistics (cf. [**1**], [**21**], [**22**], [**25**]), image processing (cf. [**15**], [**16**], [**23**]), optimal control (cf. [**6**]) can be equivalently represented as Fredholm equations of the first kind and solved as such. Discrete linear moment problems (DLMP for short) require solving a system of equations

$$\langle K_k, x\rangle = b_k, \quad \forall k \in \mathbb{N}, \tag{1.1}$$

where $\{K_k\}_{k\in\mathbb{N}}$ is a sequence of vectors contained in $X\backslash\{0\}$ and $\{b_k\}_{k\in\mathbb{N}}$ is a sequence of scalars. Using Theorem 2.2 we prove an existence criterion for solutions of the DLMP. This criterion, Theorem 3.1, shows that if Assumptions 3.1 and 3.2, given below, are satisfied, then the DLMP has solutions and that the Cimmino algorithm produces strong approximations of such solutions. As noted above, the Cimmino algorithm is relatively easy to implement when applied to Fredholm equations of the first kind and, in particular, to DLMPs in their Fredholm equation form. However, implementation of the Cimmino algorithm to the discrete linear moment problems still requires precise computation of sums for infinite series in X (see (3.18)) and this is not practically doable. Corollary 3.1 shows that, under Assumptions 3.1, 3.2 and a certain boundedness condition concerning the problem data, the Cimmino algorithm applied to the DLMP produces strong approximations of solutions to the DLMP by computing sufficiently long, but finite, partial sums of the series involved.

The Cimmino algorithm studied in this paper originates in the iterative method for solving finite systems of linear equations in $\mathbb{R}^n$ presented in [**14**]. It belongs to a wider class of iterative methods for solving feasibility problems known as projection algorithms (see [**3**] for a survey on this topic). The applicability of the Cimmino algorithm to Fredholm equations in Hilbert spaces was first shown by Kammerer and Nashed [**24**], further studied in [**7**], [**10**], [**18**] and extrapolated to a more general context in [**9**] and [**11**]. Theorem 2.1 and its Corollary 2.1 are improvements of

Theorem 5.7 in [**9**] when applied in a Hilbert space context and, in particular, to Fredholm equations of the first kind. As far as we know, Theorem 2.2, Theorem 3.1 and Corollary 3.1 are essentially new. Although spectral analysis of linear operators is commonly used in the study of Fredholm equations of the first kind and of discrete linear moment problems, this paper seems to be the first attempt to use spectral analysis of linear operators in the study of the Cimmino algorithm. It is interesting to note that in the proof of Corollary 3.1 we take advantage of results established in [**12**] and showing that the convergence behavior of orbits of nonexpansive operators is not altered by summable errors. Corollary 3.1 is yet another argument in support of the thesis that projections methods are computationally "robust" (see [**17**]).

2. Fredholm equations of the first kind and the Cimmino type algorithm

In this paper X denotes a separable real or complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $(\Omega,\mathcal{A},\boldsymbol{\mu})$ be a complete probability space and let $\{C_\omega\}_{\omega\in\Omega}$ be a family of nonempty closed convex subsets of the Hilbert space X such that the point-to-set mapping $\omega\to C_\omega$ is measurable. We say that the family $\{C_\omega\}_{\omega\in\Omega}$ is *square-integrable* (with respect to the probability space $(\Omega,\mathcal{A},\boldsymbol{\mu})$) if it has a square-integrable selector, that is, if there exists a measurable function $\xi:\Omega\to X$ such that $\|\xi(\cdot)\|^2$ is integrable and $\xi(\omega)\in C_\omega$ for μ-almost all $\omega\in\Omega$. In this case, for each $x\in X$, the function $\omega\to\|P_\omega x\|^2$ is integrable when P_ω denotes the metric projection onto the set C_ω (see [**8**, Chapter 2]). Consequently, the operator $\mathbf{P}:X\to X$ given by

$$\mathbf{P}x=\int_\Omega (P_\omega x)d\boldsymbol{\mu}(\omega), \tag{2.1}$$

as well as the the function $\mathbf{g}:X\to[0,\infty]$ given by

$$\mathbf{g}(x)=\tfrac{1}{2}\textstyle\int_\Omega \|P_\omega x-x\|^2\,d\boldsymbol{\mu}(\omega), \tag{2.2}$$

are well-defined and $\mathbf{g}$ is everywhere finite. We denote by C the collection of $\boldsymbol{\mu}$-*almost common points of the sets* C_ω, that is,

$$C:=\{x\in X: x\in C_\omega,\ \boldsymbol{\mu}\text{-a.e.}\}. \tag{2.3}$$

Clearly, this set is convex and closed.

Using the facts that the functions $g_\omega(x)=\|P_\omega x-x\|^2$ are convex and continuously differentiable (cf. [**2**, p. 24]) with the gradients

$$\nabla g_\omega(x)=x-P_\omega(x),\ \forall x\in X,$$

one can easily deduce (by applying Lebesgue's monotone convergence theorem) that $\mathbf{g}$ is convex, continuously differentiable and its gradient is

$$\nabla\mathbf{g}(x)=x-\mathbf{P}(x),\ \forall x\in X. \tag{2.4}$$

Taking into account that each P_ω is a nonexpansive operator which satisfies

$$z\in C_\omega \text{ and } x\in X\Rightarrow \|z-P_\omega x\|^2+\|x-P_\omega x\|^2\le\|z-x\|^2, \tag{2.5}$$

one can easily deduce that $\mathbf{P}$ is nonexpansive and that

$$z\in C \text{ and } x\in X\Rightarrow \|z-\mathbf{P}x\|^2+\|x-\mathbf{P}x\|^2\le\|z-x\|^2. \tag{2.6}$$

An *orbit* of the operator $\mathbf{P}$ is a sequence $\{x^k\}_{k\in\mathbb{N}}$ in X such that

$$x^0 \in X \quad \text{and} \quad x^{k+1} = \mathbf{P}x^k, \ \forall k \in \mathbb{N}. \tag{2.7}$$

The following result shows that there exist natural connections between the convergence behavior of the orbits of $\mathbf{P}$, the set of minimizers of $\mathbf{g}$ and the set C of almost common points of the sets C_ω.

Theorem 2.1. *If $\{C_\omega\}_{\omega\in\Omega}$ is a square-integrable family of nonempty, closed, convex subsets of X, then the following statements are true:*

(A) *The next five conditions are equivalent:*
- (i) *The set* $\operatorname{Arg\,min}\mathbf{g}$ *of (global) minimizers of the function* $\mathbf{g}$ *is nonempty;*
- (ii) *The set* $\operatorname{Fix}\mathbf{P}$ *is nonempty;*
- (iii) *All orbits* $\{x^k\}_{k\in\mathbb{N}}$ *of* $\mathbf{P}$ *converge weakly to points in* $\operatorname{Arg\,min}\mathbf{g}$*;*
- (iv) *All orbits* $\{x^k\}_{k\in\mathbb{N}}$ *of* $\mathbf{P}$ *are bounded;*
- (v) *The operator* $\mathbf{P}$ *has a bounded orbit* $\{x^k\}_{k\in\mathbb{N}}$.

(B) *If the set C of $\boldsymbol{\mu}$-almost common points of C_ω is nonempty, then the conditions above are also equivalent to the following one:*
- (vi) *All the orbits* $\{x^k\}_{k\in\mathbb{N}}$ *of* $\mathbf{P}$ *converge weakly to points in* C.

(C) *If $\{x^k\}_{k\in\mathbb{N}}$ is a bounded orbit of $\mathbf{P}$, then the weak limit $z = w\text{-}\lim_{k\to\infty} x^k$ exists and the following conditions are satisfied:*
- (vii) *The sequence* $\{\mathbf{g}(x^k)\}_{k\in\mathbb{N}}$ *converges to* $\mathbf{g}(z) = \inf_{x\in X}\ \mathbf{g}(x)$*;*
- $(viii)$ *If* $\inf_{x\in X}\ \mathbf{g}(x) = 0$, *then* $z \in C$.

(D) *If* $\operatorname{Arg\,min}\mathbf{g}$ $(= \operatorname{Fix}\mathbf{P})$ *has nonempty interior, then the any orbit $\{x^k\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ converges strongly to a point in* $\operatorname{Arg\,min}\mathbf{g}$. *In particular, if C has nonempty interior, then any orbit $\{x^k\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ converges strongly to a point in C.*

Proof. From (2.4) and from the convexity and differentiability of $\mathbf{g}$ it follows that $\operatorname{Arg\,min}\mathbf{g} = \operatorname{Fix}\mathbf{P}$. So, $\boxed{(i)\Leftrightarrow(ii)}$. The implication $\boxed{(ii)\Rightarrow(iii)}$ results from [**8**, Theorem 2.3.6] by taking there $f = \frac{1}{2}\|\cdot\|^2$. The implications $\boxed{(iii)\Rightarrow(iv)}$ and $\boxed{(iv)\Rightarrow(v)}$ are obvious.

We prove next that $\boxed{(v)\Rightarrow(ii)}$. Suppose that the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded. We associate to any bounded sequence $\{u^k\}_{k\in\mathbb{N}}$ in X, the convex function $F(\{u^k\}_{k\in\mathbb{N}};\cdot) : X \to \mathbb{R}$ defined by

$$F(\{u^k\}_{k\in\mathbb{N}};x) = \limsup_{k\to\infty}\left\|x - u^k\right\|.$$

The function $F(\{u^k\}_{k\in\mathbb{N}};\cdot)$ is convex and has a unique global minimizer called the asymptotic center of the sequence $\{u^k\}_{k\in\mathbb{N}}$. If $\{u^k\}_{k\in\mathbb{N}}$ is weakly convergent, then its asymptotic center and its weak limit coincide (cf. [**20**]). By the definition of $F(\{x^k\}_{k\in\mathbb{N}};\cdot)$, we have

$$F(\{x^{k+1}\}_{k\in\mathbb{N}};\cdot) = F(\{x^k\}_{k\in\mathbb{N}};\cdot). \tag{2.8}$$

Let z be the asymptotic center of $\{x^k\}_{k\in\mathbb{N}}$. By (2.8) and (2.7) we deduce that

$$\begin{aligned} F(\{x^k\}_{k\in\mathbb{N}};\mathbf{P}z) &= F(\{\mathbf{P}x^k\}_{k\in\mathbb{N}};\mathbf{P}z) \\ &= \limsup_{k\to\infty}\left\|\mathbf{P}x^k-\mathbf{P}z\right\| \\ &\leq \limsup_{k\to\infty}\left\|x^k-z\right\| = F(\{x^k\}_{k\in\mathbb{N}};z), \end{aligned}$$

where the last inequality is due to the nonexpansivity of $\mathbf{P}$. Since, as noted above, z is the unique minimizer of $F(\{x^k\}_{k\in\mathbb{N}};\cdot)$, the last inequality implies that $\mathbf{P}z=z$, i.e., $\operatorname{Fix}\mathbf{P}\neq\varnothing$. This completes the proof of (A).

According to [**9**, Theorem 5.7(C)], if $C\neq\varnothing$, then $\operatorname{Arg\,min}\mathbf{g}=C$. Thus, if $C\neq\varnothing$, then $\boxed{(vi)\Leftrightarrow(i)}$ and this proves (B). In order to prove (C) note that, as shown by (A), whenever the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded it is weakly convergent and we have that $z:=w\text{-}\lim_{k\to\infty}x^k\in\operatorname{Arg\,min}\mathbf{g}$. According to Lemma 5.6 in [**9**] applied to the function $f=\frac{1}{2}\|\cdot\|^2$, if $u\in\operatorname{Fix}\mathbf{P}$, then

$$\|u-\mathbf{P}x\|^2+\mathbf{g}(x)\leq\|u-x\|^2+\mathbf{g}(u),\ \forall x\in X.$$

Therefore, since $z\in\operatorname{Arg\,min}\mathbf{g}=\operatorname{Fix}\mathbf{P}$, we have that

$$\begin{aligned} 0 &\leq \mathbf{g}(x^k)-\mathbf{g}(z) \\ &\leq \left\|z-x^k\right\|^2-\left\|z-\mathbf{P}x^k\right\|^2 \\ &= \left\|z-x^k\right\|^2-\left\|z-x^{k+1}\right\|^2. \end{aligned}\tag{2.9}$$

By the nonexpansivity of $\mathbf{P}$ we deduce that

$$\left\|z-x^{k+1}\right\|=\left\|\mathbf{P}z-\mathbf{P}x^k\right\|\leq\left\|z-x^k\right\|,\ \forall k\in\mathbb{N},\tag{2.10}$$

showing that the sequence $\{\|z-x^k\|\}_{k\in\mathbb{N}}$ is nondecreasing and, hence, convergent. This and (2.9) imply that

$$\lim_{k\to\infty}\mathbf{g}(x^k)=\mathbf{g}(z)=\min_{x\in X}\mathbf{g}(x).$$

If $\inf_{x\in X}\mathbf{g}(x)=0$, then $\mathbf{g}(z)=0$ and, by the definition of $\mathbf{g}$, this implies that $\|P_\omega z-z\|=0$ ($\boldsymbol{\mu}$-a.e.), that is, $z=P_\omega z\in C_\omega$, $\boldsymbol{\mu}$-a.e.

Now suppose that $\operatorname{Arg\,min}\mathbf{g}$ has nonempty interior. By (2.10), we deduce that the sequence $\{x^k\}_{k\in\mathbb{N}}$ is Fejér monotone with respect to the set $\operatorname{Arg\,min}\mathbf{g}$. Consequently, Theorem 4.5.10 in [**5**] applies and it shows that $\{x^k\}_{k\in\mathbb{N}}$ converges strongly. This also happens when the interior of C is nonempty because, in this case, $C=\operatorname{Arg\,min}\mathbf{g}$ as noted above. □

Theorem 2.1 is an useful tool for analyzing and solving *Fredholm equations of the first kind* requiring to find $x\in X$ such that

$$\langle\mathcal{K}(\omega),x\rangle=b(\omega),\quad\boldsymbol{\mu}\text{-a.e.},\tag{2.11}$$

where $\mathcal{K}:\Omega\to X$ and $b:\Omega\to\mathbb{R}$ (or $\mathbb{C}$) are given measurable functions and $\mathcal{K}(\omega)\neq0$ for $\boldsymbol{\mu}$-almost all $\omega\in\Omega$. In order to show this, let

$$C_\omega:=\{u\in X:\langle\mathcal{K}(\omega),u\rangle=b(\omega)\}.\tag{2.12}$$

Clearly, this is a convex closed hyperplane in X and the (metric) projection onto it is exactly

$$P_\omega x = x + \frac{b(\omega) - \langle \mathcal{K}(\omega), x\rangle}{\|\mathcal{K}(\omega)\|^2}\mathcal{K}(\omega), \tag{2.13}$$

whenever $\mathcal{K}(\omega) \neq 0$. The following corollary, which is stated with the usual convention that that $0 \cdot \infty = 0$ and that $1/0 = \infty$, makes clear how Theorem 2.1 can be applied in order to solve (2.11).

Corollary 2.1. *Suppose that the functions $\mathcal{K} : \Omega \to X$ and $b : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) are measurable, $\mathcal{K}(\omega) \neq 0$ for μ-almost all $\omega \in \Omega$, and that the function $\omega \to |b(\omega)| / \|\mathcal{K}(\omega)\|$ is $\boldsymbol{\mu}$-square integrable, that is, the next integral exists and*

$$\int_\Omega \frac{|b(\omega)|^2}{\|\mathcal{K}(\omega)\|^2} d\boldsymbol{\mu}(\omega) < \infty.$$

Then the following statements are true:

(i) *The point-to-set mapping $\omega \to C_\omega$ is measurable and has a $\boldsymbol{\mu}$-square integrable selector;*

(ii) *The operator $\mathbf{P} : X \to X$ given by*

$$\mathbf{P}x = x + \int_\Omega \frac{b(\omega) - \langle \mathcal{K}(\omega), x\rangle}{\|\mathcal{K}(\omega)\|^2}\mathcal{K}(\omega) d\boldsymbol{\mu}(\omega), \tag{2.14}$$

as well as the function

$$\mathbf{g}(x) = \tfrac{1}{2} \textstyle\int_\Omega |b(\omega) - \langle K(\omega), x\rangle|^2 \, \|K(\omega)\|^{-2} \, d\boldsymbol{\mu}(\omega), \tag{2.15}$$

are well-defined;

(iii) *The equation (2.11) has solution if and only if there exists a bounded orbit $\{x^k\}_{k\in\mathbb{N}}$ of the operator $\mathbf{P}$ defined by (2.14) such that*

$$\liminf_{k\to\infty} \mathbf{g}(x^k) = 0. \tag{2.16}$$

(iv) *If the equation (2.11) has solution, then any orbit $\{x^k\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ converges weakly to a solution of (2.11).*

(v) *If if the function $\mathbf{g}$ is coercive (in the sense that $\lim_{\|x\|\to\infty} \mathbf{g}(x) = +\infty$), then all orbits of $\mathbf{P}$ converge weakly to fixed points of $\mathbf{P}$.*

(vi) *If $\mathbf{g}$ is coercive and there exists an orbit $\{x^k\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ such that (2.16) holds, then (2.11) has solutions and any orbit of $\mathbf{P}$ converges weakly to a solution of (2.11).*

Proof. Observe that, according to (2.13), if $\mathcal{K}(\omega) \neq 0$, then

$$P_\omega 0 = \frac{b(\omega)}{\|\mathcal{K}(\omega)\|^2}\mathcal{K}(\omega).$$

Thus, the function $\omega \to P_\omega 0$ is a selector of the point-to-set mapping $\omega \to C_\omega$ and we have

$$\|P_\omega 0\| = \frac{|b(\omega)|}{\|\mathcal{K}(\omega)\|}, \ \boldsymbol{\mu} - a.e.$$

Since the function on the right hand side is $\boldsymbol{\mu}$-square integrable, it results $\omega \to C_\omega$ has a $\boldsymbol{\mu}$-square integrable selector. This proves (i). The statement (ii) immediately results from (i) and (2.13).

In our current circumstances, the function $\mathbf{g}$ defined by (2.2) is exactly that given by (2.15). Thus, (iii) and (iv) follow from Theorem 2.1. If the function

$\mathbf{g}$ is coercive, then it has a global minimizer because $\mathbf{g}$ is also convex and lower semicontinuous. Hence, according to Theorem 2.1, all orbits of $\mathbf{P}$ converge weakly to minimizers of $\mathbf{g}$ which are also the fixed points of $\mathbf{P}$. Let $\left\{x^k\right\}_{k\in\mathbb{N}}$ be an orbit of $\mathbf{P}$ satisfying (2.16) and let z =w-$\lim_{k\to\infty} x^k$. Then, by Theorem 2.1, we have $\mathbf{g}(z) = \lim_{k\to\infty} \mathbf{g}(x^k) = 0$ and this shows that z is a solution of (2.11). $\square$

Corollary 2.1 shows that the coercivity of the function $\mathbf{g}$, defined by (2.15), implies that all orbits of $\mathbf{P}$ converge weakly. So, by establishing criteria for the coercivity of $\mathbf{g}$ we will implicitly obtain sufficient conditions for the weak convergence of the orbits of $\mathbf{P}$. To this end, we consider the linear, bounded, self-adjoint, positive semi-definite operator $M : X \to X$ given by

$$Mx = \int_\Omega \frac{\langle x, \mathcal{K}(\omega)\rangle}{\|\mathcal{K}(\omega)\|^2}\mathcal{K}(\omega) d\boldsymbol{\mu}(\omega) . \tag{2.17}$$

Note that the function M is well-defined because

$$\int_\Omega \left\| \frac{\langle x, \mathcal{K}(\omega)\rangle}{\|\mathcal{K}(\omega)\|^2}\mathcal{K}(\omega) \right\| d\boldsymbol{\mu}(\omega) \leq \|x\| , \quad \forall x \in X.$$

We denote by $\mathrm{Sp}(M)$ the spectrum of M (see [**26**, p. 371]). This is a closed set of real numbers (see, for instance, [**26**, Theorems 9.2.1, 9.2.2, 9.2.3 and 10.4.2]) contained in the closed interval $[\alpha(M), \beta(M)]$ with

$$\alpha(M) = \inf_{\|x\|=1} \langle Mx, x\rangle \text{ and } \beta(M) = \sup_{\|x\|=1} \langle Mx, x\rangle , \tag{2.18}$$

having the properties that $\alpha(M), \beta(M) \in \mathrm{Sp}(M)$ and

$$\|M\| = \beta(M) = \sup_{\|x\|=1} \langle Mx, x\rangle . \tag{2.19}$$

Recall that the eigenvalues of M are elements of $\mathrm{Sp}(M)$. Since M is positive semi-definite, it follows from (2.18) that $\alpha(M) \geq 0$. Therefore, if M has an spectral value $\lambda \neq 0$, then we also have $0 < \lambda \leq \beta(M) = \|M\|$ showing that M is not identically zero.

The following result shows that there is an intimate connection between the linear operator M and the coercivity of the function $\mathbf{g}$ defined by (2.15).

Theorem 2.2. *Suppose that the hypothesis of Corollary 2.1 is satisfied. If the linear operator M has $\alpha(M) > 0$, then the function $\mathbf{g}$ defined by (2.15) is coercive. Moreover, in this circumstances, if $\left\{x^k\right\}_{k\in\mathbb{N}}$ is an orbit of the operator $\mathbf{P}$ defined by (2.14), then $\left\{x^k\right\}_{k\in\mathbb{N}}$ is weakly convergent, its weak limit $z :=$ w-$\lim_{k\to\infty} x^k$ is a fixed point of $\mathbf{P}$ and a minimizer of $\mathbf{g}$, $\lim_{k\to\infty} \mathbf{g}(x^k) = \mathbf{g}(z)$, and one and only one of the following statements is true:*

(*i*) $\lim_{k\to\infty} \mathbf{g}(x^k) = 0$ *in which case problem (2.11) has solutions and all orbits of $\mathbf{P}$ converge strongly to solutions of (2.11);*

(*ii*) $\lim_{k\to\infty} \mathbf{g}(x^k) \neq 0$ *in which case problem (2.11) has not solution.*

Proof. Observe that, by (2.15), we have that

$$
\begin{aligned}
\mathbf{g}(x) \quad &\geq \int_\Omega \frac{\left|\langle \mathcal{K}(\omega), x\rangle\right|^2}{\left\|\mathcal{K}(\omega)\right\|^2} d\boldsymbol{\mu}(\omega) \\
&\quad - \int_\Omega \frac{\left|b(\omega)\right|^2}{\left\|\mathcal{K}(\omega)\right\|^2} d\boldsymbol{\mu}(\omega) - 2\int_\Omega \frac{\left|b(\omega)\left\langle \mathcal{K}(\omega), x\right\rangle\right|}{\left\|\mathcal{K}(\omega)\right\|^2} d\boldsymbol{\mu}(\omega) \\
&\geq \int_\Omega \frac{\left|\langle \mathcal{K}(\omega), x\rangle\right|^2}{\left\|\mathcal{K}(\omega)\right\|^2} d\boldsymbol{\mu}(\omega) \\
&\quad - \int_\Omega \frac{\left|b(\omega)\right|^2}{\left\|\mathcal{K}(\omega)\right\|^2} d\boldsymbol{\mu}(\omega) - 2\left\|x\right\| \int_\Omega \frac{\left|b(\omega)\right|}{\left\|\mathcal{K}(\omega)\right\|} d\boldsymbol{\mu}(\omega),
\end{aligned} \tag{2.20}
$$

where all three integrals exist and are finite because $b(\omega)/\left\|\mathcal{K}(\omega)\right\|$ is $\boldsymbol{\mu}$-square integrable. We have

$$
\begin{aligned}
\int_\Omega \frac{\left|\langle \mathcal{K}(\omega), x\rangle\right|^2}{\left\|\mathcal{K}(\omega)\right\|^2} d\boldsymbol{\mu}(\omega) &= \int_\Omega \frac{\langle \mathcal{K}(\omega), x\rangle \langle x, \mathcal{K}(\omega)\rangle}{\left\|\mathcal{K}(\omega)\right\|^2} d\boldsymbol{\mu}(\omega) \\
&= \left\langle \int_\Omega \frac{\langle x, \mathcal{K}(\omega)\rangle}{\left\|\mathcal{K}(\omega)\right\|^2} \mathcal{K}(\omega), x \right\rangle d\boldsymbol{\mu}(\omega) = \langle Mx, x\rangle.
\end{aligned}
$$

Denote

$$
r := \int_\Omega \frac{\left|b(\omega)\right|^2}{\left\|\mathcal{K}(\omega)\right\|^2} d\boldsymbol{\mu}(\omega) \text{ and } q := \int_\Omega \frac{\left|b(\omega)\right|}{\left\|\mathcal{K}(\omega)\right\|} d\boldsymbol{\mu}(\omega).
$$

Taking into account (2.20) we obtain

$$
\mathbf{g}(x) \geq \langle Mx, x\rangle - 2q\left\|x\right\| - r. \tag{2.21}
$$

It follows from (2.18) that

$$
\langle Mx, x\rangle \geq \alpha(M)\left|\left|x\right|\right|^2, \quad \forall x \in X. \tag{2.22}
$$

Since (2.21) holds we obtain

$$
\begin{aligned}
\mathbf{g}(x) &\geq \langle Mx, x\rangle - 2q\left\|x\right\| - r \\
&\geq \alpha(M)\left|\left|x\right|\right|^2 - 2\left|\left|q\right|\right|\left|\left|x\right|\right| - r \\
&= \left|\left|x\right|\right|^2 \left(\alpha(M) - \frac{2\left|\left|q\right|\right|}{\left|\left|x\right|\right|} - \frac{r}{\left|\left|x\right|\right|^2}\right),
\end{aligned}
$$

for any $x \in X$, $x \neq 0$. Hence,

$$
\lim_{\|x\|\to\infty} \mathbf{g}(x) \geq \lim_{\|x\|\to\infty} \left|\left|x\right|\right|^2 \left(\alpha(M) - \frac{2q}{\left|\left|x\right|\right|} - \frac{r}{\left|\left|x\right|\right|^2}\right) = +\infty,
$$

because $\alpha(M) > 0$. This shows that $\mathbf{g}$ is coercive. Applying Corollary 2.1 to the coercive function $\mathbf{g}$ we deduce that any orbit $\left\{x^k\right\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ is weakly convergent to a fixed point of $\mathbf{P}$ and that either the sequence $\left\{\mathbf{g}(x^k)\right\}_{k\in\mathbb{N}}$ converges to zero in which case the weak limit of $\left\{x^k\right\}_{k\in\mathbb{N}}$ is a solution of the DLMP, or the sequence $\left\{\mathbf{g}(x^k)\right\}_{k\in\mathbb{N}}$ converges to a positive number and, then, the DLMP has not solutions. It remains to show that if $\lim_{k\to\infty} \mathbf{g}(x^k) = 0$, then the orbit $\left\{x^k\right\}_{k\in\mathbb{N}}$ converges strongly. Let $z = w\text{-}\lim_{k\to\infty} x^k$ and suppose that $\lim_{k\to\infty} \mathbf{g}(x^k) = 0$. Observe that, according to (2.22), we have that

$$\begin{aligned}
\alpha(M)\left\|x^k - z\right\|^2 &\leq \left\langle M(x^k - z), x^k - z\right\rangle \\
&= \int_\Omega \frac{\left|\left\langle \mathcal{K}(\omega), x^k - z\right\rangle\right|^2}{\|\mathcal{K}(\omega)\|^2} d\boldsymbol{\mu}(\omega) \\
&= \int_\Omega \frac{\left|\left\langle \mathcal{K}(\omega), x^k\right\rangle - \left\langle \mathcal{K}(\omega), z\right\rangle\right|^2}{\|\mathcal{K}(\omega)\|^2} d\boldsymbol{\mu}(\omega) \\
&= \int_\Omega \frac{\left|\left\langle \mathcal{K}(\omega), x^k\right\rangle - b(\omega)\right|^2}{\|\mathcal{K}(\omega)\|^2} d\boldsymbol{\mu}(\omega) = 2\mathbf{g}(x^k),
\end{aligned}$$

for all $k \in \mathbb{N}$. Since $\alpha(M) > 0$, letting $k \to \infty$ on both sides of this inequality we obtain that $\left\{x^k\right\}_{k\in\mathbb{N}}$ converges strongly to z. □

3. Solutions of the discrete linear moment problem

In this section we consider the moment problem (1.1) described in Section 1. We make the following assumption:

Assumption 3.1: *There exists a sequence $\{\mu_k\}_{k\in\mathbb{N}}$ of positive real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k = 1 \text{ and } \sum_{k=0}^{\infty} \mu_k \left|b_k\right|^2 / \left\|K_k\right\|^2 < +\infty. \tag{3.1}$$

It is obvious that this assumption holds whenever the sequences $\left\{|b_k|^2\right\}_{k\in\mathbb{N}}$ and $\left\{\|K_k\|^2\right\}_{k\in\mathbb{N}}$ are summable because, in this case, one can take

$$\mu_k := \|K_k\|^2 \left(\sum_{j=0}^{\infty} \|K_j\|^2\right)^{-1}. \tag{3.2}$$

Our purpose is to show that, under certain circumstances, applications of Theorem 2.1 and of its corollaries presented in the previous section leads to a solution of the linear discrete moment problem. For this purpose, note that the set $\Omega := \mathbb{N}$ is a complete probability space in which all subsets of $\mathbb{N}$ are measurable and with the probability measure

$$\boldsymbol{\mu}(A) = \sum_{k\in A} \mu_k, \tag{3.3}$$

where the sequence $\{\mu_k\}_{k\in\mathbb{N}}$ is that whose existence is ensured by Assumption 3.1. Moreover, the problem (1.1) is a particular version of the Fredholm equation (2.11) with the functions $\mathcal{K}(k) = K_k$ and $b(k) = b_k$ for which the hypothesis of Theorem 2.2 is satisfied when $\mathbb{N}$ is provided with the probability measure $\boldsymbol{\mu}$ given by (3.3). In this particular case we have

$$\mathbf{P}x = x + \sum_{j=0}^{\infty} \mu_j \frac{b_j - \langle K_j, x\rangle}{\|K_j\|^2} K_j, \tag{3.4}$$

$$\mathbf{g}(x) = \sum_{j=0}^{\infty} \mu_j \left| b_j - \langle K_j, x\rangle \right|^2 \left\| K_j \right\|^{-2}, \tag{3.5}$$

and

$$Mx = \sum_{j=0}^{\infty} \mu_j \frac{\langle x, K_j\rangle}{\left\| K_j \right\|^2} K_j, \tag{3.6}$$

where all the series are convergent.

For each positive integer k we denote by G_k the Gram matrix of the vectors $\sqrt{\mu_j} \left\| K_j \right\|^{-1} K_j$, $(0 \leq j \leq k)$. This matrix is Hermitian and, thus, it has real eigenvalues only (cf. [**26**, p. 469]) and, if $w = (w_0, ..., w_k) \in \mathbb{C}^{k+1}$, then $\bar{w} G_k w^T \in \mathbb{R}$, where $\mathbb{C}$ stands for the set of complex numbers and $\bar{w} = (\bar{w}_0, ..., \bar{w}_k)$ is the vector whose coordinate $\bar{w}_j$ is the conjugate of w_j. We denote by λ_k the minimal eigenvalue of the matrix G_k. Recall that

$$\lambda_k := \inf \left\{ \bar{w} G_k w^T : w \in \mathbb{C}^{k+1}, \bar{w} \cdot w^T = 1 \right\}, \tag{3.7}$$

and that the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ is nonincreasing (cf. [**30**, p. 54]). Let $\lambda_* := \lim_{k\to\infty} \lambda_k$.

We denote by X_k the finite dimensional linear subspace of X spanned by the vectors K_j with $0 \leq j \leq k$. Recall that the sequence of vectors $\{K_k\}_{k\in\mathbb{N}}$ is called *complete* when

$$\operatorname{cl} \bigcup_{k=0}^{\infty} X_k = X. \tag{3.8}$$

Note that $\{X_k\}_{k\in\mathbb{N}}$ is a nondecreasing sequence of nonempty closed convex subsets of X and, therefore, $\operatorname{cl} \bigcup_{k=0}^{\infty} X_k$ is exactly the Mosco limit of $\{X_k\}_{k\in\mathbb{N}}$ (see [**27**, Lemma 1.2, p. 526]). With these notations and facts in mind we are now in position to formulate the second assumption on the DLMP data which we will make in the sequel:

Assumption 3.2: *The sequence of vectors $\{K_k\}_{k\in\mathbb{N}}$ is complete and the limit $\lambda_* := \lim_{k\to\infty} \lambda_k$ is positive.*

Our aim is to prove the following existence and computability result concerning the solutions of the moment problem stated above:

Theorem 3.1. *Suppose that the data of the discrete linear moment problem (1.1) satisfy Assumptions 3.1 and 3.2. Then the problem (1.1) has solutions and any orbit $\left\{x^k\right\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ converges strongly to a solution of (1.1).*

Proof. Define the bounded linear operator $M_k : X \to X$ by

$$M_k x = \sum_{j=0}^{k} \mu_j \frac{\langle x, K_j\rangle}{\left|\left| K_j \right|\right|^2} K_j. \tag{3.9}$$

This operator is self-adjoint and positive semi-definite because for any $x \in X$ we have

$$\langle M_k x, x\rangle = \sum_{j=0}^{k} \mu_j \frac{\left| \langle x, K_j\rangle \right|^2}{\left|\left| K_j \right|\right|^2} = \langle x, M_k x\rangle .$$

Consequently, the equalities in (2.18) and (2.19) still hold when M is replaced by M_k. They show that the spectral values of M_k and, in particular, the eigenvalues of M_k, if they exist, are necessarily nonnegative. Denote by $\Pi_k : X \to X_k$ the projection operator onto X_k. For each $x \in X$ we have that

$$M_k x = M_k(\Pi_k x). \tag{3.10}$$

Indeed, if $y^k \in X_k^\perp$ is the unique vector in X such that $x = \Pi_k x + y^k$, then $M_k x = M_k(\Pi_k x) + M_k y^k$, where the second term on the right hand side is null because y^k is orthogonal on all the vectors K_j with $0 \le j \le k$ and because of (3.9). The equality (3.10), combined with (3.9), shows that the range of M_k is contained in the finite dimensional space X_k. Since M_k is also bounded, it results that the linear operator M_k is compact. Being compact and self-adjoint, the operator M_k has eigenvalues (cf. [**26**, p. 469]) and all of them are real numbers. By (2.18) and (2.19) written with M_k instead of M, if λ is an eigenvalue of M_k, then

$$0 \le \alpha(M_k) \le \lambda \le \beta(M_k) = \|M_k\|.$$

Observe that

$$\|M_k x\| \le \sum_{j=0}^{k} \mu_j \frac{|\langle x, K_j\rangle|}{\|K_j\|} \le \|x\|, \quad \forall x \in X,$$

and this implies that $\|M_k\| \le 1$. Hence, all eigenvalues of M_k are contained in the interval $[0, 1]$.

Now we are going to prove the following claim:

Claim 1: If the real number $\lambda \neq 0$ is an eigenvalue of M_k, then it is also an eigenvalue of the Gram matrix G_k.

In order to prove this claim, let $\lambda \neq 0$ be an eigenvalue of M_k. Then, by (3.10), there exists $\bar{x} \in X$ such that $\bar{x} \neq 0$ and

$$\sum_{j=0}^{k} \mu_j \frac{\langle \bar{x}, K_j\rangle}{\|K_j\|^2} K_j = \lambda \bar{x}. \tag{3.11}$$

Dividing this equality by λ, it results that $\bar{x} \in X_k$. Also, the equality (3.11) implies that

$$\left\langle \frac{\sqrt{\mu_m}}{||K_m||} K_m, \sum_{j=0}^{k} \left\langle \bar{x}, \frac{\sqrt{\mu_j}}{\|K_j\|} K_j \right\rangle \frac{\sqrt{\mu_j}}{||K_j||} K_j \right\rangle = \lambda \left\langle \frac{\sqrt{\mu_m}}{||K_m||} K_m, \bar{x} \right\rangle, \tag{3.12}$$

for $m = 0, 1, ..., k$. For each $j \in \{0, 1, ..., k\}$ denote

$$c_j = \left\langle \bar{x}, \frac{\sqrt{\mu_j}}{||K_j||} K_j \right\rangle.$$

At least one of the numbers c_j is different from zero since, otherwise, $\bar{x}$ is orthogonal on the linear space X_k to which it belongs and this contradicts the fact that $\bar{x} \neq 0$. Then (3.12) is equivalent to

$$\sum_{j=0}^{k} \left\langle \frac{\sqrt{\mu_m}}{||K_m||} K_m, \frac{\sqrt{\mu_j}}{\|K_j\|} K_j \right\rangle c_j = \lambda c_m,\ 0 \le m \le k,$$

showing that $(c_0, ..., c_k)$ is an eigenvector for the matrix G_k corresponding to λ. This proves Claim 1.

According to Assumption 3.2 the number λ_* is positive and it was noted above that the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ is nonincreasing. Hence, for any nonnegative integer k, the eigenvalues of the Gram matrix G_k are positive real numbers. This implicitly means that the matrix G_k is nonsingular and, therefore, that the vectors K_j with $0 \leq j \leq k$, are linearly independent. With this in mind we are going to establish the following fact:

Claim 2: The eigenvalues of the matrix G_k are among the eigenvalues of the operator M_k.

Suppose that λ is an eigenvalue of G_k and that $w = (w_0, ..., w_k) \in \mathbb{C}^{k+1}$ is an eigenvector corresponding to λ. Then we have that

$$\sum_{j=0}^{k} \sqrt{\mu_m \mu_j} \frac{\langle K_m, \bar{w}_j K_j \rangle}{\|K_m\| \, \|K_j\|} = \lambda w_m, \quad 0 \leq m \leq k,$$

which implies that

$$\langle y, K_m \rangle = \lambda \frac{\|K_m\|}{\sqrt{\mu_m}} \bar{w}_m, \quad 0 \leq m \leq k, \tag{3.13}$$

where

$$y := \sum_{j=0}^{k} \frac{\sqrt{\mu_j} \bar{w}_j}{\|K_j\|} K_j.$$

Observe that $y \neq 0$ because the vectors K_j, $0 \leq j \leq k$, are linearly independent and, if $y = 0$, then we obtain that $\bar{w}_j = 0 = w_j$ for $0 \leq j \leq k$ contradicting the assumption made above that w is an eigenvector of G_k. Multiplying the m-th equation in (3.13) by $(\mu_m K_m) / \|K_m\|^2$ and summing up the resulting equalities we get

$$\lambda y = \sum_{m=0}^{k} \mu_m \frac{\langle y, K_m \rangle}{\|K_m\|^2} K_m = M_k y$$

and this shows that λ is an eigenvalue of M_k. Claim 2 is proved.

By Claim 1 and Claim 2, the matrix G_k and the operator M_k have the same positive eigenvalues. Obviously, the operator M_k has also zero among its eigenvalues since $M_k y = 0$ for all $y \in X_k^{\perp}$. Note that

$$\|(M - M_k)\, x\| = \left\| \sum_{j=k+1}^{\infty} \mu_j \frac{\langle x, K_j \rangle}{\|K_j\|^2} K_j \right\| \leq \|x\| \sum_{j=k+1}^{\infty} \mu_j, \quad \forall x \in X,$$

and, therefore,

$$\|M - M_k\| \leq \sum_{j=k+1}^{\infty} \mu_j, \tag{3.14}$$

for any $k \in \mathbb{N}$. This implies that the sequence of linear operators $\{M_k\}_{k \in \mathbb{N}}$ converges to M (in the dual norm).

Let M_k' be the restriction of M_k to the subspace X_k of X. By (3.9), the operator M_k' is an operator from X_k to X_k. Using (3.10) it can be easily seen that any positive eigenvalue of M_k is also an eigenvalue of the operator M_k'. Since M_k and G_k have the same positive eigenvalues, it results that M_k' and G_k have exactly the same set of eigenvalues. Being an operator on a finite dimensional space, the spectrum of M_k' consists of eigenvalues only. Consequently, the spectrum of M_k' is exactly the set of eigenvalues of G_k. Hence, the minimal spectral value of M_k' is $\alpha(M_k') = \lambda_k$. Thus, by (2.18) applied to M_k' we deduce that

$$\langle M_k'(\Pi_k x), \Pi_k x\rangle \geq \alpha(M_k') \|\Pi_k x\|^2 = \lambda_k \|\Pi_k x\|^2, \quad \forall x \in X\backslash\{0\}, \tag{3.15}$$

for any $k \in \mathbb{N}$.

According to (3.10), for any $x \in X\backslash\{0\}$ we have that

$$\begin{aligned} &|\langle Mx, x\rangle - \langle M_k'(\Pi_k x), \Pi_k x\rangle| \leq \\ &|\langle Mx, x\rangle - \langle M_k x, x\rangle| + |\langle M_k x, x\rangle - \langle M_k'(\Pi_k x), \Pi_k x\rangle| = \\ &|\langle Mx, x\rangle - \langle M_k x, x\rangle| + |\langle M_k x, x\rangle - \langle M_k x, \Pi_k x\rangle| \\ &\leq \|Mx - M_k x\| \|x\| + \|M_k x\| \|x - \Pi_k x\|, \end{aligned} \tag{3.16}$$

where the sequence $\{\|Mx - M_k x\|\}_{k\in\mathbb{N}}$ converges to zero and the sequence $\{\|M_k x\|\}_{k\in\mathbb{N}}$ is bounded because, as shown above, $\{M_k\}_{k\in\mathbb{N}}$ converges to M in the dual norm. It was noted before that, since the sequence $\{K_k\}_{k\in\mathbb{N}}$ is complete (cf. Assumption 3.2), the sequence $\{X_k\}_{k\in\mathbb{N}}$ of closed convex subsets of X converges in the sense of Mosco (see [**27**]) to X. This implies (see, for instance, [**13**, Theorem 1]) that the sequence $\{\|x - \Pi_k x\|\}_{k\in\mathbb{N}}$ converges to zero for any $x \in X$. Consequently, by (3.16), we deduce that

$$\lim_{k\to\infty} |\langle Mx, x\rangle - \langle M_k'(\Pi_k x), \Pi_k x\rangle| = 0, \quad \forall x \in X\backslash\{0\}.$$

Now, taking on both sides of the the inequality (3.15) the limit as $k \to \infty$ we obtain that

$$\langle Mx, x\rangle \geq \lambda_* \|x\|^2, \quad \forall x \in X\backslash\{0\}.$$

This and (2.18) imply that $\alpha(M) \geq \lambda_* > 0$ and, thus, Theorem 2.2 applies. It shows that any orbit $\{x^k\}_{k\in\mathbb{N}}$ of the operator $\mathbf{P}$ converges weakly and its weak limit $z = w\text{-}\lim_{k\to\infty} x^k$ is a minimizer of the function $\mathbf{g}$. We prove next that z is necessarily a solution of (1.1). If this is true then, by (3.5), we deduce that $\mathbf{g}(z) = 0$ and, thus, using Theorem 2.2 again we obtain that $\lim_{k\to\infty} \mathbf{g}(x^k) = 0$ which, in turn, implies that the convergence of $\{x^k\}_{k\in\mathbb{N}}$ is strong. In order to prove that z is a solution of (1.1), observe that, since z is a minimizer of the differentiable function $\mathbf{g}$, we have that $\nabla\mathbf{g}(z) = 0$. This, (2.4) and (3.4), taken together, imply that

$$0 = 2 \cdot \nabla\mathbf{g}(z) = 2\sum_{j=0}^{\infty} \mu_j \frac{|b_j - \langle K_j, x\rangle|^2}{\|K_j\|^2} K_j. \tag{3.17}$$

According to [**30**, Remark 1, p. 54], since $\lambda_* > 0$, it results that the sequence of vectors $\{K_k\}_{k\in\mathbb{N}}$ is minimal in the sense that

$$K_j \notin \overline{\operatorname{span}}\{K_k : k \neq j\}, \quad \forall j \in \mathbb{N}.$$

Consequently, by [**30**, Remark 1, p. 54], there exists a sequence $\{L_k\}_{k\in\mathbb{N}}$ contained in X which is biorthogonal to $\{K_k\}_{k\in\mathbb{N}}$, that is, $\langle K_k, L_j\rangle = \delta_{k,j}$, where $\delta_{k,j}$ is the Knonecker delta. Using this fact and (3.17), we deduce that

$$0 = 2\sum_{j=0}^{\infty} \mu_j \frac{|b_j - \langle K_j, x\rangle|^2}{\|K_j\|^2} \langle K_j, L_m\rangle = 2\mu_m \frac{|b_m - \langle K_m, x\rangle|^2}{\|K_m\|^2},$$

for any $m \in \mathbb{N}$. This can not happen unless $b_m - \langle K_m, z\rangle = 0$ for all $m \in \mathbb{N}$, that is, unless z is a solution of (1.1). □

Theorem 3.1 indicates a method of solving the DLMP by computing a large number of iterates x^k of an arbitrary orbit of $\mathbf{P}$. An intrinsic difficulty of this method is that it requires precise computation of the iterates x^k given by the rule

$$x^{k+1} = x^k + \sum_{j=0}^{\infty} \mu_j \frac{b_j - \langle K_j, x^k \rangle}{\|K_j\|^2} K_j, \tag{3.18}$$

which involves infinitely long summations. Obviously, effectively computing the infinite sum occurring in (3.18) is not practically doable. This leads to the question whether Theorem 3.1 remains true if one replaces the iterates x^k given by (3.18) by "inexact" iterates y^k of the form

$$y^{k+1} = y^k + \sum_{j=0}^{n(k)} \mu_j \frac{b_j - \langle K_j, y^k \rangle}{\|K_j\|^2} K_j, \tag{3.19}$$

where, for each $k \in \mathbb{N}$, the nonnegative integer $n(k)$ is sufficiently large. The following result shows that this is indeed the case when the sequence $\{|b_k| / \|K_k\|\}_{k\in\mathbb{N}}$ has a known positive upper bound.

Corollary 3.1. *Suppose that Assumptions 3.1 and 3.2 are satisfied and let γ be a positive upper bound of the sequence $\{|b_k| / \|K_k\|\}_{k\in\mathbb{N}}$. If, for a summable sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ of positive real numbers, and for each $k \in \mathbb{N}$, the number $n(k)$ is chosen such that*

$$1 - \sum_{j=0}^{n(k)} \mu_j \le \frac{\varepsilon_k}{\gamma + \|y^k\|}, \tag{3.20}$$

then any sequence $\{y^k\}_{k\in\mathbb{N}}$ given by (3.19) converges strongly to a solution of the discrete linear moment problem (1.1).

Proof. If $\{y^k\}_{k\in\mathbb{N}}$ is a sequence given by (3.19) with the numbers $n(k)$ satisfying (3.20), then we have

$$\|y^{k+1} - \mathbf{P}y^k\| = \left\| \sum_{j=n(k)+1}^{\infty} \mu_j \frac{b_j - \langle K_j, y^k \rangle}{\|K_j\|^2} K_j \right\|$$

$$\le \sum_{j=n(k)+1}^{\infty} \mu_j \frac{|b_j - \langle K_j, y^k \rangle|}{\|K_j\|} \le (\gamma + \|y^k\|) \sum_{j=n(k)+1}^{\infty} \mu_j$$

$$= (\gamma + \|y^k\|) \left[1 - \sum_{j=0}^{n(k)} \mu_j \right] \le \varepsilon_k,$$

for all $k \in \mathbb{N}$. This implies that the sequence $\{\|y^{k+1} - \mathbf{P}y^k\|\}_{k\in\mathbb{N}}$ is summable, that is $\{y^k\}_{k\in\mathbb{N}}$ is an inexact orbit with summable errors for $\mathbf{P}$. As noted in Section 2, the operator $\mathbf{P}$ is nonexpansive and, by Theorem 3.1, all orbits of $\mathbf{P}$ are converging strongly to solutions of (1.1). Clearly, the set of solutions of (1.1) and the set of fixed points of $\mathbf{P}$ coincide. So, the orbits of the nonexpansive operator $\mathbf{P}$ converge strongly to fixed points of $\mathbf{P}$. By Theorem 3, Corollary 4 and Theorem 4 in [**12**], it follows that all inexact orbits with summable errors of $\mathbf{P}$ and, in particular, the sequence $\{y^k\}_{k\in\mathbb{N}}$, converges strongly to a solution of the DLMP (1.1). □

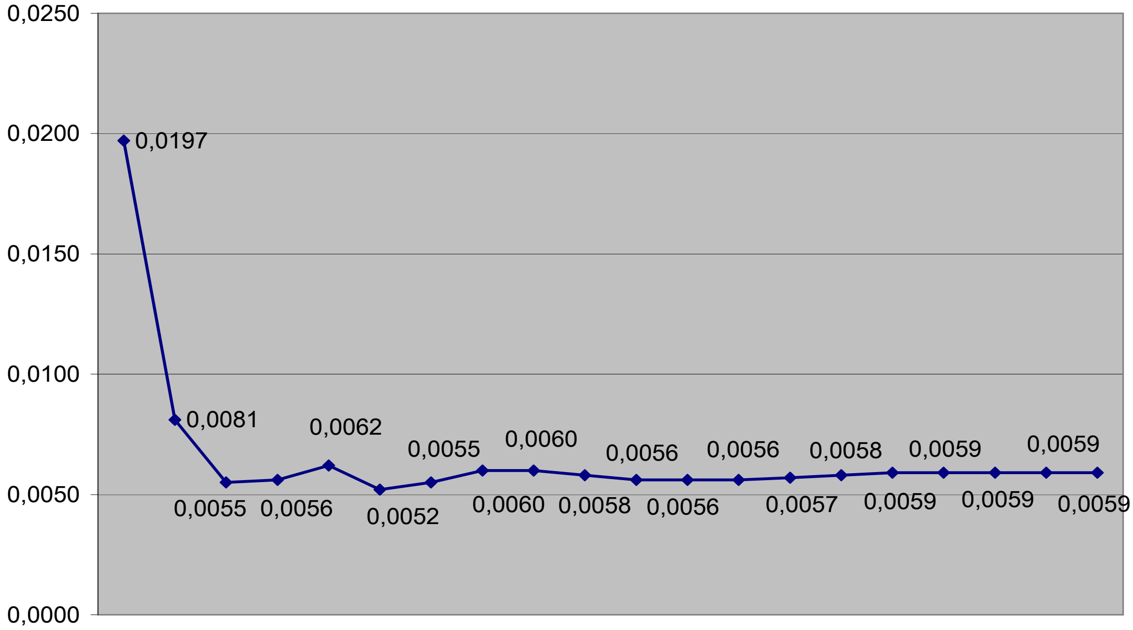

FIGURE 1. The minimal eigenvalues of the Gram matrices G_k

The following example shows that our assumptions in Theorem 3.1 and in Corollary 3.1 are consistent. We present a sequence of functions $\{K_k\}_{k\in\mathbb{N}}$ which satisfies the hypothesis of those results.

Example 3.1: Consider the real Hilbert space $X = L^2[0,5]$ of (Lebesgue) square-summable functions on $[0,5]$ provided with the inner product

$$\langle x, y\rangle = \int_0^5 x(t)y(t)dt.$$

Let

$$K_k(t) = 10\sqrt{10}\exp\left[-(k+1)^2 t\right], \quad \forall k \in \mathbb{N}. \tag{3.21}$$

It is well known that this sequence is complete in the Hilbert space X which we consider here. The behavior of the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ associated with the functions $K_k(t)$ is shown in Figure 1. Observe that after 100 computational steps the values of λ_k became stable around the positive number $\lambda_* = 0.0059$. Hence, our functions K_k satisfy Assumption 3.2 above.

Note that the norms of the functions K_k are

$$\|K_k\|^2 = \frac{1}{2(k+1)^2}\left(1 - e^{-10(k+1)^2}\right), \quad \forall k \in \mathbb{N},$$

showing that $\sum_{k=0}^{\infty} \|K_k\|^2 < \infty$. Take, for example, $b_k = 1/(k+1)$ for any $k \in \mathbb{N}$. As pointed out at the beginning of this section, since both sequences $\{b_k\}_{k\in\mathbb{N}}$ and $\{\|K_k\|\}_{k\in\mathbb{N}}$ are square-summable, we can take the numbers μ_k as defined by (3.2) and Assumption 1 holds too.

Acknowledgement

The authors are grateful to Roman Shklyar, a Ph. D. Student from Bar Ilan University in Ramat Gan, Israel, for his help with the numerical computations presented in this work. His readiness to learn our results and his creative implementation of them is greatly appreciated.

References

[1] Akhiezer, N. and Glazman, I.: Theory of Linear Operators in Hilbert Spaces, *Dover Publications*, 1993.
[2] Aubin, J.P. and Cellina, A.: Differential Inclusions, *Springer Verlag*, Heidelberg, 1984.
[3] Bauschke, H. H. and Borwein, J.M.: On projection algorithms for solving convex feasibility problems, *SIAM Rev.* **38** (1996), 367–426.
[4] Bauschke, H.H., Matoušková, E. and Reich, S.: Projection and proximal point methods: convergence results and counterexamples, *Nonlinear Anal.* **56** (2004), 715-738.
[5] Borwein, J.M. and Zhu, Q.J.: Techniques of Variational Analysis, *Springer Verlag*, New York, 2005.
[6] Butkovskii, A.G.: Methods of Control for Distributed Parameter Systems (Russian), *Nauka*, Moscow, 1975.
[7] Butnariu, D.: The expected-projection methods: Its behavior and applications to linear operator equations and convex optimization, *J. Appl. Anal.* **1** (1995), 95-108.
[8] Butnariu, D. and Iusem, A.N.: Totally Convex Functions for Fixed Point Computation and Infinite Dimensional Optimization, *Kluwer Academic Publishers*, Dordrecht, 2000.
[9] Butnariu, D., Iusem, A. and Burachik, R.: Iterative methods for solving stochastic convex feasibility problems and applications, *Comput. Optim. Appl.*, **15** (2000), 269-307.
[10] Butnariu, D. and Flåm, S.J.: Strong convergence of expected projection methods in Hilbert spaces, *Numer. Funct. Anal. Optim.* **16**, (1995), 601-636.
[11] Butnariu, D. and Markowicz, I.: A relaxed Cimmino type method for computing almost common fixed points of totally nonexpansive families of operators, *Semin. Fixed Point Theory Cluj-Napoca* **3** (2002), 149–155.
[12] Butnariu, D., Reich, S. and Zaslavski, A.: Convergence to fixed points of inexact orbits of Bregman-monotone and nonexpansive operators in Banach spaces, in: H. Fetter (ed.) "Proc. Intl. Conf. of Fixed Point Theory and Applications", Guadajuato, Mexico, 2005, - to appear.
[13] Butnariu, D. and Resmerita, E.: Mosco stability of proximal mappings in reflexive Banach spaces , preprint 2006.
[14] Cimmino, G.: Calcolo approsimato per le soluzioni di sistemi di ecuazioni lineari, *La Riserca Scientifica, Roma* **2** (1938), 326-333.
[15] Combettes, P. L.: The convex feasibility problem in image recovery, in: "Advances in Imaging and Electron Physics", Vol. 97, pp. 155-270, *Academic Press*, New York, 1995.
[16] Combettes, P. L.: Hilbertian convex feasibility problem: convergence of projection methods. *Appl. Math. Optim.* **35** (1997), 311–330.
[17] Combettes, P.L.: On the numerical robustness of a parallel projection method in signal synthesis, *IEEE Signal Processing Letters* **8** (2001), 45-47.
[18] Flåm, S.D.: Successive averages of firmly nonexpansive mappings, *Math. Oper. Res.* **20** (1995), 497-512, .
[19] Gantmaher, F. R.: The Theory of Matrices, *Chelsea Publishing Co.,* Providence, RI, 1959.

[20] Goebel, K. and Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, *Marcel Dekker*, New York and Basel, 1984.

[21] Hausdorff, F.: Summationsmethoden und Momentfolgen I, *Math. Z.* **9** (1921), 74-109.

[22] Hausdorff, F.: Summationsmethoden und Momentfolgen II, *Math. Z.* **9** (1921), 280-299.

[23] Herman, G. and Tuy, H.G.: Image reconstruction from projections: An approach from mathematical analysis, in: P.C. Sabatier (ed.), "Basic Methods of Tomography and Inverse Problems", *Adam Hilger*, Bristol, 1987.

[24] Kammerer, W.J. and Nashed, M.Z.: A generalization of a matrix iterative method of G. Cimmino to best approximate solution of linear integral equations of the first kind, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur* **51** (1971), 20–25.

[25] Korobeinik, Yu.: The problem of moments, of interpolation and of being a basis, (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **42** (1978), 989–1020.

[26] Kreyszig, E.: Introductory Functional Analysis with Applications, *John Willey&Sons*, New York, 1978.

[27] Mosco, U.: Convergence of convex sets and of solutions of variational inequalities, *Advances in Mathematics* **4** (1969), 519-585.

[28] Sedletskii, A.M.: Biorthogonal expansions of functions in exponential series on intervals of the real axis, (Russian), *Uspekhi Mat. Nauk* **37** (1982), 51–95.

[29] Shohat, J.A. and Tamarkin, J.D.: The Problem of Moments, *Amer. Math. Soc. Surveys* 1, 1963.

[30] Taldykin, A.T.: Vector-Functions and Equations (with Applications to Control Theory), *Leningrad University Publisher*, Leningrad, 1977 (in Russian).

Dan Butnariu: Department of Mathematics, University of Haifa, 31905 Haifa, Israel
E-mail address: dbutnaru@math.haifa.ac.il

Ben Zion Schlyar: Holon Institute of Technology, 52 Golomb Street, 58102 Holon, Israel
E-mail address: shk_b@hit.ac.il