ASYMPTOTIC BEHAVIOR OF INEXACT ORBITS FOR A CLASS OF OPERATORS IN COMPLETE METRIC SPACES

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Abstract. We exhibit a class of nonlinear operators with the property that their iterates converge to their unique fixed points even when computational errors are present.

1. Introduction

Assume that (X, ρ) is a complete metric space and let the operator $A : X \to X$ have the following properties:

(A1) there exists a unique $x_A \in X$ such that $Ax_A = x_A$;

(A2) $A^n x \to x_A$ as $n \to \infty$, uniformly on all bounded subsets of X;

(A3) A is uniformly continuous on bounded subsets of X;

(A4) A is bounded on bounded subsets of X.

Many operators with these properties can be found, for example, in [1-5, 7-9]. Note that if X is either a closed, convex subset of a Banach space or a closed, ρ-convex subset of a complete hyperbolic metric space [6], then (A4) follows from (A3).

In view of (A2), it is natural to ask if the convergence of the orbits of A will be preserved even in the presence of computational errors. In this paper we provide affirmative answers to this question. More precisely, we have the following results.

Theorem 1.1. *Let K be a nonempty, bounded subset of X and $\epsilon > 0$. Then there exist $\delta = \delta(\epsilon, K) > 0$ and a natural number N such that for each natural number $n \geq N$, and each sequence $\{x_i\}_{i=0}^n \subset X$ which satisfies*

$$x_0 \in K \text{ and } \rho(Ax_i, x_{i+1}) \leq \delta, \ i = 0, \dots, n-1,$$

the following inequality holds:

$$\rho(x_i, x_A) \leq \epsilon, \ i = N, \dots, n.$$

Corollary 1.1. *Assume that $\{x_i\}_{i=0}^\infty \subset X$, $\{x_i\}_{i=0}^\infty$ is bounded, and that*

$$\lim_{i \to \infty} \rho(Ax_i, x_{i+1}) = 0.$$

Then $\rho(x_i, x_A) \to 0$ as $i \to \infty$.

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Theorem 1.2. *Let $\epsilon > 0$. Then there exists $\delta = \delta(\epsilon) > 0$ such that for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies*

$$\rho(x_0, x_A) \leq \delta \text{ and } \rho(x_{i+1}, Ax_i) \leq \delta,\ i = 0, 1, \ldots,$$

the following inequality holds:

$$\rho(x_i, x_A) \leq \epsilon,\ i = 0, 1, \ldots$$

These results show that, roughly speaking, in order to achieve an ϵ-approximation of x_A, it suffices to compute *inexact orbits* of A, that is, sequences $\{x_i\}_{i=0}^{\infty}$ such that

$$x_0 \in X \text{ and } \rho(x_{i+1}, Ax_i) \leq \delta \text{ for any } i \geq 0,$$

where δ is a sufficiently small positive number.

However, sometimes the operator A is not given explicitly and only some approximation of it, B_i, is available at each step i of the inexact orbit computing procedure. The next result shows that for certain operators A, the procedure of approximating x_A by inexact orbits is stable in the sense that, even in this case, the orbits determined by the sequence of operators B_i approach x_A provided that each B_i is a sufficiently accurate approximation of A in the topology of uniform convergence on bounded subsets of X. To be precise, we set, for each $x \in X$ and $E \subset X$,

$$\rho(x, E) = \inf\{\rho(x, y) :\ y \in E\}.$$

Denote by $\mathcal{A}$ the set of all self-mappings $A : X \to X$ which have properties (A3) and (A4). Fix $\theta \in X$. For each natural number n, set

$$E_n = \{(A, B) \in \mathcal{A} \times \mathcal{A} :\ \rho(Ax, Bx) \leq 1/n \text{ for all } x \in B(\theta, n)\}. \tag{1.1}$$

We equip the set $\mathcal{A}$ with the uniformity determined by the base E_n, $n = 1, 2, \ldots$. This uniformity is metrizable by a complete metric.

Denote by $\mathcal{A}_{reg}$ the set of all mappings $A \in \mathcal{A}$ which satisfy (A1) and (A2), and by $\bar{\mathcal{A}}_{reg}$ the closure of $\mathcal{A}_{reg}$ in $\mathcal{A}$.

Theorem 1.3. *Assume that $A \in \mathcal{A}_{reg}$ and x_A is a fixed point of A. Let $m, \epsilon > 0$. Then there exist a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ and a natural number N such that for each $x \in B(\theta, m)$, each integer $n \geq N$, and each sequence $\{B_i\}_{i=1}^{n} \subset \mathcal{U}$,*

$$\rho(B_i \ldots B_1 x, x_A) \leq \epsilon \text{ for } i = N, \ldots, n.$$

As a matter of fact, it turns out that the stability property established in this theorem is generic. That is, it holds for most (in the sense of Baire category) operators in the closure of $\mathcal{A}_{reg}$.

Theorem 1.4. *The set $\mathcal{A}_{reg}$ contains an everywhere dense G_δ subset of $\bar{\mathcal{A}}_{reg}$.*

The proofs of Theorems 1.1, 1.2, 1.3 and 1.4 are given in Sections 2, 3, 4 and 5, respectively.

2. Proof of Theorem 1.1

For $x \in X$ and $r > 0$, set

$$B(x,r) = \{y \in X : \ \rho(x,y) \le r\}.$$

We may assume without loss of generality that

$$\epsilon \le 1 \text{ and } B(x_A, 4) \subset K. \tag{2.1}$$

By (A2), there exists a natural number $N \ge 4$ such that

$$\rho(A^n x, x_A) \le \epsilon/4 \text{ for all integers } n \ge N \text{ and all } x \in K. \tag{2.2}$$

By (A4), the set $A^m(K)$ is bounded for all natural numbers m. Hence there exists a positive number $S > 0$ such that

$$A^i(K) \subset B(x_A, S), \ i = 0, \dots, 2N. \tag{2.3}$$

(Here we use the convention that A^0 is the identity operator.) By induction and (A3), we define a finite sequence of positive numbers $\{\gamma_i\}_{i=0}^{2N}$ so that

$$\gamma_i \le 1, \ i = 0, 1, \dots, 2N, \ \gamma_{2N} = \epsilon/4, \ \gamma_i \le \gamma_{i+1}, \ i = 0, \dots, 2N-1, \tag{2.4}$$

and for each $j \in \{0, \dots, 2N-1\}$,

$$\rho(Ax, Ay) \le 2^{-1}\gamma_{j+1} \text{ for all } x, y \in B(x_A, S+4) \text{ with } \rho(x,y) \le \gamma_j. \tag{2.5}$$

Set

$$\delta = \min\{\gamma_i : \ i = 0, \dots, 2N\}/2. \tag{2.6}$$

First, we prove the following auxiliary result.

Lemma 2.1. *Suppose that $\{z_i\}_{i=0}^{2N} \subset X$ satisfies*

$$z_0 \in K \textit{ and } \rho(z_{i+1}, Az_i) \le \delta, \ i = 0, \dots, 2N-1. \tag{2.7}$$

Then

$$\rho(z_i, x_A) \le \epsilon, \ i = N, \dots, 2N.$$

Proof. We will show that for $i = 1, \dots, 2N$,

$$\rho(z_i, A^i z_0) \le \gamma_i. \tag{2.8}$$

Clearly, (2.8) holds for $i = 1$ by (2.7) and (2.6).

Assume that $i \in \{2, \dots, 2N\}$ and

$$\rho(z_{i-1}, A^{i-1} z_0) \le \gamma_{i-1}. \tag{2.9}$$

Then (2.7) implies that

$$\rho(z_i, A^i z_0) \le \rho(z_i, Az_{i-1}) + \rho(Az_{i-1}, A(A^{i-1}z_0))$$

$$\le \delta + \rho(Az_{i-1}, A(A^{i-1}z_0)). \tag{2.10}$$

It follows from the definition of γ_{i-1} (see (2.4)), (2.9), (2.7) and (2.3) that

$$A^{i-1}z_0,\ z_{i-1} \in B(x_A, S+1).$$

By these inclusions, the definition of γ_{i-1} (see (2.5) with $j = i-1$) and (2.9),

$$\rho(A(A^{i-1}z_0), Az_{i-1}) \le \gamma_i/2.$$

When combined with (2.10) and (2.6), this inequality implies that

$$\rho(z_i, A^i z_0) \le \delta + \gamma_i/2 \le \gamma_i.$$

Therefore (2.8) is valid for all $i \in \{1, \dots, 2N\}$. Together with (2.2), (2.4), (2.7) and (2.8), this last inequality implies that for all $i \in \{N, \dots, 2N\}$, we have

$$\rho(z_i, x_A) \le \rho(z_i, A^i z_0) + \rho(A^i z_0, x_A) \le \gamma_i + \epsilon/4 \le \epsilon/2.$$

Lemma 2.1 is proved.

Now we are ready to complete the proof of Theorem 1.1.

To this end, assume that $n \ge N$ is a natural number and that the sequence $\{x_i\}_{i=0}^n \subset X$ satisfies

$$x_0 \in K \text{ and } \rho(Ax_i, x_{i+1}) \le \delta,\ i = 0, \dots, n-1.$$

We will show that

$$\rho(x_i, x_A) \le \epsilon,\ i = N, \dots, n. \tag{2.11}$$

If $n \le 2N$, then (2.11) follows from Lemma 2.1. Therefore we may confine our attention to the case where $n > 2N$. Again by Lemma 2.1,

$$\rho(x_i, x_A) \le \epsilon,\ i = N, \dots, 2N. \tag{2.12}$$

Assume by way of contradiction that there exists an integer $q \in (2N, n]$ such that

$$\rho(x_q, x_A) > \epsilon. \tag{2.13}$$

In view of (2.12), we may assume without loss of generality that

$$\rho(x_i, x_A) \le \epsilon,\ i \in \{2N, \dots, q-1\}. \tag{2.14}$$

Define $\{z_i\}_{i=0}^{2N} \subset X$ by

$$z_i = x_{i+q-N},\ i = 0, \dots, N,\ z_{i+1} = Az_i,\ i = N, \dots, 2N-1. \tag{2.15}$$

We will show that the sequence $\{z_i\}_{i=0}^{2N}$ satisfies (2.7). To meet this goal, we only need to show that $z_0 \in K$. By (2.15), (2.14) and (2.12),

$$z_0 = x_{q-N} \text{ and } \rho(z_0, x_A) \le \epsilon.$$

The last inequality and (2.1) imply that $z_0 \in K$. Therefore (2.7) holds. It now follows from Lemma 2.1 and (2.15) that

$$\rho(x_A, x_q) = \rho(x_A, z_N) \le \epsilon.$$

This, however, contradicts (2.13). The contradiction we have reached proves (2.11) and this completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

We may assume without loss of generality that $\epsilon \le 1$. By Theorem 1.1, there exist a natural number N and $\delta_0 \in (0, \epsilon)$ such that the following property holds.

(P1) For each natural number $n \ge N$ and each sequence $\{y_i\}_{i=0}^n \subset X$ which satisfies

$$y_0 \in B(x_A, 4) \text{ and } \rho(y_{i+1}, Ay_i) \le \delta_0,\ i = 0, \dots, n-1, \tag{3.1}$$

the following inequality holds:

$$\rho(y_i, x_A) \le \epsilon,\ i = N, \dots, n. \tag{3.2}$$

By property (A4), the set $A^i(B(x_A, 4))$ is bounded for any integer $i \ge 1$. Choose a number $s > 1$ such that

$$\cup_{i=0}^N A^i(B(x_A, 4)) \subset B(x_A, s). \tag{3.3}$$

By induction and (A3), we define a finite sequence of positive numbers $\{\gamma_i\}_{i=0}^N$ so that

$$\gamma_i \le 1,\ i = 0, \dots, N,\ \gamma_N \le \delta_0/4,\ \gamma_i \le \gamma_{i+1},\ i = 0, \dots, N-1, \tag{3.4}$$

and for each $j \in \{0, \dots, N-1\}$,

$$\rho(Ax, Ay) \le 2^{-1}\gamma_{j+1} \text{ for all } x, y \in B(x_A, s+4) \text{ with } \rho(x, y) \le \gamma_j. \tag{3.5}$$

Set

$$\delta = \gamma_0/4. \tag{3.6}$$

Assume that $\{x_i\}_{i=0}^\infty \subset X$,

$$\rho(x_0, x_A) \le \delta \text{ and } \rho(x_{i+1}, Ax_i) \le \delta,\ i = 0, 1, \dots. \tag{3.7}$$

We will show that

$$\rho(x_i, x_A) \le \epsilon \tag{3.8}$$

for all integers $i \ge 0$. By (3.7), (3.6) and (P1), inequality (3.8) holds for all integers $i \ge N$. Therefore we only need to prove (3.8) for $i < N$. Clearly, (3.8) holds for $i = 0$.

We will show that for $i = 0, \dots, N$, we have

$$\rho(x_i, x_A) = \rho(x_i, A^i x_A) \le \gamma_i. \tag{3.9}$$

By (3.7), (3.6) and (3.4), this is true for $i = 0$. Assume that $i \in \{1, \dots, N\}$ and

$$\rho(x_{i-1}, A^{i-1}x_A) = \rho(x_{i-1}, x_A) \le \gamma_{i-1}. \tag{3.10}$$

Then (3.7) implies that

$$\rho(x_i, x_A) \le \rho(x_i, Ax_{i-1}) + \rho(Ax_{i-1}, x_A) \le \delta + \rho(Ax_{i-1}, x_A). \tag{3.11}$$

It follows from (3.10) and (3.4) that

$$x_{i-1} \in B(x_A, s). \tag{3.12}$$

By (3.12), (3.10) and the definition of γ_{i-1} (see (3.5) with $j = i-1$),

$$\rho(Ax_{i-1}, x_A) \le 2^{-1}\gamma_i. \tag{3.13}$$

Using (3.11), (3.13), (3.6) and (3.4), we obtain

$$\rho(x_i, x_A) \le \delta + 2^{-1}\gamma_i \le \gamma_i.$$

Thus (3.9) indeed holds for all $i \in \{0, \dots, N\}$. This fact, when combined with (3.4), implies that (3.8) is true for all $i \in \{0, \dots, N\}$. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

We may assume, without loss of generality, that $\epsilon < 1$ and that $m \geq 1$ is an integer such that

$$m \geq \rho(x_A, \theta) + 4. \tag{4.1}$$

By Theorem 1.1, there exist $\delta \in (0, \epsilon)$ and a natural number N such that the following property holds.

(P2) For each natural number $n \geq N$ and each sequence $\{x_i\}_{i=0}^n \subset X$ which satisfies

$$x_0 \in B(\theta, m) \text{ and } \rho(Ax_i, x_{i+1}) \leq \delta,\ i = 0, \dots, n-1, \tag{4.2}$$

the following inequality holds:

$$\rho(x_i, x_A) \leq \epsilon,\ i = N, \dots, n. \tag{4.3}$$

Set

$$K_0 = B(\theta, m) \text{ and } K_{i+1} = \{z \in X :\ \rho(z, A(K_i)) \leq 1\},\ i = 0, 1, \dots. \tag{4.4}$$

Clearly, the set K_i is bounded for any integer $i \geq 0$. Choose a natural number $q \geq 8$ such that

$$\cup_{i=0}^{2N} K_i \subset B(\theta, q) \text{ and } 1/q < \delta/8. \tag{4.5}$$

We are going to use the following technical result.

Lemma 4.1. *Assume that*

$$z \in B(\theta, m) \text{ and } \{B_i\}_{i=1}^{2N} \subset \{C \in \mathcal{A} :\ (C, A) \in E_q\}, \tag{4.6}$$

where E_q is given by (1.1). Then

$$\rho(B_i \dots B_1 z, x_A) \leq \epsilon,\ i = N, \dots, 2N. \tag{4.7}$$

Proof. Set

$$z_0 = z \text{ and } z_i = B_i z_{i-1},\ i = 1, \dots, 2N. \tag{4.8}$$

We will show that

$$z_i \in K_i \tag{4.9}$$

for $i = 0, \dots, 2N$. Clearly, (4.9) holds for $i = 0$. Assume that $i \in \{0, \dots, 2N-1\}$ and (4.9) is valid. Inclusions (4.9) and (4.5) imply that

$$z_i \in K_i \subset B(\theta, q). \tag{4.10}$$

When combined with (4.6), (1.1) and (4.8), this last inclusion implies that

$$\rho(Az_i, z_{i+1}) = \rho(Az_i, B_{i+1}z_i) \leq 1/q. \tag{4.11}$$

Consequently, (4.11), (4.10) and (4.4) imply that $z_{i+1} \in K_{i+1}$. Therefore (4.9) is true for all $i = 0, \dots, 2N$. This implies (see (4.5)) that

$$\{z_i\}_{i=0}^{2N} \subset B(\theta, q).$$

It follows from this inclusion, (4.5), (4.6) and (4.8) that for $i = 0, \dots, 2N-1$,

$$\rho(z_{i+1}, Az_i) = \rho(B_{i+1}z_i, Az_i) \leq 1/q < \delta.$$

By (P2), we see that

$$\rho(B_i \dots B_1 z, x_A) = \rho(z_i, x_A) \leq \epsilon, \ i = N, \dots, 2N.$$

Lemma 4.1 is proved.

Now we are ready to complete the proof of Theorem 1.3. To this end, set

$$\mathcal{U} = \{C \in \mathcal{A} : \ (C, A) \in E_q\}. \tag{4.12}$$

Let $n \geq N$ be an integer, $x \in B(\theta, m)$, and $\{B_i\}_{i=1}^n \subset \mathcal{U}$. We will show that

$$\rho(B_i, \cdots B_1 x, x_A) \leq \epsilon \text{ for } i = N, \dots, n. \tag{4.13}$$

If $n \leq 2N$, then (4.13) follows from Lemma 4.1. Therefore we may restrict our attention to the case $n > 2N$. By Lemma 4.1,

$$\rho(B_i \cdots B_1 x, x_A) \leq \epsilon, \ i = N, \dots, 2N. \tag{4.14}$$

Suppose now that there exists an integer $p > 2N$, $p \leq n$, such that

$$\rho(B_p \dots B_1 x, x_A) > \epsilon. \tag{4.15}$$

According to (4.14), we may assume, without loss of generality, that

$$\rho(B_i, \dots B_1 x, x_A) \leq \epsilon, \ i = 2N, \dots, p-1. \tag{4.16}$$

Define $\{D_i\}_{i=0}^{2N} \subset \mathcal{A}$ by

$$D_i = B_{i+p-N}, \ i = 0, \dots, N, \ D_i = A, \ i = N+1, \dots, 2N, \tag{4.17}$$

and let

$$z = B_{p-N} \dots B_1 x.$$

It follows from (4.14), (4.16), (4.17) and (4.1) that

$$\rho(z, x_A) \leq \epsilon \text{ and } z \in B(\theta, m).$$

Applying now Lemma 4.1 to the mappings $\{D_i\}_{i=0}^{2N}$ defined by (4.17), we deduce that

$$\epsilon \geq \rho(D_N \dots D_1 z, x_A) = \rho(x_A, B_p \dots B_{p-N+1} z) = \rho(x_A, B_p \dots B_1 x),$$

which contradicts (4.15). Hence (4.13) is true and Theorem 1.3 is established.

5. Proof of Theorem 1.4

Let $A \in \mathcal{A}_{reg}$ and let $k \geq 1$ be an integer. There is $x_A \in K$ such that

$$Ax_A = x_A. \tag{5.1}$$

According to Theorem 1.3, there exist a natural number $N(A,k)$ and an open neighborhood $\mathcal{U}(A,k)$ of A in $\mathcal{A}$ such that the following property holds.

(P3) For each $x \in B(\theta,k)$, each natural number $n \geq N(A,k)$ and each $B \in \mathcal{U}(A,k)$, we have $\rho(B^n, x_A) \leq 1/k$.

Define

$$\mathcal{F} = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(A,k) : \ A \in \mathcal{A}_{reg}, \ \ \text{an integer } k \geq q\}] \cap \bar{\mathcal{A}}_{reg}. \tag{5.2}$$

Clearly, $\mathcal{F}$ is an everywhere dense G_δ subset of $\bar{\mathcal{A}}_{reg}$.

Let $B \in \mathcal{F}$. We claim that $B \in \mathcal{A}_{reg}$. Indeed, let q be a natural number. There exists a mappping $A_q \in \mathcal{A}_{reg}$ with a fixed point x_{A_q} and a natural number $k_q \geq q$ such that

$$B \in \mathcal{U}(A_q, k_q). \tag{5.3}$$

This inclusion together with (P3) imply that the following property holds.

(P4) For each $x \in B(\theta, q) \subset B(\theta, k_q)$ and each natural number $n \geq N(A_q, k_q)$,

$$\rho(B^n x, x_{A_q}) \leq k_q^{-1} \leq 1/q.$$

Since q is an arbitrary natural number, we obtain that for any $x \in X$, the sequence $\{B^n x\}_{n=1}^{\infty}$ is a Cauchy sequence and its limit is the unique fixed point x_B of B. Thus

$$\lim_{n\to\infty} B^N z = x_B \text{ for any } z \in X.$$

Property (P4) implies that

$$\rho(x_{A_q}, x_B) \leq 1/q. \tag{5.4}$$

Finally, it follows from property (P4) and (5.4) that for any $x \in B(\theta, q)$ and any $n \geq N(A_q, k_q)$,

$$\rho(B^n x, x_B) \leq 2/q.$$

This implies that $B^n x \to x_B$ as $n \to \infty$, uniformly on any bounded subset of X. This completes the proof of Theorem 1.4.

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