Triangular norm-based measures

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1 Introduction

Triangular norm-based measures, or simply T-measures, are special real valuations defined on T-tribes, the latter being classes of fuzzy sets (generalizing the concept of a σ-algebra) based on a triangular norm T. They attracted the interest of scientists because of their applications in fields like mathematical statistics (see [29, 30]), pattern recognition [76, 81], possibility and plausibility theory [41,82] and many-valued logics [32,33,70]. Butnariu & Klement [18, Chapter IV] have used T-measures to identify classes of games for which Aumann-Shapley values in the spirit of [2] exist. Among the basic tools employed in this respect are integral representations of T-measures.

This survey first presents the necessary preliminaries about triangular norms and fuzzy sets, including a concepts of disjointness, emphasizing the special role of Frank t-norms T_λ [31]. Next, T-tribes are introduced, and their most prominent properties are listed. Three different representations for T-measures (for measures with respect to the minimum $T_{\mathbf{M}}$ and to the Łukasiewicz t-norm $T_{\mathbf{L}}$, and for general T_λ-measures) as integral with respect to a suitable Markov kernel follow, as well as a decomposition of monotone T_λ-measures.

The next group of sections concentrates on measures with respect to the Łukasiewicz t-norm $T_{\mathbf{L}}$ and some of their most important properties: Jordan decomposition, absolute continuity, Darboux property, nonatomicity.

The two last sections provide a Liapounoff type theorem concerning the compactness and convexity of the range of vector measures, first for $T_{\mathbf{L}}$-measures (this turned out to be one of the basic tools to show the existence of an Aumann-Shapley value on the space of games with fuzzy coalitions spanned by positive integer powers of monotone $T_{\mathbf{L}}$-measures [18, Chapters IV–VI]) and then for general T_λ-measures, as recently proved by Barbieri & Weber [5].

2 Triangular norms, fuzzy subsets

Fuzzy sets, as introduced by Zadeh [80], are a many-valued generalization of Cantorian sets. For a Cantorian set A the possible truth values of the statement 'x is an element of A' are 0 and 1 (false and true), whereas for a fuzzy set A the truth values of the statement 'x is an element of A' can be

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any number in the interval $[0,1]$. We shall describe fuzzy sets in more detail below — first we want to provide the tools for extending operations such as intersection and union of sets.

In order to do so properly, we consider suitable functions $T : [0,1]^2 \longrightarrow [0,1]$ and $S : [0,1]^2 \longrightarrow [0,1]$, whose restrictions to $\{0,1\}^2$ coincide with conjunction and disjunction, respectively, and which satisfy such general properties like associativity, commutativity and monotonicity.

Such functions are known as triangular norms and triangular conorms, respectively; they proved to be useful in the theory of probabilistic metric spaces, as initiated by Menger [57] and extensively studied by Schweizer & Sklar [74]. In such spaces the distance between two points is measured by probability distribution functions rather than by real numbers, and the triangular norms allow the triangle inequality to be properly generalized.

In continuum physics Menger [58–60] interpreted triangular norms as rules for generating new probabilistically determined objects from existing ones in the psychophysical continuum space. In accordance with one of the main ideas underlying the foundation of modern game theory in Neumann & Morgenstern [67], we shall use triangular norms as models for interaction rules of economic agents in a way which is similar to the modelling of the interaction of physical particles.

Triangular norms are also closely related to the still unsolved general associativity functional equation [22]. In the setting of many-valued logics with truth values in the whole interval $[0,1]$ (generalizing the Boolean two-valued logic whose set of truth values equals $\{0,1\}$), triangular norms are natural models for the interpretation of the conjunction (naturally extending the classical conjunction $\wedge$ of the Boolean logic) [32,33,51,56].

Definition 2.1 (i) A function $T\colon [0,1]^2 \longrightarrow [0,1]$ is called *triangular norm* (*t-norm* for short) if it satisfies the following conditions for all $x,y,z \in [0,1]$:

(T1) $\quad T(x,1) = x,$ $\quad$ (*boundary condition*)

(T2) $\quad T(x,y) \leq T(x,z)$ whenever $y \leq z,$ $\quad$ (*monotonicity*)

(T3) $\quad T(x,y) = T(y,x),$ $\quad$ (*commutativity*)

(T4) $\quad T(x,T(y,z)) = T(T(x,y),z).$ $\quad$ (*associativity*)

(ii) If T is a triangular norm, then the *corresponding triangular conorm* (*t-conorm*) is the function $S\colon [0,1]^2 \longrightarrow [0,1]$ defined by

$$S(x,y) = 1 - T(1-x, 1-y).$$

It is obvious that, given a t-norm T, the corresponding t-conorm S of T satisfies conditions (T2), (T3) and (T4) and, dually to (T1),

(S1) $\quad S(x,0) = x.$ $\quad$ (*boundary condition*)

A function $S\colon [0,1]^2 \longrightarrow [0,1]$ fulfilling (S1) and (T2)–(T4) is called a *triangular conorm* (t-*conorm*). It is easily seen that given a t-conorm S, then $T\colon [0,1]^2 \longrightarrow [0,1]$ defined by

$$T(x,y) = 1 - S(1-x, 1-y).$$

is a t-norm whose corresponding t-conorm is exactly the t-conorm S we started with.

Example 2.2 Some of the most important pairs of t-norms and corresponding t-conorms are the minimum $T_{\mathbf{M}}$ and the maximum $S_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the probabilistic sum $S_{\mathbf{P}}$, the Lukasiewicz t-norm $T_{\mathbf{L}}$ and the Łukasiewicz t-conorm $S_{\mathbf{L}}$, and the drastic product $T_{\mathbf{D}}$ and the drastic sum $S_{\mathbf{D}}$

given by, respectively:

$$\begin{aligned}
T_{\mathbf{M}}(x,y) &= \min(x,y), & S_{\mathbf{M}}(x,y) &= \max(x,y),\\
T_{\mathbf{P}}(x,y) &= x\cdot y, & S_{\mathbf{P}}(x,y) &= x+y-x\cdot y,\\
T_{\mathbf{L}}(x,y) &= \max(0,x+y-1), & S_{\mathbf{L}}(x,y) &= \min(1,x+y),\\
T_{\mathbf{D}}(x,y) &= \begin{cases} x & \text{if } y=1,\\ y & \text{if } x=1,\\ 0 & \text{otherwise,}\end{cases} & S_{\mathbf{D}}(x,y) &= \begin{cases} x & \text{if } y=0,\\ y & \text{if } x=0,\\ 1 & \text{otherwise.}\end{cases}
\end{aligned}$$

From the boundary condition (T1) and the monotonicity (T2) it readily follows that $T_{\mathbf{D}}$ is the smallest and $T_{\mathbf{M}}$ is the largest t-norm, i.e., for each t-norm T we have

$$T_{\mathbf{D}} \le T \le T_{\mathbf{M}}.$$

A t-norm is a rather special semigroup operation on the unit interval $[0,1]$. It is therefore possible to consider additional algebraic properties such as the Archimedean property and strictness.

A t-norm T is called *Archimedean* if for each $(x,y) \in\,]0,1[^2$ there is an $n\in\mathbb{N}$ such that $x_T^{(n)} < y$, where $x_T^{(n)}$ is the n-th power of x with respect to T defined inductively by $x_T^{(0)} = 1$ and

$$x_T^{(n+1)} = T\big(x_T^{(n)}, x\big).$$

A t-norm T is called *strict* if it is continuous and if it satisfies $T(x,y) < T(x,z)$ whenever $x \in\,]0,1]$ and $y,z\in[0,1]$ with $y<z$.

Obviously, each strict t-norm is Archimedean. The Łukasiewicz t-norm $T_{\mathbf{L}}$ continuous and Archimedean but not strict, the drastic product $T_{\mathbf{D}}$ is Archimedean but not continuous (and hence not strict). The minimum $T_{\mathbf{M}}$ is continuous but not Archimedean (and hence not strict), and the product $T_{\mathbf{P}}$ is strict (and therefore Archimedean).

A very special role is played in this survey by the family of Frank t-norms $(T_\lambda)_{\lambda\in[0,\infty]}$ (they were called fundamental t-norms) in [18]. The starting point for the investigation of Frank t-norms was the following problem in connection with the associativity of duals of copulas [66,75] in the framework of distribution functions and probabilistic metric spaces [74]: characterize all continuous (or, equivalently, nondecreasing) associative functions $F\colon [0,1]^2 \longrightarrow [0,1]$, which satisfy for each $x\in[0,1]$ the boundary conditions $F(0,x) = F(x,0)$ and $F(x,1) = F(1,x) = x$, such that the function $G\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$G(x,y) = x+y-F(x,y)$$

is also associative. Frank [31] has shown that then F has to be an ordinal sum [21,55,73] of members of the following family of t-norms. In particular this means that each pair (T_λ, S_λ) of mutually corresponding Frank t-norms and t-conorms satisfies the functional equation

$$T(x,y) + S(x,y) = x+y. \tag{1}$$

Example 2.3 (i) The family of *Frank t-norms* $(T_\lambda)_{\lambda\in[0,\infty]}$ is given by:

$$T_\lambda(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \lambda=0,\\ T_{\mathbf{P}}(x,y) & \text{if } \lambda=1,\\ T_{\mathbf{L}}(x,y) & \text{if } \lambda=\infty,\\ \log_\lambda\left(1+\dfrac{(\lambda^x-1)\cdot(\lambda^y-1)}{\lambda-1}\right) & \text{if } \lambda\in\,]0,\infty[\,\setminus\{1\}.\end{cases}$$

(ii) The family of *Frank t-conorms* $(S_\lambda)_{\lambda\in[0,\infty]}$ is defined as follows:

$$S_\lambda(x,y) = \begin{cases} S_{\mathbf{M}}(x,y) & \text{if } \lambda = 0, \\ S_{\mathbf{P}}(x,y) & \text{if } \lambda = 1, \\ S_{\mathbf{L}}(x,y) & \text{if } \lambda = \infty, \\ 1 - \log_\lambda \left(1 + \dfrac{(\lambda^{1-x}-1)\cdot(\lambda^{1-y}-1)}{\lambda - 1}\right) & \text{if } \lambda \in \,]0,\infty[\, \setminus \{1\}. \end{cases}$$

All Frank t-norms are continuous, and a Frank t-norm T_λ is strict if and only if $\lambda \in \,]0,\infty[$. Recall that T_∞ $(= T_{\mathbf{L}})$ is Archimedean but not strict, and that T_0 $(= T_{\mathbf{M}})$ is neither Archimedean nor strict. The family $(T_\lambda)_{\lambda\in[0,\infty]}$ is a continuous family in the sense that for each $\lambda_0 \in [0,\infty]$ we have

$$\lim_{\lambda\to\lambda_0} T_\lambda = T_{\lambda_0}.$$

A nontrivial result is that the family of Frank t-norms $(T_\lambda)_{\lambda\in[0,\infty]}$ is strictly decreasing with respect to the parameter λ, i.e., for $\lambda_1, \lambda_2 \in [0,\infty]$ we have $T_{\lambda_1} < T_{\lambda_2}$ if and only if $\lambda_1 > \lambda_2$. A first (and rather complicated) proof of this monotonicity of the family of Frank t-norms was given by Butnariu & Klement [18, Proposition 1.12], a significantly shorter proof based on the additive generators of the Frank t-norms was published in [48, Example 3.2]. Many more details about Frank t-norms are contained in [31] and in Section 4.4 of Klement, Mesiar & Pap [49].

Turning back to an arbitrary t-norm T, the associativity (T4) allows us to extend this binary operation in a unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1, x_2, \ldots, x_n) \in [0,1]^n$

$$\mathop{\mathrm{T}}_{i=1}^{n} x_i = T\Big(\mathop{\mathrm{T}}_{i=1}^{n-1} x_i, x_n\Big) = T(x_1, x_2, \ldots, x_n).$$

The fact that each t-norm T is bounded from above by $T_{\mathbf{M}}$ makes it possible to extend it to a (countably) infinitary operation, putting for each sequence $(x_i)_{i\in\mathbb{N}}$ of elements of $[0,1]$

$$\mathop{\mathrm{T}}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathop{\mathrm{T}}_{i=1}^{n} x_i.$$

Note that the limit on the right-hand side always exists since the sequence

$$\Big(\mathop{\mathrm{T}}_{i=1}^{n} x_i\Big)_{n\in\mathbb{N}}$$

is nonincreasing and bounded from below.

Coming back to sets, a well-known fact about Cantorian sets (for which we shall use the term *crisp set* to distinguish them from the fuzzy sets to be defined below) is that, given a universe of discourse X, each subset A of X can be identified with its characteristic function $A\colon X \longrightarrow \{0,1\}$, assigning the value 1 to all elements of X which belong to A, and the value 0 to all remaining elements of X.

It was exactly this concept of a characteristic function which was generalized by Zadeh [80] who replaced the two-element set $\{0,1\}$ by the unit interval $[0,1]$, therefore considering a continuum of truth values rather than the two classical truth values true and false (which correspond to 1 and 0, respectively) only.

Therefore, given a (crisp) universe of discourse X, a *fuzzy subset* A of X (or, briefly, a *fuzzy set* A) is characterized by its *membership function*

$$A\colon X \longrightarrow [0,1],$$

where for $x \in X$ the value $A(x)$ is interpreted as the *degree of membership* of x in the fuzzy set A. The class of all fuzzy subsets of X will be denoted $X^{[0,1]}$.

In a logical setting, the degree of membership $A(x)$ can also be seen as the *truth value* of the statement '*x is element of A*'.

Since the membership function of a fuzzy subset of X generalizes the characteristic function of a crisp subset of X, each crisp subset of X is just a special case of a fuzzy subset of X, and its characteristic function is a special membership function (assuming the values 0 and 1 only). Subsequently, the universe X and the empty set $\emptyset$ are identified with the constant functions mapping each $x \in X$ to 1 and 0, respectively.

Given a fuzzy subset A of a universe X we can associate with it several crisp subsets of X, among them the *kernel* $\ker(A)$ and the *support* $\mathrm{supp}(A)$ defined by, respectively,

$$\ker(A) = \{x \in X \mid \mu_A(x) = 1\}, \tag{2}$$
$$\mathrm{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}. \tag{3}$$

Let T be a t-norm and S its corresponding t-conorm. In order to define the intersection and the union of two fuzzy subsets A and B of a universe X, we extend the binary operations T and S on $[0,1]$ to $[0,1]^X$ pointwise, i.e.,

$$(A \mathsf{T} B)(x) = T(A(x), B(x)),$$
$$(A \mathsf{S} B)(x) = S(A(x), B(x)).$$

Finite and countable intersection and unions are introduced in a straightforward way. Note that the duality of T and S implies the validity of the De Morgan laws, i.e., for a sequence $(A_n)_{n\in\mathbb{N}}$ of fuzzy subsets of X we have

$$\complement\Big(\mathop{\mathrm{T}}_{n=1}^{\infty} A_n\Big) = \mathop{\mathrm{S}}_{n=1}^{\infty} \complement A_n,$$
$$\complement\Big(\mathop{\mathrm{S}}_{n=1}^{\infty} A_n\Big) = \mathop{\mathrm{T}}_{n=1}^{\infty} \complement A_n,$$

where the complement $\complement A$ of a fuzzy subset A of X is defined by $\complement A(x) = 1 - A(x)$.

Restricted to crisp sets, these operations coincide with intersection, union and complement, respectively, regardless which t-norm and t-conorm are considered. The class $[0,1]^X$ of all fuzzy subsets of X, together with either one of the operations T and S, forms a partially ordered commutative semigroup having $\emptyset$ as smallest element and X as largest element. However, $[0,1]^X$ provided with the operations T, S and the complement $\complement$ is not a Boolean algebra. In general, T and S are not distributive with respect to each other, and $A \mathsf{T} \complement A$ may be different from $\emptyset$ and $A \mathsf{S} \complement A$ may be different from X.

In order to classify T-measures and to distinguish between T-measures and other kinds of valuations which can be defined on T-tribes (see, for instance, [72]), we also need a concept of disjointness. There are different approaches to disjointness (mostly in the context of fuzzy partitions [23, 26–28, 38–40, 43, 52, 64, 71, 77]. Our concept is based on Butnariu [12] (see also [18]).

Definition 2.4 Let X be a nonempty set, T a t-norm and S its corresponding t-conorm.

(i) A finite family $(A_1, A_2, \ldots, A_n)$ of fuzzy sets in $[0,1]^X$ is said to be *T-disjoint* if for each $k \in \{1, \ldots, n\}$

$$\text{(TD)} \qquad \Big(\mathop{\mathrm{S}}_{j\neq k} A_j\Big) \mathsf{T} A_k = \emptyset.$$

(ii) A countable family $(A_n)_{n\in\mathbb{N}} \subseteq [0,1]^X$ is called T-disjoint if, for each $n \in \mathbb{N}$, the finite family $(A_1, \ldots, A_n)$ is T-disjoint.

(iii) A sequence $(A_n)_{n\in\mathbb{N}} \subseteq [0,1]^X$ is called a *T-partition* of a fuzzy set A if it is T-disjoint and if

(TP) $\qquad A = \mathop{\mathbf{S}}_{n=1}^{\infty} A_n.$

Each subfamily $(A_n)_{n\in I}$ of an at most countable T-disjoint family $(A_n)_{n\in J}$ is also T-disjoint: it suffices to show this for finite index sets J and $I \subseteq J$. Indeed, for each $i \in I$ we get

$$\Big(\mathop{\mathbf{S}}_{n\in I\setminus\{i\}} A_n \Big) \mathbf{T} A_i \leq \Big(\mathop{\mathbf{S}}_{n\in J\setminus\{i\}} A_n \Big) \mathbf{T}\, A_i = \emptyset.$$

Also, the definition of the T-disjointness does not depend on the order in which the fuzzy sets $(A_n)_{n\in\mathbb{N}}$ are numbered, i.e., if π is a permutation of $\mathbb{N}$ and the countable family $(A_n)_{n\in\mathbb{N}}$ is T-disjoint, then the permuted family $(A_{\pi(n)})_{n\in\mathbb{N}}$ is T-disjoint too.

Different t-norms may lead to different concepts of disjointness. However, for some classes of t-norms the corresponding disjointness concepts do not depend on the choice of the t-norm in that class.

Example 2.5 Let $(A_n)_{n\in\mathbb{N}}$ be a countable family of fuzzy subsets of X.

(i) If all A_n are crisp subsets of X, then for each t-norm T the T-disjointness is equivalent to the pairwise disjointness of the crisp sets.

(ii) For each t-norm T, the T-disjointness implies pairwise T-disjointness, but the converse is not generally true: if we take the family of constant fuzzy sets (A_1, A_2, A_3) given by $A_1(x) = A_2(x) = A_3(x) = 0.5$, then A_1, A_2, A_3 are pairwise $T_{\mathbf{L}}$-disjoint, but the family (A_1, A_2, A_3) is not $T_{\mathbf{L}}$-disjoint.

(iii) For $\lambda \in [0, \infty[$ we get: $(A_n)_{n\in\mathbb{N}}$ is T_λ-disjoint if and only if each element $x \in X$ is "contained" in at most one A_k (i.e., if and only if $A_k(x) > 0$ for at most one $k \in \mathbb{N}$).

(iv) The $T_{\mathbf{D}}$-disjointness of $(A_n)_{n\in\mathbb{N}}$ means that for each $x \in X$ exactly one of the following conditions holds:

  (a) There is exactly one $k \in \mathbb{N}$ such that $A_k(x) = 1$, and $A_n(x) = 0$ for all $n \neq k$.

  (b) There are at most two indices $k, l \in \mathbb{N}$ such that $\{A_k(x), A_l(x)\} \subset\,]0, 1[$, and we have $A_n(x) = 0$ for $n \notin \{k, l\}$.

The following results (see Lemma 4.4 and Corollary 4.5 in [18]) emphasize the crucial role of the functional equation (1) with respect to disjointness.

Lemma 2.6 *Let T be a t-norm and S be its corresponding t-conorm such that the functional equation (1) holds, and let $(A_1, A_2 \ldots, A_n)$ be a finite family of fuzzy subsets of X. Then for each $n \geq 2$ the following are equivalent:*

(i) The family $(A_1, A_2, \ldots, A_n)$ is T-disjoint.

(ii) For all $k \in \{2, \ldots, n\}$ we have $\Big(\mathop{\mathbf{S}}_{i=1}^{k-1} A_i \Big) \mathbf{T}\, A_k = \emptyset.$

(iii) For all $k \in \{2, \ldots, n\}$ we have $\mathop{\mathbf{S}}_{i=1}^{k} A_i = \sum_{i=1}^{k} A_i.$

(iv) For each subset $I \subseteq \{1, 2, \ldots, n\}$ containing at least $n-1$ elements we have $\mathop{\mathbf{S}}_{i\in I} A_i = \sum_{i\in I} A_i.$

Corollary 2.7 *Let T be a t-norm and S its corresponding t-conorm such that the functional equation (1) holds, and let $(A_n)_{n\in\mathbb{N}}$ be a countable family of fuzzy subsets of X. Then the following are equivalent:*

(i) The family $(A_n)_{n\in\mathbb{N}}$ is T-disjoint.

(ii) For all $k \geq 2$ we have $\Big(\mathop{\mathbf{S}}\limits_{i=1}^{k-1} A_i\Big) \mathbf{T}\, A_k = \emptyset$.

(iii) For all $k \geq 2$ we have $\mathop{\mathbf{S}}\limits_{i=1}^{k} A_i = \sum\limits_{i=1}^{k} A_i$.

(iv) For each finite subset $I \subseteq \mathbb{N}$ we have $\mathop{\mathbf{S}}\limits_{i\in I} A_i = \sum\limits_{i\in I} A_i$.

As an immediate consequence we have that an at most countable family of fuzzy sets $(A_n)_{n\in J}$ is $T_{\mathbf{L}}$-disjoint if and only if $\sum_{n\in J} A_n \leq 1$.

From Lemma 2.6 and Corollary 2.7 we know that if T and its corresponding t-conorm S satisfy (1) and if $(A_n)_{n\in J}$ is T-disjoint, then necessarily $\sum_{n\in J} A_n \leq 1$. However, the converse is not generally true (see Example 2.5(iii)). Also, the requirement that T and S satisfy (1) cannot be dropped in Lemma 2.6 and in Corollary 2.7. For instance, for the t-norm $T_{\mathbf{D}}$ and its corresponding t-conorm $S_{\mathbf{D}}$ the conditions (i), (ii) and (iii) are no longer equivalent.

3 T-tribes

We now study certain subclasses of $[0,1]^X$, the class of all fuzzy subsets of a (crisp) universe of discourse X, which are closed under intersections and unions based on t-norms and t-conorms, respectively. In many situations it is also desirable to have also a kind of σ-completeness. The following concept of T-tribes was presented in [18], it basically was introduced in Klement [47] for arbitrary t-norms T, in Klement [45] for $T = T_{\mathbf{M}}$ and in Butnariu [12] for $T = T_{\mathbf{L}}$ (see also [24]). For all lattice theoretical notions we usually refer to Birkhoff [9], in the context of algebras and σ-algebras of crisp sets to Halmos [36] and Bauer [6].

Definition 3.1 Let T be a t-norm.

(i) A subclass $\mathcal{C}$ of $[0,1]^X$ is called a T-*clan* on X if the following properties are satisfied:

- (TC1) $\emptyset \in \mathcal{C}$.
- (TC2) For all $A \in \mathcal{C}$ we have $\complement A \in \mathcal{C}$.
- (TC3) For all $A, B \in \mathcal{C}$ we have $A \mathbf{T} B \in \mathcal{C}$.

(ii) A T-clan $\mathcal{T} \subseteq [0,1]^X$ is called a T-*tribe* if

- (TT) for all sequences $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ we have $\mathop{\mathbf{T}}\limits_{n=1}^{\infty} A_n \in \mathcal{T}$.

Trivially, each algebra of crisp subsets of X [6,36] is a T-clan with respect to an arbitrary t-norm T. Moreover, a subclass $\mathcal{A}$ of $\mathcal{P}(X)$, i.e., a class of crisp subsets of X, is a σ-algebra if and only if it is a T-tribe with respect to an arbitrary t-norm T.

Each $T_{\mathbf{M}}$-clan is a De Morgan algebra [42], and each $T_{\mathbf{L}}$-clan is a semi-simple MV-algebra as introduced in Chang [19] (compare also [3,7,8,25,42]). Also, each $T_{\mathbf{M}}$-tribe is a σ-complete De Morgan algebra, and each $T_{\mathbf{L}}$-tribe is both a σ-complete semi-simple MV-algebra and a clan in the sense of Wyler [79].

Obviously, not each T-clan is a T-tribe (a rather simple counterexample is given by the $T_{\mathbf{L}}$-clan $\{A \mid \text{for all } x \in X : A(x) \in \mathbb{Q}\}$ which is not a $T_{\mathbf{L}}$-tribe).

Example 3.2 (i) Define for each $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n-1\}$ the fuzzy subset $A_{\frac{k}{n}}$ of X by $A_{\frac{k}{n}}(x) = \frac{k}{n}$. Then the class $\mathcal{T}_n = \big\{\emptyset, A_{\frac{1}{n}}, A_{\frac{2}{n}}, \ldots, A_{\frac{n-1}{n}}, X\big\}$ is both a $T_{\mathbf{M}}$-clan and a $T_{\mathbf{L}}$-clan, but it is never a T-clan when T is a strict t-norm.

(ii) Given a Borel measurable t-norm T and a σ-algebra $\mathcal{A}$ of crisp subsets of X, then the class $\mathcal{A}^\wedge$ of all fuzzy subsets of X with $\mathcal{A}$-measurable membership functions is a T-tribe. Note that $\mathcal{A}^\wedge$ was called a *generated (T-)tribe* in [18,45,47], and that it was used in [81] as the class of fuzzy events.

(iii) Given a T-tribe $\mathcal{T}$, the set $\mathcal{T}^\vee$ of all crisp sets in $\mathcal{T}$ is a σ-algebra, and hence a T-tribe.

(iv) The class $\mathcal{T}$ consisting of all fuzzy subsets of X, whose membership functions are either constant or have all their values in the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, is both a $T_{\mathbf{D}}$-tribe and a $T_{\mathbf{M}}$-tribe, but no T-tribe whenever T is a continuous Archimedean t-norm [47].

(v) If in (iv) we additionally require that all the membership functions of the elements of $\mathcal{T}$ be continuous, then $\mathcal{T}$ is a $T_{\mathbf{D}}$-tribe, but no longer a $T_{\mathbf{M}}$-tribe [47].

(vi) Consider a crisp subset Y of X. Given a σ-algebra $\mathcal{A}$ of crisp subsets of X, the class $\mathcal{A}_Y^\wedge$ of all fuzzy subsets of X, whose membership functions are $\mathcal{A}$-measurable and assume on Y only the values 0 and 1, is a T-tribe with respect to an arbitrary Borel measurable t-norm T. Note however that, up to the case $Y = X$, $\mathcal{A}_Y^\wedge$ does not contain any fuzzy subset of X with a nontrivial constant membership function (compare Theorem 3.3(iii)). The tribe $\mathcal{A}_Y^\wedge$ was called a *semigenerated (T-)tribe* in Mesiar [61].

The following result for tribes based on Frank t-norms (see Propositions 2.7 and 3.3 and Theorem 3.2 in [18]) demonstrates the special role of the Frank t-norms in this context. The crucial argument is, of course, that the Frank t-norms and their dual t-conorms are solutions of the Frank functional equation (1).

Theorem 3.3 *For all $\lambda \in \,]0, \infty]$ the following assertions hold:*

(i) Each T_λ-tribe is a $T_{\mathbf{M}}$-tribe (and, hence, a σ-complete De Morgan algebra) as well as a $T_{\mathbf{L}}$-tribe (and, hence, a σ-complete semi-simple MV-algebra).

(ii) For each T_λ-tribe $\mathcal{T}$ we have $\mathcal{T} \subseteq (\mathcal{T}^\vee)^\wedge$, i.e., each element in $\mathcal{T}$ has a $\mathcal{T}^\vee$-measurable membership function.

(iii) A T_λ-tribe $\mathcal{T}$ is generated if and only if it contains all fuzzy subsets of X which have a constant membership function.

The interesting problem to characterize all subclasses of $[0,1]^X$ which are T-tribes with respect to an arbitrary Borel measurable t-norm T was solved in Navara [65] (compare also Mesiar [61,62] and Mesiar & Navara [63]). The result turns out to be a generalization of the fact that each class of crisp subsets of X is a σ-algebra if and only if it is a T-tribe with respect to an arbitrary t-norm T, and of Theorem 3.3(i) (for the definition of $\mathrm{supp}(A)$ and $\ker(A)$ see (3) and (2)).

Theorem 3.4 *Let X be a nonempty set and $\mathcal{T}$ a subclass of $[0,1]^X$. Then the following are equivalent:*

(i) $\mathcal{T}$ is a T-tribe for all Borel measurable t-norms T.

(ii) There is a $\lambda \in \,]0, \infty[$ such that $\mathcal{T}$ is a T_λ-tribe.

(iii) $\mathcal{T}$ is a weakly generated tribe, i.e., there is a σ-ideal $\mathcal{D}$ of $\mathcal{T}^\vee$ such that

$$\mathcal{T} = \{A \in (\mathcal{T}^\wedge)^\vee \mid \mathrm{supp}(A) \setminus \ker(A) \in \mathcal{D}\}.$$

This means that a T_λ-tribe for $\lambda \in \,]0, \infty[$ on X is not only a σ-complete lattice (in particular, it is closed under $T_{\mathbf{M}}$) and a σ-complete semi-simple MV-algebra (in particular, it is closed under $T_{\mathbf{L}}$), but that it is also closed under each Borel measurable t-norm. Observe that this is not true for $T_{\mathbf{L}}$-tribes:

for example, the $T_{\mathbf{L}}$-clan mentioned in Example 3.2(i) is a $T_{\mathbf{L}}$-tribe, but no T-tribe if T is a strict t-norm (see also Example 3.6 below).

We already mentioned that each $T_{\mathbf{L}}$-tribe is a σ-complete semi-simple MV-algebra. In fact, each semi-simple MV-algebra is isomorphic to some $T_{\mathbf{L}}$-clan (see Belluce [7,8], a complete characterization of all MV-algebras, involving nonstandard real numbers, can be found in Di Nola [25]). Another characterization of $T_{\mathbf{L}}$-tribes was given in Klement & Navara [50]. Both results are summarized below.

Theorem 3.5 *Let $\mathcal{T}$ be a subclass of $[0,1]^X$. Then the following are equivalent:*

(i) $\mathcal{T}$ is a $T_{\mathbf{L}}$-tribe.

(ii) $\mathcal{T}$ is a σ-complete semi-simple MV-algebra.

(iii) There is a σ-algebra $\mathcal{A}$ of crisp subsets of X and a sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of σ-filters in $\mathcal{A}$, with $\mathcal{E}_m \subseteq \mathcal{E}_n$ whenever n is a divisor of m, such that

$$\mathcal{T} = \big\{A \in \mathcal{A}^\wedge \;\big|\; \{x \in X \mid n \cdot A(x) \in \mathbb{N}\} \in \mathcal{E}_n \text{ for all } n \in \mathbb{N}\big\}.$$

Combining all these results, it is possible to give, as a simple example, a rather complete picture of all possible tribes with respect to Frank t-norms, to arbitrary strict t-norms, and to the weakest t-norm $T_{\mathbf{D}}$, respectively.

Example 3.6 Let $\{x_0\}$ be a singleton. Then the only generated tribe on $\{x_0\}$, let us call it $\mathcal{T}_\infty$, is just isomorphic to the unit interval $[0,1]$. There are two semigenerated tribes, namely, $\mathcal{T}_\infty$ and $\mathcal{T}_1$ (which is isomorphic to $\{0,1\}$). Since $\{x_0\}$ is finite, these are also the only weakly generated tribes. Due to Theorem 3.4 each T_λ-tribe with $\lambda \in\,]0,\infty[$ equals either $\mathcal{T}_\infty$ or $\mathcal{T}_1$, the same being true for each strict t-norm. The only $T_{\mathbf{L}}$-tribes on $\{x_0\}$ are $\mathcal{T}_\infty$ and $\mathcal{T}_n$ with $n \in \mathbb{N}$ (as mentioned in Example 3.2(i)). Each $T_{\mathbf{M}}$-tribe on $\{x_0\}$ is isomorphic to some closed subset of $[0,1]$ which contains 0 and which is symmetric with respect to the midpoint 0.5, and each $T_{\mathbf{D}}$-tribe on $\{x_0\}$ is isomorphic to some arbitrary subset of $[0,1]$ which contains 0 and which is symmetric with respect to 0.5.

4 T-measures and their representation by Markov kernels

Definition 4.1 Let T be a t-norm, S its corresponding t-conorm, let $\mathcal{T}$ be a T-clan and let $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty,\infty]$ be a function which assumes at most one of the values $-\infty$ and ∞.

(i) The function $\mathbf{m}$ is called a *T-valuation on* $\mathcal{T}$) if it satisfies the following conditions:

(TM1) $\quad \mathbf{m}(\emptyset) = 0.$

(TM2) $\quad \mathbf{m}(A \mathbin{\mathsf{T}} B) + \mathbf{m}(A \mathbin{\mathsf{S}} B) = \mathbf{m}(A) + \mathbf{m}(B).$

(ii) The function $\mathbf{m}$ is said to be *finitely T-additive* if it satisfies (TM1) and

(TM3) $\quad \mathbf{m}(A \mathbin{\mathsf{S}} B) = \mathbf{m}(A) + \mathbf{m}(B) \quad$ whenever $\quad A \mathbin{\mathsf{T}} B = \emptyset.$

(iii) The function $\mathbf{m}$ is said to be *monotone* if

(TM4) $\quad \mathbf{m}(A) \leq \mathbf{m}(B) \quad$ whenever $\quad A \leq B.$

Remark 4.2 (i) If $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty,\infty]$ is a T-valuation on the T-clan $\mathcal{T}$, then $\mathbf{m}$ is also finitely T-additive, the converse not being generally true: if, for instance, $\mathcal{T}$ consists of all the constant functions in $[0,1]^X$ and if $\lambda \in [0,\infty[$, then because of the absence of any nontrivial T_λ-disjoint elements in the T_λ-clan $\mathcal{T}$, each function $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty,\infty]$ which satisfies (TM1) is finitely

T_λ-additive without being necessarily a T_λ-valuation. This shows that T-valuations are particular additive functions in the sense of Schmidt [72, p. 558] and, consequently, if they are finite, they can be represented as differences of monotone T-additive functions (compare [72, Theorem 2.2]). However, it is not possible to conclude directly that the T-valuations have Jordan decompositions.

(ii) If $\mathcal{T}$ is a $T_{\mathbf{L}}$-clan and $\mathbf{m}$ is a finite $T_{\mathbf{L}}$-additive function on $\mathcal{T}$, then $\mathbf{m}$ is also a $T_{\mathbf{L}}$-valuation.

(iii) If $\mathcal{T}$ is an algebra of crisp subsets of X and if T^* is some t-norm, then the finite T^*-additive functions are T-valuations for each t-norm T.

Definition 4.3 (i) Let $\mathcal{T}$ be a T-clan. A function $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty,\infty]$ is called a *T-measure* if it is a T-valuation and if it is continuous from the left in the following sense:

$$\text{(TM5)} \qquad \lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(A) \quad \text{whenever} \quad (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}, (A_n)_{n\in\mathbb{N}} \nearrow A \text{ and } A \in \mathcal{T}.$$

(ii) If a T-measure $\mathbf{m}$ on $\mathcal{T}$ assumes only values in [0,1]and if m(X) = 1, we shall call $\mathbf{m}$ a *probability T-measure.*

(iii) Let $\mathcal{T}$ be a T-tribe. The function $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty,\infty]$ is said to be *countably T-additive* if it satisfies (TM1) and if for each sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}$ we have

$$\text{(TM6)} \qquad \mathbf{m}\Big(\mathop{\mathrm{S}}_{i=1}^{\infty} A_n\Big) = \sum_{i=1}^{\infty} \mathbf{m}(A_n) \quad \text{whenever} \quad (A_n)_{n\in\mathbb{N}} \text{ is } T\text{-disjoint.}$$

It is clear that countably T-additive functions are always finitely T-additive. Moreover, T-measures on T-tribes are countably T-additive since if we take a T-disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in the T-tribe $\mathcal{T}$ and define $B_n = \mathrm{S}_{i=1}^{n} A_n$, then $(B_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{T}$ which converges nondecreasingly to $\mathrm{S}_{n=1}^{\infty} A_n$ and, therefore,

$$\mathbf{m}\Big(\mathop{\mathrm{S}}_{n=1}^{\infty} A_n\Big) = \mathbf{m}\big(\lim_{n\to\infty} B_n\big) = \lim_{n\to\infty} \mathbf{m}(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{m}(A_n) = \sum_{n=1}^{\infty} \mathbf{m}(A_n).$$

Remark 4.4 In general, for arbitrary t-norms T and T-tribes $\mathcal{T}$, the countable T-additivity does not imply the continuity from the left condition (TM5), and thus countably T-additive functions are not necessarily T-measures. For example, if $\lambda \in [0,\infty[$ and $\mathcal{T} = [0,1]^X$, then for each fixed $x_0 \in X$ the function $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty,\infty]$ defined by

$$\mathbf{m}(A) = \begin{cases} 1 & \text{if } A(x_0) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is countably T_λ-additive, but it is not a T_λ-measure since it does not satisfy (TM5).

However, it will be shown later (see Proposition 8.1) that countably $T_{\mathbf{L}}$-additive functions are $T_{\mathbf{L}}$-measures, that is, the countable $T_{\mathbf{L}}$-additivity and the continuity from the left are equivalent in this case. The same is true when T is an arbitrary t-norm and $\mathcal{T}$ consists of crisp subsets of X only. If $\mathcal{T}$ is a σ-algebra of crisp subsets of X, then each (σ-additive) measure on $\mathcal{T}$ is a T-measure with respect to any t-norm T. In this case, the countable T-additivity and the continuity from the left are equivalent too.

The following result will turn out to be useful when we represent T_λ-measures by Markov kernels.

Lemma 4.5 *If T is a t-norm and if $\mathcal{T}$ is both a T-clan and a $T_{\mathbf{M}}$-clan, then each T-valuation on $\mathcal{T}$ is also a $T_{\mathbf{M}}$-valuation on $\mathcal{T}$.*

Proof: For $A, B \in \mathcal{T}$ we get

$$\begin{aligned}\mathbf{m}(A \mathsf{T}_\mathsf{M} B) + \mathbf{m}(A \mathsf{S}_\mathsf{M} B) &= \mathbf{m}((A \mathsf{T}_\mathsf{M} B) \mathsf{T} (A \mathsf{S}_\mathsf{M} B) + \mathbf{m}((A \mathsf{T}_\mathsf{M} B) \mathsf{S} (A \mathsf{S}_\mathsf{M} B)) \\ &= \mathbf{m}(A \mathsf{T} B) + \mathbf{m}(A \mathsf{S} B) \\ &= \mathbf{m}(A) + \mathbf{m}(B),\end{aligned}$$

showing that $\mathbf{m}$ is indeed a $T_\mathbf{M}$-valuation. □

Remark 4.6 (i) Lemma 4.5 shows that if $\mathbf{m}$ is a T-measure on $\mathcal{T}$ ($\mathcal{T}$ being both a T- and a $T_\mathbf{M}$-clan), then $\mathbf{m}$ is also a $T_\mathbf{M}$-measure.

(ii) Taking into account Theorem 3.3(i), each $\mathbf{m}$ is a T_λ-measure on a T_λ-tribe $\mathcal{T}$ with $\lambda \in \,]0, \infty]$ is also a $T_\mathbf{M}$-measure.

(iii) The converse of Lemma 4.5 does not generally hold: for instance, let $\mathcal{T}$ be the family of all Borel measurable fuzzy sets on $X = [0, 1]$. Then the function $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty, \infty]$ defined by

$$\mathbf{m}(A) = \int_{\{A>0\}} (1 + A(x))\, \mathrm{d}x$$

is a $T_\mathbf{M}$-valuation (even a $T_\mathbf{M}$-measure) but not a $T_\mathbf{L}$-valuation.

The $T_\mathbf{M}$-measures play a fundamental role in the following exposition. In order to give an integral representation for them we need the notion of Markov kernels (for more details see, e.g., Bauer [6]).

Definition 4.7 Let $(X, \mathcal{A})$ be a measurable space, $\mathcal{B}$ be the family of all Borel subsets of $[0, 1]$, and $\mathcal{B}_1 = \mathcal{B} \cap [0, 1[$. A function $K\colon X \times \mathcal{B}_1 \longrightarrow \mathbb{R}$ is called an $\mathcal{A}$-*Markov kernel* if it satisfies the following conditions:

(i) For each $x \in X$, the function $K(x, \cdot)\colon \mathcal{B}_1 \longrightarrow \mathbb{R}$ defined by $K(x, \cdot)(B) = K(x, B)$ is a probability measure on $\mathcal{B}_1$.

(ii) For each $B \in B_1$, the function $K(\cdot, B)\colon X \longrightarrow \mathbb{R}$ defined by $K(\cdot, B)(x) = K(x, B)$ is $\mathcal{A}$-measurable.

From Example 3.2(ii) we know that if $(X, \mathcal{A})$ is a measurable space, then the family $\mathcal{A}^\wedge$ of all $\mathcal{A}$-measurable functions from X to $[0, 1]$ is a $T_\mathbf{M}$-tribe. The next result, from Klement [44], shows that $T_\mathbf{M}$-measures on $\mathcal{A}^\wedge$ can be represented as integrals of Markov kernels in the following way.

Theorem 4.8 *Let $\mathcal{A}$ be a σ-algebra of crisp subsets of X and $\mathcal{A}^\wedge$ the family of all $\mathcal{A}$-measurable functions in $[0, 1]^X$. If $\mathbf{m}$ is a finite monotone $T_\mathbf{M}$-measure on $\mathcal{A}^\wedge$, then there exists a unique measure $\check{\mathbf{m}}$ on $\mathcal{A}$, namely, the restriction of $\mathbf{m}$ to $\mathcal{A}$, and an $\check{\mathbf{m}}$-almost everywhere uniquely determined $\mathcal{A}$-Markov kernel $K\colon X \times \mathcal{B}_1 \longrightarrow \mathbb{R}$ such that for all $A \in \mathcal{A}$*

$$\mathbf{m}(A) = \int_X K(x, [0, A(x)[)\, \mathrm{d}\check{\mathbf{m}}(x). \tag{4}$$

Proof: The proof is carried out in several steps. First, observe that if (4) holds for some measure $\check{\mathbf{m}}$ on $\mathcal{A}$, then $\check{\mathbf{m}}$ has to be necessarily the restriction of $\mathbf{m}$ to $\mathcal{A}$ since for $M \in \mathcal{A}$ equality (4) gives

$$\mathbf{m}(M) = \int_M K(x, [0, 1[)\, \mathrm{d}\check{\mathbf{m}}(x) = \check{\mathbf{m}}(M). \tag{5}$$

Without loss of generality we may restrict our proof to the case when $\mathbf{m}(X) = 1$.

Claim 1. For each rational number $\alpha \in [0,1]$, the function $\mathbf{m}_\alpha \colon \mathcal{A} \longrightarrow [0,1]$, defined by

$$\mathbf{m}_\alpha(M) = \mathbf{m}(\alpha \cdot M), \tag{6}$$

is a nonnegative finite measure on $\mathcal{A}$, and it satisfies $\mathbf{m}_\alpha \leq \check{\mathbf{m}}$.

Obviously, $\mathbf{m}_\alpha(\emptyset) = 0$ and $\mathbf{m}_\alpha$ is finite and monotone because of the corresponding properties of $\mathbf{m}$. To prove σ-additivity of $\mathbf{m}_\alpha$, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of mutually disjoint sets in $\mathcal{A}$. Then

$$\begin{aligned}
\mathbf{m}_\alpha\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) &= \mathbf{m}\Big(\sup\Big\{\Big(\alpha \cdot \Big(\bigcup_{i=1}^{n} A_i\Big)\Big) \;\Big|\; n \in \mathbb{N}\Big\}\Big) \\
&= \sup\Big\{\mathbf{m}\Big(\alpha \cdot \Big(\bigcup_{i=1}^{n} A_i\Big)\Big) \;\Big|\; n \in \mathbb{N}\Big\} \\
&= \sup\Big\{\sum_{i=1}^{n} \mathbf{m}(\alpha \cdot A_i) \;\Big|\; n \in \mathbb{N}\Big\} \\
&= \sum_{n=1}^{\infty} \mathbf{m}(\alpha \cdot A_n).
\end{aligned}$$

The last equality follows from the monotonicity of $\mathbf{m}$. Therefore, Claim 1 is proved.

Claim 1 implies that each measure $\mathbf{m}_\alpha$ defined by (6) is $\check{\mathbf{m}}$-absolutely continuous. Hence, for each $\alpha \in [0,1] \cap \mathbb{Q}$ there exists a Radon-Nikodym derivative

$$f_\alpha = \frac{\mathrm{d}\mathbf{m}_\alpha}{\mathrm{d}\check{\mathbf{m}}}.$$

Since the Radon-Nikodym derivative of an $\check{\mathbf{m}}$-absolutely continuous measure is $\check{\mathbf{m}}$-a.e. unique, it follows that for all $\alpha \in]0,1[$

$$\begin{aligned}
f_0 &= 0 && \check{\mathbf{m}} - \text{a.e.} && (7)\\
f_1 &= 1 && \check{\mathbf{m}} - \text{a.e.} && (8)\\
f_\alpha &= \sup\{f_\beta \mid \beta \in \mathbb{Q} \cap [0,\alpha[\} && \check{\mathbf{m}} - \text{a.e.} && (9)
\end{aligned}$$

Therefore, changing (if necessary) the values of the functions f_α on an $\check{\mathbf{m}}$-null set, we may assume that (7) holds everywhere on X, and we still have

$$\mathbf{m}_\alpha(M) = \int_M f_\alpha \,\mathrm{d}\check{\mathbf{m}}. \tag{10}$$

For an arbitrary (not necessarily rational) $\alpha \in [0,1]$ define

$$g_\alpha = \sup\{f_\beta \mid \beta \in \mathbb{Q} \cap [0,\alpha[\}. \tag{11}$$

Since g_α is the supremum of a countable family of $\mathcal{A}$-measurable functions, it is also $\mathcal{A}$-measurable, and for each $M \in \mathcal{A}$ we again have

$$\mathbf{m}_\alpha(M) = \int_M g_\alpha \,\mathrm{d}\check{\mathbf{m}} \tag{12}$$

because of (7) and the Monotone Convergence Theorem. Furthermore, for each $x \in X$ the function $h_x \colon [0,1] \longrightarrow \mathbb{R}$ defined by

$$h_x(\alpha) = g_\alpha(x) \tag{13}$$

is a probability distribution function which determines a unique probability measure $\mathbf{q}_x$ on $([0,1[\,,\mathcal{B}_1)$ which, whenever $0 \leq \alpha < \beta \leq 1$, satisfies

$$\mathbf{q}_x([\alpha,\beta[) = h_x(\beta) - h_x(\alpha).$$

Claim 2. The function $K\colon X \times \mathcal{B}_1 \longrightarrow \mathbb{R}$ defined by

$$K(x, B) = \mathbf{q}_x(B) \tag{14}$$

is an $\mathcal{A}$-Markov kernel, i.e., $K(x, \cdot)$ is a probability measure on $\mathcal{B}_1$ for each $x \in X$ and $K(\cdot, B)$ is $\mathcal{A}$-measurable for each $B \in \mathcal{B}_1$.

The first statement of the claim is trivial. The second one will be proved now. Consider the set $\mathcal{D} = \{C \in \mathcal{B}_1 \mid K(\cdot, C) \text{ is } \mathcal{A}\text{-measurable}\}$. This is a Dynkin system (see, e.g., [6]) on $[0, 1[$. Indeed, $[0, 1[\in \mathcal{D}$ since for each $x \in X$

$$K(x, [0, 1[) = \mathbf{q}_x([0, 1[) = 1.$$

Also, if $C, D \in \mathcal{D}$ and $C \subseteq D$, then for each $x \in X$

$$K(x, D \setminus C) = \mathbf{q}_x(D) - \mathbf{q}_x(C) = K(x, D) - K(x, C),$$

and this shows that $K(x, D \setminus C)$ is $\mathcal{A}$-measurable too. Finally, if $(C_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{D}$, then we have for each $x \in X$

$$K\Big(x, \bigcup_{n=1}^{\infty} C_n\Big) = \mathbf{q}_x\Big(x, \bigcup_{n=1}^{\infty} C_n\Big) = \sum_{n=1}^{\infty} \mathbf{q}_x(C_n) = \sum_{n=1}^{\infty} K(x, C_n),$$

and, subsequently, $K(\cdot, \bigcup_{n=1}^{\infty} C_n)$ is also measurable. Thus, $\mathcal{D}$ is a Dynkin system on $[0, 1[$. Now, observe that for all $\alpha, \beta \in [0, 1]$ with $\alpha < \beta$ and for each $x \in X$ we have

$$K(x, [\alpha, \beta[) = \mathbf{q}_x([\alpha, \beta[) = h_x(\beta) - h_x(\alpha) = g_\beta(x) - g_\alpha(x),$$

and this implies that $K(\cdot, [\alpha, \beta[)$ is $\mathcal{A}$-measurable, i.e., $[\alpha, \beta[\in \mathcal{D}$. Thus the Dynkin system $\mathcal{D}$ contains all intervals $[\alpha, \beta[\subseteq [0, 1[$ which means $\mathcal{D} = \mathcal{B}_1$, and so Claim 2 is proved.

Now we have to show that the $\mathcal{A}$-Markov kernel K defined in (14) indeed satisfies (4). To this end, let A be an $\mathcal{A}$-measurable step function $A = \sum_{i=1}^{n} \alpha_i \cdot A_i$, where all $\alpha_i \in [0, 1]$ and the sets $A_1, A_2, \ldots, A_n$ form a partition of X. Then

$$\begin{aligned}
\mathbf{m}(A) &= \sum_{i=1}^{n} \mathbf{m}(\alpha_i \cdot A_i) \\
&= \sum_{i=1}^{n} \int_{A_i} g_{\alpha_i} \, \mathrm{d}\check{\mathbf{m}} \\
&= \sum_{i=1}^{n} \int_{A_i} K(x, [0, \alpha_i[) \, \mathrm{d}\check{\mathbf{m}}(x) \\
&= \sum_{i=1}^{n} \int_{X} K(x, [0, \alpha_i[) \cdot A_i(x) \, \mathrm{d}\check{\mathbf{m}}(x) \\
&= \int_X K(x, [0, A(x)[) \, \mathrm{d}\check{\mathbf{m}}(x).
\end{aligned}$$

If A is an $\mathcal{A}$-measurable fuzzy set and $(A_n)_{n \in \mathbb{N}}$ is a nondecreasing sequence of $\mathcal{A}$-measurable step functions converging pointwise to A, then (4) holds because of

$$\begin{aligned}
\mathbf{m}(A) &= \sup\{\mathbf{m}(A_n) \mid n \in \mathbb{N}\} \\
&= \sup\Big\{ \int_X K(x, [0, A_n(x)[) \, \mathrm{d}\check{\mathbf{m}}(x) \;\Big|\; n \in \mathbb{N}\Big\} \\
&= \int_X K(x, [0, A(x)[) \, \mathrm{d}\check{\mathbf{m}}(x).
\end{aligned}$$

Finally, we have to show the uniqueness of K. Since for $\mathcal{A}$-measurable functions $u, v\colon X \longrightarrow \mathbb{R}$ we can conclude $u = v$ $\check{\mathbf{m}}$-a.e. if for all $M \in \mathcal{A}$ we have

$$\int_M u \, d\check{\mathbf{m}} = \int_M v \, d\check{\mathbf{m}},$$

it follows that K is $\check{\mathbf{m}}$-a.e. unique in the sense that any other $\mathcal{A}$-Markov kernel satisfying (4) may differ from K on an $\check{\mathbf{m}}$-null subset of X only. □

Taking into account Theorem 3.3(i), Proposition 4.5 and Remark 4.6(ii) we can restate Theorem 4.8 in the following form for measures based on Frank t-norms.

Corollary 4.9 *If T_λ is a Frank t-norm with $\lambda \in [0, \infty]$, $\mathcal{T}$ is a generated T_λ-tribe and $\mathbf{m}$ is a finite monotone T_λ-measure on $\mathcal{T}$, then there exists a unique measure $\check{\mathbf{m}}$ on $\mathcal{T}^\vee$, namely, the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$, and an $\check{\mathbf{m}}$-a.e. uniquely determined $\mathcal{T}^\vee$-Markov kernel $K\colon X \times \mathcal{B}_1 \longrightarrow \mathbb{R}$ such that the representation (4) holds for all $A \in \mathcal{T}$.*

5 Integral representation of $T_{\mathbf{L}}$-measures

Theorem 4.8 and Corollary 4.9 show that monotone finite measures based on Frank t-norms T_λ and defined on generated T_λ-tribes can be represented as integrals of Markov kernels. It is clear that this holds for $T = T_{\mathbf{L}}$ too. However, in this particular case the result is true even if the underlying $T_{\mathbf{L}}$-tribe $\mathcal{T}$ is not generated. This is based on the following properties proved by Butnariu [15], which are specific for $T_{\mathbf{L}}$-measures.

Proposition 5.1 *If $\mathcal{T}$ is a $T_{\mathbf{L}}$-tribe and if $\mathbf{m}$ is a finite $T_{\mathbf{L}}$-measure on $\mathcal{T}$, then we have:*

(i) $\mathbf{m}$ is monotone if and only if $\mathbf{m}$ is nonnegative.

(ii) If $\mathbf{m}$ is nonnegative, then $\mathbf{m}$ is continuous in the sense that

$$\lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(A) \qquad \textit{whenever} \qquad (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T} \quad \textit{and} \quad \lim_{n\to\infty} A_n = A. \tag{15}$$

Proof: In (i) the "only if" part is trivial. Assume that $\mathbf{m}$ is nonnegative, $A, B \in \mathcal{T}$ and $A \le B$. Then $B = A \, \mathsf{S}_{\mathsf{L}} (B \, \mathsf{T}_{\mathsf{L}} \, \complement A)$ and $A \, \mathsf{T}_{\mathsf{L}} (B \, \mathsf{T}_{\mathsf{L}} \, \complement A) = \emptyset$. Thus

$$\mathbf{m}(B) = \mathbf{m}\big(A \, \mathsf{S}_{\mathsf{L}} \, (B \, \mathsf{T}_{\mathsf{L}} \, \complement A)\big) + \mathbf{m}\big(A \, \mathsf{T}_{\mathsf{L}} \, (B \, \mathsf{T}_{\mathsf{L}} \, \complement A)\big) = \mathbf{m}(A) + \mathbf{m}\big(B \, \mathsf{T}_{\mathsf{L}} \, \complement A\big),$$

i.e., $\mathbf{m}(B) \ge \mathbf{m}(A)$.

In order to prove (ii), let $(A_n)_{n\in\mathbb{N}}$ be a nonincreasing sequence in $\mathcal{T}$ converging pointwise to A. Then

$$\lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(X) - \lim_{n\to\infty} \mathbf{m}\big(\complement A_n\big) = \mathbf{m}(X) - \mathbf{m}\big(\complement A\big) = \mathbf{m}(A)$$

because of the continuity from the left of $\mathbf{m}$ and since for all $C \in \mathcal{T}$, C and $\complement C$ are $T_{\mathbf{L}}$-disjoint and satisfy $C \, \mathsf{S}_{\mathsf{L}} \, \complement C = X$. Now, if $(A_n)_{n\in\mathbb{N}}$ is a (not necessarily monotone) sequence in $\mathcal{T}$ converging to A, then

$$\mathop{\mathrm{T}_{\mathbf{M}}}_{k=1}^{\infty} \Big(\mathop{\mathrm{S}_{\mathbf{M}}}_{n=k}^{\infty} A_n \Big) = A = \mathop{\mathrm{S}_{\mathbf{M}}}_{k=1}^{\infty} \Big(\mathop{\mathrm{T}_{\mathbf{M}}}_{n=k}^{\infty} A_n \Big),$$

and $\mathcal{T}$ is a T_0-tribe (compare Theorem 3.3(i)). Therefore, we have

$$\mathbf{m}\Big(\mathop{\mathrm{T}_{\mathbf{M}}}_{k=n}^{\infty} A_k \Big) \le \mathbf{m}(A_n) \le \mathbf{m}\Big(\mathop{\mathrm{S}_{\mathbf{M}}}_{k=n}^{\infty} A_k \Big)$$

and, taking here the limit $n \to \infty$ we obtain $\mathbf{m}(A) = \lim_{n\to\infty} \mathbf{m}(A_n)$. □

The following representation of $T_{\mathbf{L}}$-measures is essentially Theorem 2.6(c) of Butnariu [16]. We present it here with an alternative proof (see Butnariu & Klement [18, Theorem 6.2])

Theorem 5.2 *If $\mathcal{T}$ is a $T_{\mathbf{L}}$-tribe and $\mathbf{m}$ is a finite nonnegative $T_{\mathbf{L}}$-measure on it, then there exists a unique measure $\check{\mathbf{m}}$ on $\mathcal{T}^{\vee}$, namely, the restriction of $\mathbf{m}$ to $\mathcal{T}^{\vee}$, such that for each A in $\mathcal{T}$*

$$\mathbf{m}(A) = \int_X A(x)\, \mathrm{d}\check{\mathbf{m}}(x). \tag{16}$$

Proof: It is clear that (5.2) implies that $\check{\mathbf{m}}$ has to be the restriction of $\mathbf{m}$ to $\mathcal{T}^{\vee}$.

Claim 1. If $A \in \mathcal{T}$, $\alpha, \beta \in [0,1]$ and $\alpha < \beta$, then the set

$$A_{\alpha,\beta} = \{x \in X \mid \alpha < A(x) \leq \beta\} \tag{17}$$

belongs to $\mathcal{T}^{\vee}$, and the product $A \cdot A_{\alpha,\beta} = A \,\mathsf{T}_{\mathsf{L}}\, A_{\alpha,\beta} \in \mathcal{T}$, and

$$\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) \leq \mathbf{m}(A \cdot A_{\alpha,\beta}). \tag{18}$$

The first assertion follows from Theorem 3.3(ii). The second results from the fact that for each $M \in \mathcal{T}^{\vee}$ we have $A \cdot M = A \,\mathsf{T}_{\mathsf{L}}\, M \in \mathcal{T}$. Now, in order to prove (18), it is sufficient to show that this inequality holds for all A in $\mathcal{T}$ and for all $\alpha, \beta \in [0,1]$, such that $\alpha < \beta$ and α has a finite binary representation

$$\alpha = \sum_{i=1}^{k} \frac{a_i}{2^i}$$

with $k \in \mathbb{N}$ and $a_i \in \{0,1\}$ for each $i \in \{1,\ldots,k\}$. Suppose that (18) holds whenever α has a finite binary representation. For all $0 < \alpha < \beta \leq 1$ we can find a sequence $(\alpha_n)_{n\in\mathbb{N}}$ which is nonnegative, nonincreasing and converges to α, and such that each α_n has a finite binary representation as above. Using the continuity of $\mathbf{m}$ (see Proposition 5.1(ii)) we get

$$\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) = \lim_{n\to\infty} \alpha_n \cdot \check{\mathbf{m}}(A_{\alpha_n,\beta}) \leq \lim_{n\to\infty} \mathbf{m}(A \cdot A_{\alpha_n,\beta}) = \mathbf{m}(A \cdot A_{\alpha,\beta}),$$

since $\left(A_{\alpha_n,\beta}\right)_{n\in\mathbb{N}} \searrow A_{\alpha,\beta}$. Now assume that α has the form as above. If $\alpha = 0$ there is nothing to prove. Suppose $\alpha > 0$. In this situation we proceed by induction upon the number k in the binary representation of α. If $k = 1$, then $\alpha = 0$ or $\alpha = \frac{1}{2}$. In the first case (18) clearly holds. In the second case we have $A_{\alpha,\beta} = (A \cdot A_{\alpha,\beta}) \,\mathsf{S}_{\mathsf{L}}(A \cdot A_{\alpha,\beta})$, and this implies

$$\begin{aligned}
\check{\mathbf{m}}(A_{\alpha,\beta}) &= \mathbf{m}(A_{\alpha,\beta}) \\
&= \mathbf{m}(A \cdot A_{\alpha,\beta}) + \mathbf{m}(A \cdot A_{\alpha,\beta}) - \mathbf{m}[(A \cdot A_{\alpha,\beta}) \,\mathsf{T}_{\mathsf{L}}(A \cdot A_{\alpha,\beta})] \\
&\leq 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta}),
\end{aligned}$$

which is exactly (18) with $\alpha = \frac{1}{2}$. Suppose that (18) holds for all $A \in \mathcal{T}$ and for all α, β in $[0,1]$ such that $\alpha < \beta$ and α has a binary representation with at most k binary digits. Put

$$\alpha = \sum_{i=1}^{k+1} \frac{a_i}{2^i}$$

with $\alpha_i \in \{0,1\}$ for each $i \in \{1,\ldots,m+1\}$ and suppose that $\alpha_{k+1} \neq 0$. Then $\alpha = \frac{\delta}{2}$ where

$$\delta = \sum_{i=1}^{k+1} \frac{a_i}{2^{i-1}} \tag{19}$$

Case 1.1. $\delta < 1$ and $\beta < \frac{1}{2}$. Then $A_{\alpha,\beta} = (A \,\mathsf{S}_{\mathsf{L}}\, A)_{\delta,2\beta}$, and using the inductive assumption for the set $(A \,\mathsf{S}_{\mathsf{L}}\, A)_{\delta,2\beta}$ (this is possible since the sum in (19) has at most k nonzero terms in our case) we get $\delta \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) \leq \mathbf{m}[(A \mathsf{S_L} A) \cdot A_{\alpha,\beta}]$. Observing that $(A \mathsf{S_L} A) \cdot A_{\alpha,\beta} = (A \cdot A_{\alpha,\beta}) \mathsf{S_L} (A \cdot A_{\alpha,\beta})$, we obtain

$$\begin{aligned}
\delta \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) &\leq \mathbf{m}[(A \mathsf{S_L} A) \cdot A_{\alpha,\beta}] \\
&\leq \mathbf{m}[(A \cdot A_{\alpha,\beta}) \mathsf{S_L} (A \cdot A_{\alpha,\beta})] \\
&= 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta}) - \mathbf{m}[(A \cdot A_{\alpha,\beta}) \mathsf{T_L} (A \cdot A_{\alpha,\beta})] \\
&\leq 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta})
\end{aligned}$$

by the monotonicity and additivity of $\mathbf{m}$ (compare Proposition 5.1(i)). This implies (18) in this case.

Case 1.2. $\delta < 1$ and $\beta = \frac{1}{2}$. Consider a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $]\alpha, \beta[$ which converges increasingly to β. The sets A_{α,γ_n} satisfy the conditions of Case 1.1. Thus we have for each $n \in \mathbb{N}$

$$\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\gamma_n}) \leq \mathbf{m}(A \cdot A_{\alpha,\gamma_n}). \tag{20}$$

The continuity of $\mathbf{m}$ (compare Proposition 5.1(ii)) and the convergence $\big(A_{\alpha,\gamma_n}\big)_{n\in\mathbb{N}} \nearrow \underset{n=1}{\overset{\infty}{\mathrm{S}_{\mathbf{M}}}} A_{\alpha,\gamma_n}$ imply

$$\alpha \cdot \check{\mathbf{m}}\Big(\underset{n=1}{\overset{\infty}{\mathrm{S}_{\mathbf{M}}}} A_{\alpha,\gamma_n} \Big) \leq \mathbf{m}\Big[A \cdot \Big(\underset{n=1}{\overset{\infty}{\mathrm{S}_{\mathbf{M}}}} A_{\alpha,\gamma_n} \Big) \Big], \tag{21}$$

where

$$\Big(\bigcup_{n=1}^{\infty} A_{\alpha,\gamma_n} \Big) \cup \{x \in X \mid A(x) = \beta\} = A_{\alpha,\beta}.$$

Since the sets in the latter union are disjoint, we may write

$$\begin{aligned}
\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) &= \alpha \cdot \check{\mathbf{m}}\Big(\bigcup_{n=1}^{\infty} A_{\alpha,\gamma_n} \Big) + \alpha \cdot \check{\mathbf{m}}(\{A = \beta\}) \\
&\leq \mathbf{m}\Big[A \cdot \Big(\underset{n=1}{\overset{\infty}{\mathrm{S}_{\mathbf{M}}}} A_{\alpha,\gamma_n} \Big) \Big] + \alpha \cdot \check{\mathbf{m}}(\{A = \beta\}).
\end{aligned} \tag{22}$$

Since for $\beta = \frac{1}{2}$ the fuzzy sets B_1 and B_2 defined by

$$B_1(x) = B_2(x) = \begin{cases} 0 & \text{if } A(x) \neq \beta, \\ \beta & \text{if } A(x) = \beta, \end{cases}$$

are $T_{\mathbf{L}}$-disjoint and are contained in $\mathcal{T}$, we have $\mathbf{m}(B_1 \mathsf{S_L} B_2) = 2 \cdot \mathbf{m}(\beta \cdot \{A = \beta\})$ (compare Remark 4.2(i)) and $B_1 \mathsf{S_L} B_2 = \{A = \beta\}$. Hence,

$$\mathbf{m}(\beta \cdot \{A = b\}) = \beta \cdot \check{\mathbf{m}}(\{A = \beta\}) \geq \alpha \cdot \mathbf{m}(\{A = \beta\}).$$

Combining this with (22) and using the $T_{\mathbf{L}}$-additivity of $\mathbf{m}$ we deduce

$$\begin{aligned}
\alpha \cdot \mathbf{m}(A_{\alpha,\beta}) &\leq \mathbf{m}\Big[A \cdot \Big(\underset{n=1}{\overset{\infty}{\mathrm{S}_{\mathbf{M}}}} A_{\alpha,\gamma_n} \Big) \Big] + \mathbf{m}(\beta \cdot \{A = \beta\}) \\
&= \mathbf{m}\Big[A \cdot \Big(\{A = \beta\} \cup \Big(\underset{n=1}{\overset{\infty}{\mathrm{S}_{\mathbf{M}}}} A_{\alpha,\gamma_n} \Big) \Big) \Big] \\
&= \mathbf{m}(A \cdot A_{\alpha,\beta}).
\end{aligned}$$

This proves (18) in this case.

Case 1.3. $\delta < 1$ and $\beta > \frac{1}{2}$. Then $A \cdot A_{\alpha,\beta} = (A \cdot A_{\alpha,\frac{1}{2}}) \mathsf{S_L} (A \cdot A_{\frac{1}{2},\beta})$, where the fuzzy sets on the right hand side are $T_{\mathbf{L}}$-disjoint. Hence, according to Remark 4.2(i) we have

$$\mathbf{m}(A \cdot A_{\alpha,\beta}) = \mathbf{m}(A \cdot A_{\alpha,\frac{1}{2}}) + \mathbf{m}(A \cdot A_{\frac{1}{2},\beta}).$$

The first term on the right hand side satisfies the conditions of Case 1.2 and the second one meets the inductive assumption. Hence,

$$\mathbf{m}(A \cdot A_{\alpha,\beta}) \geq \alpha \cdot \check{\mathbf{m}}(A_{\frac{1}{2},\beta}) + \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\frac{1}{2},\beta}) \geq \alpha \cdot [\check{\mathbf{m}}(A_{\alpha,\frac{1}{2}}) + \check{\mathbf{m}}(A_{\frac{1}{2},\beta})] = \alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta})$$

showing that (18) holds in this case too.

Case 1.4. $\delta \geq 1$. In this case α can be written as $\alpha = \frac{\varepsilon+1}{2}$ where

$$\varepsilon = \sum_{i=1}^{k} \frac{a_{i+1}}{2^i} \in [0,1].$$

We also have $A_{\alpha,\beta} = \big[A \mathsf{T}_{\mathsf{L}} A\big]_{\varepsilon,2\rho}$ with $\rho = \beta - \frac{1}{2}$. Using the inductive assumption for the set $\big[A \mathsf{T}_{\mathsf{L}} A\big]_{\varepsilon,2\rho}$ we obtain

$$\begin{aligned}
\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) &= \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \frac{\varepsilon}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) \\
&= \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \frac{\varepsilon}{2} \cdot \check{\mathbf{m}}\Big([A \mathsf{T}_{\mathsf{L}} A]_{\varepsilon,2\rho}\Big) \\
&\leq \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \frac{1}{2} \cdot \check{\mathbf{m}}\Big([A \mathsf{T}_{\mathsf{L}} A]_{\varepsilon,2\rho}\Big) \\
&= \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \frac{1}{2} \cdot \mathbf{m}\Big(A_{\alpha,\beta} \cdot [A \mathsf{T}_{\mathsf{L}} A]_{\varepsilon,2\rho}\Big).
\end{aligned} \tag{23}$$

Observe that $A_{\alpha,\beta} \cdot (A \mathsf{T}_{\mathsf{L}} A) = D \mathsf{T}_{\mathsf{L}} \complement C$ with $D = A_{\alpha,\beta}$ and $C = A_{\alpha,\beta} \cdot (\complement A \mathsf{S}_{\mathsf{L}} \complement A)$. According to Remark 4.2(i), it follows that whenever $E, F \in \mathfrak{T}$ and $E \geq F$ then

$$\mathbf{m}(E \mathsf{T}_{\mathsf{L}} \complement F) = \mathbf{m}(E) - \mathbf{m}(F). \tag{24}$$

Since we clearly have $D \geq C$, (23) combined with (24) gives

$$\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) \leq \frac{1}{2} \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) + \frac{1}{2} \cdot [\mathbf{m}(D) - \mathbf{m}(C)] = \check{\mathbf{m}}(A_{\alpha,\beta}) - \frac{1}{2} \cdot \mathbf{m}(C). \tag{25}$$

Now, taking into account $\delta \geq 1$, we deduce

$$C = (\complement A \cdot A_{\alpha,\beta}) \mathsf{S}_{\mathsf{L}} (\complement A \cdot A_{\alpha,\beta}) \quad \text{and} \quad (\complement A \cdot A_{\alpha,\beta}) \mathsf{T}_{\mathsf{L}} (\complement A \cdot A_{\alpha,\beta}) = \emptyset.$$

Thus $\mathbf{m}(C) = 2 \cdot \mathbf{m}(\complement A \cdot A_{\alpha,\beta})$ by the $T_{\mathbf{L}}$-additivity of $\mathbf{m}$. Since $\complement A \cdot A_{\alpha,\beta} = A_{\alpha,\beta} \mathsf{T}_{\mathsf{L}} \complement(A \cdot A_{\alpha,\beta})$, we get $\mathbf{m}(C) = 2 \cdot [\mathbf{m}(A_{\alpha,\beta}) - \mathbf{m}(A \cdot A_{\alpha,\beta})]$ because of (24). Substituting this into (25) we obtain (18), and Claim 1 is completely proved.

Claim 2. If $A \in \mathfrak{T}$, $0 \leq \alpha < \beta \leq 1$ and

$$\bar{A}_{\alpha,\beta} = \{x \in X \mid \alpha < A(x) < \beta\} \tag{26}$$

then $\bar{A}_{\alpha,\beta} \in \mathfrak{T}^{\vee}$, $A \cdot \bar{A}_{\alpha,\beta} \in \mathfrak{T}$ and

$$\mathbf{m}(A \cdot \bar{A}_{\alpha,\beta}) \leq \beta \cdot \check{\mathbf{m}}(\bar{A}_{\alpha,\beta}). \tag{27}$$

The first two assertions are obvious. We only have to prove (27). To this end, observe that

$$A_{\alpha,\beta} = \overline{\complement A}_{\beta',\alpha'} \tag{28}$$

where $\alpha' = 1-\alpha$ and $\beta' = 1-\beta$. Then

$$\begin{aligned}\mathbf{m}[\complement(A\cdot\bar{A}_{\alpha,\beta})] &= \check{\mathbf{m}}(X) - \mathbf{m}(A\cdot\bar{A}_{\alpha,\beta})\\ &= \check{\mathbf{m}}(\bar{A}_{\alpha,\beta}) + \check{\mathbf{m}}(\complement\bar{A}_{\alpha,\beta}) - \mathbf{m}(A\cdot\bar{A}_{\alpha,\beta})\\ &= \mathbf{m}(\complement A\cdot\bar{A}_{\alpha,\beta}) + \check{\mathbf{m}}(\complement\bar{A}_{\alpha,\beta})\\ &= \mathbf{m}(\complement A\cdot\complement A_{\beta',\alpha'}) + \check{\mathbf{m}}(\complement\bar{A}_{\alpha,\beta})\\ &\geq \beta'\cdot\check{\mathbf{m}}(X) + \check{\mathbf{m}}(\complement\bar{A}_{\alpha,\beta})\\ &= \check{\mathbf{m}}(X) - \beta\cdot\check{\mathbf{m}}(\bar{A}_{\alpha,\beta}),\end{aligned}$$

where the forth equality is a consequence of (28), and the inequality follows from Claim 1. This implies (27), and Claim 2 is proved.

In order to complete the proof, choose $A \in \mathcal{T}$. For a given $n \in \mathbb{N}$ denote

$$G_{n,i} = \begin{cases}\{x\in X \mid A(x)=0\} & \text{if } i=0,\\ A_{\frac{i-1}{2^n},\frac{i}{2^n}} & \text{if } i\in\{1,2,\ldots,2^n\},\end{cases}$$

$$H_{n,i} = \begin{cases}\bar{A}_{\frac{i}{2^n},\frac{i+1}{2^n}} & \text{if } i\in\{0,1,\ldots,2^n-1\},\\ \{x\in X \mid A(x)=1\} & \text{if } i=2^n.\end{cases}$$

From Claims 1 and 2 it follows that the step functions

$$s_n = \sum_{i=1}^{2^n}\frac{i-1}{2^n}\cdot G_{n,i},$$
$$t_n = \sum_{i=1}^{2^n}\frac{i}{2^n}\cdot H_{n,i},$$

are all $\mathcal{T}^{\vee}$-measurable. It is clear that $(s_n)\nearrow A$ and $(t_n)\searrow A$. Taking into account (18) and (27) we deduce

$$\int_X s_n\,\mathrm{d}\check{\mathbf{m}} = \sum_{i=1}^{2^n}\frac{i-1}{2^n}\cdot\check{\mathbf{m}}(G_{n,i}) \leq \sum_{i=0}^{2^n}\mathbf{m}(A\cdot G_{n,i}) = \mathbf{m}(A)$$

and

$$\int_X t_n\,\mathrm{d}\check{\mathbf{m}} = \sum_{i=1}^{2^n}\frac{i}{2^n}\cdot\check{\mathbf{m}}(H_{n,i}) \geq \sum_{i=0}^{2^n}\mathbf{m}(A\cdot H_{n,i}) = \mathbf{m}(A).$$

Taking the limit $n\to\infty$ in the last two inequalities we obtain (16), completing the proof of the Theorem. □

Comparing Theorem 5.2 with Corollary 4.9 it is easily seen that the $T_{\mathbf{L}}$-measures on a generated T_λ-tribe with T_λ a Frank t-norm are exactly those T_λ-measures for which the corresponding Markov kernel K, in the representation (4), for each $x\in X$ can be written in the form

$$K(x,[\alpha,\beta[) = \beta-\alpha.$$

This suggests that the concept of Lebesgue integral could be axiomatically defined as a $T_{\mathbf{L}}$-measure.

6 Integral representation of monotone T_λ-measures

Theorem 5.2 not only drops the condition that $\mathcal{T}$ be generated in Corollary 4.9 in the case $T = T_{\mathbf{L}}$, but also specifies the form of the Markov kernel occuring in the representation (4) in this particular case. For Frank t-norms T_λ with $\lambda\in\,]0,\infty[$ and generated T_λ-tribes it is also possible to specify the form of the Markov kernel in (4). To be precise, we get the following result of Klement [46].

Theorem 6.1 *If T_λ is a Frank t-norm with $\lambda \in]0,\infty[$, $\mathcal{T}$ is a generated T_λ-tribe and $\mathbf{m}$ is a monotone finite T_λ-measure on $\mathcal{T}$, then there exists a unique measure $\check{\mathbf{m}}$ on $\mathcal{T}^\vee$, namely, the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$, and an $\check{\mathbf{m}}$-a.e. uniquely determined $\mathcal{T}^\vee$-measurable function $f\colon X \longrightarrow [0,1]$ such that for all $A \in \mathcal{T}$*

$$\mathbf{m}(A) = \int_{\{A>0\}} [f + (1-f)\cdot A]\,\mathrm{d}\check{\mathbf{m}}. \tag{29}$$

Proof: First, observe that if (29) holds for some measure $\check{\mathbf{m}}$ on $\mathcal{T}^\vee$ and for some $\mathcal{T}^\vee$-measurable function f on X, then, necessarily, $\check{\mathbf{m}}$ has to be the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$. Now let us show that the required function f exists. To this end, consider the $\mathcal{T}^\vee$-Markov kernel K corresponding to $\mathbf{m}$ in the representation (4). For each a in $[0,1]$ define the function $f_a\colon X \longrightarrow [0,1]$ by

$$f_a(x) = K(x,[0,a[). \tag{30}$$

Each f_a is $\mathcal{T}^\vee$-measurable since K is a $\mathcal{T}^\vee$-Markov kernel. We claim that there exists a function $f\colon X \longrightarrow [0,1]$ such that for all $a,b \in]0,1[$ with $a+b \le 1$,

$$f_{a+b} = f_a + f_b - f \qquad \check{\mathbf{m}}\text{-a.e.} \tag{31}$$

In order to prove that, consider the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ defined as follows:

$$\begin{aligned} a_1 &= a, & b_1 &= b,\\ a_{n+1} &= S_\lambda(a_n,b_n), & b_n &= T_\lambda(a_n,b_n). \end{aligned}$$

Since $\mathcal{T}$ is generated, these sequences belong to $\mathcal{T}$ (compare Theorem 3.3(iii)). Exactly as in the proof of Theorem 3.3(i) we show that the sequence $(a_n)_{n\in\mathbb{N}}$ is nondecreasing, $(b_n)_{n\in\mathbb{N}}$ is nonincreasing and converges to zero, and for each $n \in \mathbb{N}$ we have $a_n + b_n = a + b$. Define the $\mathcal{T}^\vee$-measurable function

$$f = \inf\{f_{b_n} \mid n \in \mathbb{N}\}. \tag{32}$$

For each M in $\mathcal{T}^\vee$, the fuzzy sets $A_n = a_n\cdot M$ and $B_n = b_n \cdot M$ are contained in $\mathcal{T}$ (compare Theorem 3.3(i)), and $(A_n)_{n\in\mathbb{N}} \nearrow A_1 \mathsf{S_L}\, B_1$ and $(B_n)_{n\in\mathbb{N}} \searrow \emptyset$. Also, for each $n\in\mathbb{N}$, we get $A_n + B_n = (a+b)\cdot M$. Induction upon n yields that $\mathbf{m}(A_n) + \mathbf{m}(B_n) = \mathbf{m}(a\cdot M) + \mathbf{m}(b \cdot M)$ for all $n \in \mathbb{N}$. Consequently, we have

$$\begin{aligned} \int_M f_{a+b}\,\mathrm{d}\check{\mathbf{m}} &= \mathbf{m}[(a+b)\cdot M]\\ &= \mathbf{m}(\sup\{A_n \mid n\in\mathbb{N}\})\\ &= \sup\{\mathbf{m}(A_n) \mid n \in \mathbb{N}\}\\ &= \mathbf{m}(a\cdot M) + \mathbf{m}(b\cdot M) - \inf\{\mathbf{m}(B_n) \mid n\in\mathbb{N}\}\\ &= \int_M (f_a + f_b)\,\mathrm{d}\check{\mathbf{m}} - \inf\{\mathbf{m}(B_n) \mid n \in \mathbb{N}\} \end{aligned}$$

by (30), (4) and the continuity from the left of $\mathbf{m}$. From (30), (4) and the Monotone Convergence Theorem we also have

$$\inf\{\mathbf{m}(B_n) \mid n\in\mathbb{N}\} = \inf\left\{ \int_M f_{b_n}\,\mathrm{d}\check{\mathbf{m}} \;\middle|\; n \in\mathbb{N}\right\} = \int_M \inf\{f_{b_n} \mid n\in\mathbb{N}\}\,\mathrm{d}\check{\mathbf{m}} = \int_M f\,\mathrm{d}\check{\mathbf{m}}.$$

Hence,

$$\int_M f_{a+b}\,\mathrm{d}\check{\mathbf{m}} = \int_M (f_a + f_b - f)\,\mathrm{d}\check{\mathbf{m}}$$

for each M in $\mathcal{T}^\vee$. This implies (31), and our claim is proved. Equation (31) is a Gaussian functional equation (see Aczél [1]) with the border conditions $f_1 = 1$ and $f_0 = f$. The unique solution of such an equation has to be $f_a = f + (1-f)\cdot a \;\; \check{\mathbf{m}}$-a.e. for all a in $]0,1[$. Taking into account (30) this gives

$$K(x,[0,a[) = f + (1-f)\cdot a \qquad \check{\mathbf{m}}\text{-a.e.}$$

for $a \in [0,1]$. Inserting this into (4), we get (29) and the proof is complete. $\square$

Theorem 29 is proven for T_λ-measures on generated T_λ-tribes when $\lambda \in\,]0,\infty[$. Combining this result with Theorem 5.2 we get the following.

Corollary 6.2 *If T_λ is a Frank t-norm with $\lambda \in\,]0,\infty]$, $\mathcal{T}$ a generated T_λ-tribe and $\mathbf{m}$ a monotone finite T_λ-measure on $\mathcal{T}$, then there exists a unique finite measure $\check{\mathbf{m}}$ on $\mathcal{T}^\vee$, namely, the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$, and an $\check{\mathbf{m}}$-a.e. uniquely determined $\mathcal{T}^\vee$-measurable function $f\colon X \longrightarrow [0,1]$, such that for all $A \in \mathcal{T}$ the equality (29) holds.*

It is obvious (compare Theorems 3.3(i) and 5.2) that $T_\mathbf{L}$-measures on T_λ-tribes with $\lambda \in\,]0,\infty[$ are also T_λ-measures (even if the T_λ-tribe is not generated). However, not each T_λ-measure is necessarily a $T_\mathbf{L}$-measure, even on generated T_λ-tribes. The following shows that a T_λ-measure on a generated T_λ-tribe is a $T_\mathbf{L}$-measure (and hence an integral) only if it is continuous from the right at $\emptyset$ (and then, clearly, at each $A \in \mathcal{T}$).

Corollary 6.3 *Let $\lambda \in\,]0,\infty[$, T_λ be the corresponding Frank t-norm and $\mathcal{T}$ be a T_λ-tribe. A monotone finite T_λ-measure $\mathbf{m}$ on $\mathcal{T}$ is a $T_\mathbf{L}$-measure on $\mathcal{T}$ if and only if it satisfies the following condition:*

$$\lim_{n\to\infty} \mathbf{m}(A_n) = 0 \quad \textit{whenever} \quad (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T} \quad \textit{and} \quad (A_n)_{n\in\mathbb{N}} \searrow \emptyset. \tag{33}$$

Proof: From Proposition 5.1 we already know that $T_\mathbf{L}$-measures are continuous and, therefore, satisfy (33). Conversely, assume that $\mathbf{m}$ is a T_λ-measure on the generated T_λ-tribe $\mathcal{T}$ satisfying (33). Then $\mathbf{m}$ can be represented in the form (29) and for each M in $\mathcal{T}^\vee$ the sequence $(M_n)_{n\in\mathbb{N}} = (\frac{1}{n}\cdot M)_{n\in\mathbb{N}}$ is contained in $\mathcal{T}$. Using the Monotone Convergence Theorem we deduce that for each $M \in \mathcal{T}^\vee$

$$0 = \lim_{n\to\infty} \int_M [f + (1-f)\cdot \frac{1}{n}]\, \mathrm{d}\check{\mathbf{m}} = \int_M f\, \mathrm{d}\check{\mathbf{m}}.$$

This implies that $f = 0$ $\check{\mathbf{m}}$-a.e., and rewriting (29) in this case we obtain for $A \in \mathcal{T}$

$$\mathbf{m}(A) = \int_{\{A>0\}} A\, \mathrm{d}\check{\mathbf{m}} = \int_X A\, \mathrm{d}\check{\mathbf{m}}$$

which immediately implies that $\mathbf{m}$ is a $T_\mathbf{L}$-measure. $\square$

7 Decompositon of monotone T_λ-measures

Theorem 6.1 shows that for $\lambda \in\,]0,\infty[$ finite monotone T_λ-measures $\mathbf{m}$ which are defined on generated tribes $\mathcal{T}$ differ from $T_\mathbf{L}$-measures (i.e., from integrals according to Theorem 5.2) only by functions of the form

$$A \longmapsto \int_{\{A>0\}} f\, \mathrm{d}\check{\mathbf{m}},$$

which are themselves monotone finite T_λ-measures on $\mathcal{T}$. The question we deal with in this section is how much a T_λ-measure defined on a nongenerated T_λ-tribe differs from a $T_\mathbf{L}$-measure (i.e., from an integral). The results are drawn mostly from Butnariu & Klement [17], some important completions of the proofs there are due to Mesiar [61] and Mesiar & Navara [63] (see also Section 8 in Butnariu & Klement [18]).

Proposition 7.1 *Let $\lambda \in\,]0,\infty]$, $\mathcal{T}$ be a T_λ-tribe and $\mathbf{m}\colon \mathcal{T} \longrightarrow \mathbb{R}$ a finite monotone T_λ-measure on $\mathcal{T}$. Then there exists a unique pair $(\mathbf{m}_\mathbf{L}, \mathbf{m}_\lambda)$ of functions from $\mathcal{T}$ to $[0,\infty[$ such that:*

(i) $\mathbf{m}_\mathbf{L}$ is a $T_\mathbf{L}$-measure on $\mathcal{T}$.

(ii) $\mathbf{m}_\lambda$ *is a* T_λ*-measure on* $\mathcal{T}$*.*

(iii) $\mathbf{m} = \mathbf{m_L} + \mathbf{m}_\lambda$*.*

(iv) $\mathbf{m_L}$ *is the greatest* $T_\mathbf{L}$*-measure in the sense that we have* $\mathbf{m_L} \geq \mathbf{n}$ *whenever* $\mathbf{n}\colon \mathcal{T} \longrightarrow [0,\infty[$ *is a* $T_\mathbf{L}$*-measure such that* $\mathbf{m} - \mathbf{n}$ *is monotone.*

(v) There exists a unique measure $\check{\mathbf{m}}$ *on* $\mathcal{T}^\vee$*, namely, the restriction of* $\mathbf{m}$ *to* $\mathcal{T}^\vee$*, and an* $\check{\mathbf{m}}$*-a.e. unique* $\mathcal{T}^\vee$*-measurable function* $f\colon X \longrightarrow [0,1]$ *such that for all* $A \in \mathcal{T}$

$$\mathbf{m_L}(A) = \int_X (1-f)\cdot A \,\mathrm{d}\check{\mathbf{m}} \tag{34}$$

and for all $M \in \mathcal{T}^\vee$

$$\mathbf{m}_\lambda(M) = \int_M f \,\mathrm{d}\check{\mathbf{m}}. \tag{35}$$

Proof: Let $\mathcal{M}(\mathbf{m})$ be the set of all $T_\mathbf{L}$-measures $\mathbf{p}\colon \mathcal{T} \longrightarrow [0,\infty[$, such that $\mathbf{m}-\mathbf{p}$ is monotone (and, therefore, a T_λ-measure due to Remark 4.6(ii) and 6.3). The set $\mathcal{M}(\mathbf{m})$ is nonempty since it contains the zero measure. $\mathcal{M}(\mathbf{m})$ is naturally equipped with the partial order

$$\mathbf{p} \leq \mathbf{q} \quad \text{if and only if} \quad \mathbf{p}(A) \leq \mathbf{q}(A) \quad \text{for all} \quad A \in \mathcal{T}. \tag{36}$$

Claim 1. There exists a maximal element in $\mathcal{M}(\mathbf{m})$.
If $(\mathbf{p}_j)_{j\in J}$ is a chain in $\mathcal{M}(\mathbf{m})$, then the function $\mathbf{p}\colon \mathcal{T} \longrightarrow [0,\infty]$ defined by

$$\mathbf{p}(A) = \sup\{\mathbf{p}_j(A) \mid j \in J\} \tag{37}$$

is a $T_\mathbf{L}$-valuation. Indeed, $\mathbf{p}(\emptyset) = 0$ clearly holds and for all $A, B \in \mathcal{T}$ we have

$$\begin{aligned}
\mathbf{p}(A \,\mathsf{S}_\mathsf{L}\, B) + \mathbf{p}(A \,\mathsf{T}_\mathsf{L}\, B) &= \sup\{\mathbf{p}_j(A \,\mathsf{S}_\mathsf{L}\, B) \mid j \in J\} + \sup\{\mathbf{p}_j(A \,\mathsf{T}_\mathsf{L}\, B) \mid j \in J\} \\
&= \sup\big\{\mathbf{p}_j(A \,\mathsf{S}_\mathsf{L}\, B) + \mathbf{p}_j(A \,\mathsf{T}_\mathsf{L}\, B) \bigm| j \in J\big\} \\
&= \sup\big\{\mathbf{p}_j(A) + \mathbf{p}_j(B) \bigm| j \in J\big\} \\
&= \mathbf{p}(A) + \mathbf{p}(B)
\end{aligned}$$

where the second and the fourth equality follow from the monotonicity of the chain $(\mathbf{p}_j(C))_{j\in J}$ for each $C \in \mathcal{T}$. It is clear that $\mathbf{p}$ is monotone. Hence, $0 \leq \mathbf{p}(A) \leq \mathbf{m}(A) \leq \mathbf{m}(X)$ for all $A \in \mathcal{T}$, implying that $\mathbf{p}$ is also finite. If $(A_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence in $\mathcal{T}$, then

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{p}(A_n) &= \sup\{\mathbf{p}(A_n) \mid n \in \mathbb{N}\} \\
&= \sup\big\{\sup\{\mathbf{p}_j(A_n) \mid j \in J\} \bigm| n \in \mathbb{N}\big\} \\
&= \sup\big\{\sup\{\mathbf{p}_j(A_n) \mid n \in \mathbb{N}\} \bigm| j \in J\big\} \\
&= \sup\big\{\lim_{n\to\infty} \mathbf{p}_j(A_n) \bigm| j \in J\big\} \\
&= \mathbf{p}\big(\lim_{n\to\infty} A_n\big),
\end{aligned}$$

showing that $\mathbf{p}$ is continuous from the left. Hence, $\mathbf{p}$ is a finite nonnegative $T_\mathbf{L}$-measure on $\mathcal{T}$. Moreover, $\mathbf{m}-\mathbf{p}$ is also monotone since for all $A, B \in \mathcal{T}$ with $A \leq B$ it follows from $p_j \in \mathcal{M}(\mathbf{m})$ that

$$(\mathbf{m}-\mathbf{p})(A) = \inf\{(\mathbf{m}-\mathbf{p}_j)(A) \mid j \in J\} \leq \inf\{(\mathbf{m}-\mathbf{p}_j)(B) \mid j \in J\} = (\mathbf{m}-\mathbf{p})(B),$$

thus showing that $\mathbf{p} \in \mathcal{M}(\mathbf{m})$. In other words, each chain in $\mathcal{M}(\mathbf{m})$ has an upper bound in $\mathcal{M}(\mathbf{m})$ and, by Zorn's Lemma, $\mathcal{M}(\mathbf{m})$ contains a maximal element denoted $\mathbf{m_L}$. The $T_\mathbf{L}$-measure $\mathbf{m_L}$ is also a T_λ-measure, and so is the difference

$$\mathbf{m}_\lambda = \mathbf{m} - \mathbf{m_L}. \tag{38}$$

Because of $\mathbf{m_L} \in \mathcal{M}(\mathbf{m})$ we know that $\mathbf{m}_\lambda$ is monotone and finite, and the pair $(\mathbf{m_L}, \mathbf{m}_\lambda)$ satisfies properties (i)–(iii). The following claim [61,63] shows that $\mathcal{M}(\mathbf{m})$ is a directed set, thus immediately implying that $\mathbf{m_L}$ (and subsequently $\mathbf{m}_\lambda$) are uniquely determined.

Claim 2. For all $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{M}(\mathbf{m})$ there exists a $\mathbf{q} \in \mathcal{M}(\mathbf{m})$ such that $\mathbf{q}_1 \leq \mathbf{q}$ and $\mathbf{q}_2 \leq \mathbf{q}$. Since $\mathbf{q}_1, \mathbf{q}_2$ are $T_\mathbf{L}$-measures, we may write for $i \in \{1,2\}$

$$\mathbf{q}_i(A) = \int_X A \, \mathrm{d}\check{\mathbf{q}}_i,$$

where the finite σ-additive measure $\check{\mathbf{q}}_i$ is the restriction of $\mathbf{q}_i$ to $\mathcal{T}^\vee$. Then $\mathbf{q}_0 = \mathbf{q}_1 + \mathbf{q}_2$ is also a finite-additive measure on $\mathcal{T}^\vee$, and $\mathbf{q}_i$ is absolutely continuous with respect to $\mathbf{q}$ for $i \in \{1,2\}$. Then the Radon-Nikodym Theorem provides $\mathcal{T}^\vee$-measurable functions $h_1, h_2 \colon X \longrightarrow \mathbb{R}$ such that

$$\check{\mathbf{q}}_i(L) = \int_L h_i \, \mathrm{d}\mathbf{q}_0$$

for all $L \in \mathcal{T}^\vee$ and $i \in \{1,2\}$. Define the $\mathcal{T}^\vee$-measurable function $h \colon X \longrightarrow \mathbb{R}$ by $h = \max(h_1, h_2)$ and the finite σ-additive measure $\check{\mathbf{q}}$ on $\mathcal{T}^\vee$ by

$$\check{\mathbf{q}}(L) = \int_L h \, \mathrm{d}\mathbf{q}_0.$$

Then the function $\mathbf{q} \colon \mathcal{T} \longrightarrow [0, \infty[$ given by

$$\mathbf{q}(A) = \int_X A \, \mathrm{d}\check{\mathbf{q}}$$

is a finite monotone $T_\mathbf{L}$-measure such that $\mathbf{q}_i \leq \mathbf{q}$ for $i \in \{1,2\}$. The monotonicity of $\mathbf{m} - \mathbf{q}$ can be seen as follows. Consider the crisp sets $P = \{h = h_1\}$ and $Q = \{h > h_1\}$. Obviously $P, Q \in \mathcal{T}^\vee$ and $P \cup Q = X$ and $P \cap Q = \emptyset$. Note that for any fuzzy set $C \in [0,1]^X$ and for any crisp subset $L \subseteq X$ we always have $C \top_\lambda L = C \cdot L$. Therefore, using the fact that for fuzzy sets $D, E \in \mathcal{T}$ with $D \leq P$ and $E \leq Q$ we always have $\mathbf{q}(D) = \mathbf{q}_1(D)$ and $\mathbf{q}(E) = \mathbf{q}_2(E)$, we obtain for all $A, B \in \mathcal{T}$ with $A \leq B$

$$\begin{aligned}
(\mathbf{m} - \mathbf{q})(A) &= (\mathbf{m} - \mathbf{q})(A \top_\lambda P) + (\mathbf{m} - \mathbf{q})(A \top_\lambda Q) \\
&= (\mathbf{m} - \mathbf{q}_1)(A \top_\lambda P) + (\mathbf{m} - \mathbf{q}_2)(A \top_\lambda Q) \\
&\leq (\mathbf{m} - \mathbf{q}_1)(B \top_\lambda P) + (\mathbf{m} - \mathbf{q}_2)(B \top_\lambda Q) \\
&= (\mathbf{m} - \mathbf{q})(B \top_\lambda P) + (\mathbf{m} - \mathbf{q})(B \top_\lambda Q) \\
&= (\mathbf{m} - \mathbf{q})(B),
\end{aligned}$$

showing that $\mathbf{m} - \mathbf{q}$ is monotone. This completes the proof of Claim 2 which together with Claim 1 implies that $(\mathbf{m_L}, \mathbf{m}_\lambda)$ satisfies (iv) too.

In order to prove (v), we take $\check{\mathbf{m}}$ for the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$. We have to find a $\mathcal{T}^\vee$-measurable function $f \colon X \longrightarrow [0,1]$ such that (34) and (35) hold. To this end, observe that $\mathcal{T}$ is a $T_\mathbf{L}$-tribe by Theorem 3.3(i). Hence, according to Theorem 5.2, the $T_\mathbf{L}$-measure $\mathbf{m_L}$ can be written as

$$\mathbf{m_L}(A) = \int_X A \, \mathrm{d}\check{\mathbf{m}}_\mathbf{L} \tag{39}$$

for each $A \in \mathcal{T}$ where $\check{\mathbf{m}}_\mathbf{L}$ is the restriction of $\mathbf{m_L}$ to $\mathcal{T}^\vee$. From $\mathbf{m_L} \leq \mathbf{m}$ it follows that $\check{\mathbf{m}}_\mathbf{L}$ is absolutely continuous with respect to $\check{\mathbf{m}}$. Because of $\check{\mathbf{m}}_\mathbf{L} \leq \mathbf{m}$ there is a Radon-Nikodym derivative

$$\frac{\mathrm{d}(\check{\mathbf{m}})^\vee}{\mathrm{d}\check{\mathbf{m}}}$$

which maps X into $[0,1]$. Putting $f = 1 - \mathrm{d}\check{\mathbf{m}}_{\mathbf{L}}/\mathrm{d}\check{\mathbf{m}}$, then f is also a $\mathcal{T}^{\vee}$-measurable function with values in $[0,1]$, and (39) implies (34). Now, taking into account (38), we can write for all crisp sets $L \in \mathcal{T}^{\vee}$

$$\mathbf{m}_{\lambda}(L) = \mathbf{m}(L) = \check{\mathbf{m}}(L) - \check{\mathbf{m}}_{\mathbf{L}}(L) = \check{\mathbf{m}}(L) - \int_{L} (1-f)\,\mathrm{d}\check{\mathbf{m}} = \int_{L} f\,\mathrm{d}\check{\mathbf{m}},$$

and therefore, (35) also holds. Formulae (34), (35) and (38) imply that no other choice for $\check{\mathbf{m}}$ (and, subsequently, for f) is possible. □

Remark 7.2 (i) The component $\mathbf{m}_{\lambda}$ of a T_{λ}-measure $\mathbf{m}$, which exists due to Proposition 7.1, is a pure T_{λ}-measure in the sense that it has a zero $T_{\mathbf{L}}$-component: $(\mathbf{m}_{\lambda})_{\mathbf{L}} = 0$. In fact, assuming the contrary contradicts the maximality of $\mathbf{m}_{\mathbf{L}}$ in Proposition 7.1.

(ii) If the T_{λ}-tribe $\mathcal{T}$ is generated, then (35) somehow holds for each $A \in \mathcal{T}$ (and not only for crisp sets in the σ-algebra $\mathcal{T}^{\vee}$). To be precise, we always have

$$\mathbf{m}_{\lambda}(A) = \int_{\{A>0\}} f\,\mathrm{d}\check{\mathbf{m}}.$$

This follows from (29) in combination with (34) and (35), and the $\check{\mathbf{m}}$-a.e. uniqueness of f in (29).

Let $\lambda \in\,]0,\infty[$ and $\mathcal{T}$ be a T_{λ}-tribe. If $\mathbf{p}$ is a σ-additive measure on the σ-algebra $\mathcal{T}^{\vee}$ and if $g,h\colon X \longrightarrow [0,\infty]$ are two $\mathcal{T}^{\vee}$-measurable functions, then the function $\mathbf{m}\colon \mathcal{T} \longrightarrow [0,\infty]$ defined by

$$\mathbf{m}(A) = \int_{\{A>0\}} (g + h\cdot A)\,\mathrm{d}\mathbf{p} \tag{40}$$

is a monotone T_{λ}-measure on $\mathcal{T}$. Indeed, monotonicity is obvious as is $\mathbf{m}(\emptyset) = 0$; continuity from the left follows from the Monotone Convergence Theorem, taking into account that for a nondecreasing sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}$ which converges pointwise to A we have

$$\bigcup_{n=1}^{n} \{A_n > 0\} = \{A > 0\}.$$

The T_{λ}-additivity is shown as follows:

$$\begin{aligned}
&\mathbf{m}(A \mathsf{T}_{\lambda} B) + \mathbf{m}(A \mathsf{S}_{\lambda} B)\\
&\quad= \int_{\{A>0\}\cap\{B>0\}} [g + h\cdot(A \mathsf{T}_{\lambda} B)]\,\mathrm{d}\mathbf{p} + \int_{\{A>0\}\cup\{B>0\}} [g + h\cdot(A \mathsf{S}_{\lambda} B)]\,\mathrm{d}\mathbf{p}\\
&\quad= \int_{\{A>0\}\cap\{B>0\}} [2\cdot g + h\cdot((A \mathsf{T}_{\lambda} B) + (A \mathsf{S}_{\lambda} B))]\,\mathrm{d}\mathbf{p}\\
&\qquad+ \int_{\{A>0\}\cap\{B=0\}} [g + h(A \mathsf{S}_{\lambda} B)]\,\mathrm{d}\mathbf{p} + \int_{\{A=0\}\cap\{B>0\}} [g + h(A \mathsf{S}_{\lambda} B)]\,\mathrm{d}\mathbf{p}\\
&\quad= \int_{\{A>0\}\cap\{B>0\}} [2\cdot g + h\cdot(A+B)]\,\mathrm{d}\mathbf{p} + \int_{\{A>0\}\cap\{B=0\}} (g + h\cdot A)\,\mathrm{d}\mathbf{p}\\
&\qquad+ \int_{\{A=0\}\cap\{B>0\}} (g + h\cdot B)\,\mathrm{d}\mathbf{p}\\
&\quad= \int_{\{A>0\}} (g + h\cdot A)\,\mathrm{d}\mathbf{p} + \int_{\{B>0\}} (g + h\cdot B)\,\mathrm{d}\mathbf{p}\\
&\quad= \mathbf{m}(A) + \mathbf{m}(B).
\end{aligned}$$

Note that the measure $\mathbf{p}$ involved in the representation of $\mathbf{m}$ via (40) does not necessarily coincide with the restriction of $\mathbf{m}$ to $\mathcal{T}^{\vee}$.

Definition 7.3 A T_λ-measure $\mathbf{m}$ on $\mathcal{T}$ which can be represented in the form (40) for some nonnegative σ-additive measure $\mathbf{p}$ on $\mathcal{T}^\vee$ and for some $\mathcal{T}^\vee$-measurable functions $g, h \colon X \longrightarrow [0, \infty]$ is said to be *generated* (*by* $\mathbf{p}, g$ and h).

Theorem 6.1 shows that if $\mathcal{T}$ is generated, then all finite monotone T_λ-measures on $\mathcal{T}$ are generated. From Theorem 5.2 we already know that $T_\mathbf{L}$-measures on $\mathcal{T}$ are always generated, even if $\mathcal{T}$ is not generated. Thus, it is natural to ask whether, in general, T_λ-measures on T_λ-tribes are automatically generated. In order to answer this question we introduce the following notion.

Definition 7.4 A T_λ-measure $\mathbf{m}$ on a T_λ-tribe $\mathcal{T}$ is called *monotonically irreducible* if it is monotone and if the zero measure on $\mathcal{T}$ is the only generated T_λ-measure $\mathbf{q}$ on $\mathcal{T}$ such that $\mathbf{m} - \mathbf{q}$ is monotone.

It is obvious that a T_λ-measure $\mathbf{m}$ is generated if and only if it can be extended to a T_λ-measure on the generated T_λ-tribe $(\mathcal{T}^\vee)^\wedge$: if $\mathbf{m}$ is generated, then it is always defined on the whole generated tribe $(\mathcal{T}^\vee)^\wedge$; the converse follows from Theorem 6.1. On the other hand, monotonically irreducible T_λ-measures, with the exception of the zero measure, are not generated and, therefore, cannot be extended to $(\mathcal{T}^\vee)^\wedge$.

Theorem 7.5 *Let $\lambda \in \,]0, \infty]$, $\mathcal{T}$ be a T_λ-tribe and $\mathbf{m}$ a finite monotone T_λ-measure on $\mathcal{T}$. Then $\mathbf{m}$ can be uniquely decomposed into the sum of a monotonically irreducible and a generated T_λ-measure, i.e., there exist a unique monotonically irreducible T_λ-measure $\mathring{\mathbf{m}}$ on $\mathcal{T}$, a measure $\check{\mathbf{m}}$ on $\mathcal{T}^\vee$ (namely, the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$) and two $\check{\mathbf{m}}$-a.e. uniquely determined $\mathcal{T}^\vee$-measurable functions $g, h \colon X \longrightarrow [0, 1]$, such that for all $A \in \mathcal{T}^\vee$*

$$\mathbf{m}(A) = \mathring{\mathbf{m}}(A) + \int_{\{A>0\}} (g + h \cdot A) \, \mathrm{d}\check{\mathbf{m}}. \tag{41}$$

Proof: If $\lambda = \infty$, then the result follows from Theorem 5.2 taking for g and h the constant functions $g(x) = 0$ and $h(x) = 1$, respectively, and for $\mathring{\mathbf{m}}$ the zero measure. Assume, therefore, that $\lambda \in \,]0, \infty[$. In this case we proceed in several steps.

Claim 1. If $\mathbf{p}$ is a finite monotone T_λ-measure on $\mathcal{T}$, then there exists a finite monotone T_λ-measure $\tilde{\mathbf{p}}$ on the generated T_λ-tribe $(\mathcal{T}^\vee)^\wedge$ which is maximal in the sense that the difference $\tilde{\mathbf{p}} - \mathbf{q}$ is also monotone on $(\mathcal{T}^\vee)^\wedge$ whenever $\mathbf{q}$ is a T_λ-measure on $(\mathcal{T}^\vee)^\wedge$ such that $\mathbf{p} - \mathbf{q}$ is monotone on $\mathcal{T}$.

In order to verify this, let $\mathcal{N}(\mathbf{p})$ be the set of all monotone T_λ-measures $\mathbf{q}$ on $(\mathcal{T}^\vee)^\wedge$ such that $\mathbf{p} - \mathbf{q}$ is monotone on $\mathcal{T}$. This set is partially ordered by the *dominance relation* defined by

$$\mathbf{q} \gg \mathbf{r} \qquad \text{if and only if} \qquad \mathbf{q} - \mathbf{r} \quad \text{is monotone on } (\mathcal{T}^\vee)^\wedge. \tag{42}$$

Let $(\mathbf{q}_j)_{j \in J}$ be a chain in $\mathcal{N}(\mathbf{p})$, and for each $A \in (\mathcal{T}^\vee)^\wedge$ define $\mathbf{q}(A) = \sup\{\mathbf{q}_j(A) \mid j \in J\}$. Exactly as in the proof of Proposition 35, it follows that $\mathbf{q}$ is a monotone finite T_λ-measure on $(\mathcal{T}^\vee)^\wedge$. For $A, B \in \mathcal{T}$ with $A \leq B$ we have

$$\begin{aligned}
\mathbf{p}(A) - \mathbf{q}(A) &= \inf\{\mathbf{p}(A) - \mathbf{q}_j(A) \mid j \in J\} \\
&\leq \inf\{\mathbf{p}(B) - \mathbf{q}_j(B) \mid j \in J\} \\
&= \mathbf{p}(B) - \mathbf{q}(B),
\end{aligned}$$

i.e., $\mathbf{p} - \mathbf{q}$ is monotone on $\mathcal{T}$. For each $k \in J$ and for $A, B \in (\mathcal{T}^\vee)^\wedge$ with $A \leq B$, the fact that $(\mathbf{q}_j)_{j \in J}$ is a chain induces

$$(\mathbf{q} - \mathbf{q}_k)(A) = \sup\{(\mathbf{q}_j - \mathbf{q}_k)(A) \mid j \in J\} \leq \sup\{(\mathbf{q}_j - \mathbf{q}_k)(B) \mid j \in J\} = (\mathbf{q} - \mathbf{q}_k)(B),$$

which means that $\mathbf{q} - \mathbf{q}_k$ is monotone for each $k \in J$, i.e., $\mathbf{q} \gg \mathbf{q}_k$. Hence, each chain in $\mathcal{N}(\mathbf{p})$ has an upper bound in $\mathcal{N}(\mathbf{p})$ and, by Zorn's Lemma, this implies that $\mathcal{N}(\mathbf{p})$, equipped with the dominance relation $\gg$, contains a maximal element, completing the proof of Claim 1.

The following claim shows that $\mathcal{N}(\mathbf{p})$ is also a directed set, implying that the maximal element is indeed the unique greatest element.

Claim 2. For all $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{N}(\mathbf{p})$ there exists a $\mathbf{q} \in \mathcal{N}(\mathbf{p})$ such that $\mathbf{q} \gg \mathbf{q}_1$ and $\mathbf{q} \gg \mathbf{q}_2$.
Let $\check{\mathbf{q}}_1, \check{\mathbf{q}}_2$ be the restrictions of $\mathbf{q}_1$ and $\mathbf{q}_2$ to $\mathcal{T}^\vee$, respectively, let (P, Q) be a Hahn decomposition of the σ-additive signed measure $\check{\mathbf{q}}_1 - \check{\mathbf{q}}_2$, and define $\mathbf{q} \colon (\mathcal{T}^\vee)^\wedge \longrightarrow [0, \infty]$ by

$$\mathbf{q}(A) = \mathbf{q}_1(A \mathsf{T}_\lambda P) + \mathbf{q}_2(A \mathsf{T}_\lambda Q).$$

Obviously, $\mathbf{q}(\emptyset) = 0$ and $\mathbf{q}$ is continuous from the left. To see that $\mathbf{q}$ is also a T_λ-valuation, take into account that P and Q are crisp subsets of X, and then we get for all $A, B \in (\mathcal{T}^\vee)^\wedge$:

$$\begin{aligned}
\mathbf{q}(A \mathsf{T}_\lambda B) + \mathbf{q}(A \mathsf{S}_\lambda B) &= \mathbf{q}_1(A \mathsf{T}_\lambda B \mathsf{T}_\lambda P) + \mathbf{q}_2(A \mathsf{T}_\lambda B \mathsf{T}_\lambda Q) \\
&\quad + \mathbf{q}_1((A \mathsf{S}_\lambda B) \mathsf{T}_\lambda P) + \mathbf{q}_2((A \mathsf{S}_\lambda B) \mathsf{T}_\lambda Q) \\
&= \mathbf{q}_1((A \mathsf{T}_\lambda P) \mathsf{T}_\lambda (B \mathsf{T}_\lambda P)) + \mathbf{q}_1((A \mathsf{T}_\lambda P) \mathsf{S}_\lambda (B \mathsf{T}_\lambda P)) \\
&\quad + \mathbf{q}_2((A \mathsf{T}_\lambda Q) \mathsf{T}_\lambda (B \mathsf{T}_\lambda Q)) + \mathbf{q}_2((A \mathsf{T}_\lambda Q) \mathsf{S}_\lambda (B \mathsf{T}_\lambda Q)) \\
&= \mathbf{q}_1(A \mathsf{T}_\lambda P) + \mathbf{q}_1(B \mathsf{T}_\lambda P) + \mathbf{q}_2(A \mathsf{T}_\lambda Q) + \mathbf{q}_2(B \mathsf{T}_\lambda Q) \\
&= \mathbf{q}(A) + \mathbf{q}(B).
\end{aligned}$$

The T_λ-measure $\mathbf{q}$ is monotone since for fuzzy sets $A, B \in \mathcal{T}$ with $A \leq B$ we have

$$\mathbf{q}(A) = \mathbf{q}_1(A \mathsf{T}_\lambda P) + \mathbf{q}_2(A \mathsf{T}_\lambda Q) \leq \mathbf{q}_1(B \mathsf{T}_\lambda P) + \mathbf{q}_2(B \mathsf{T}_\lambda Q) = \mathbf{q}(B).$$

The monotone T_λ-measure $\mathbf{q}$ is an element of $\mathcal{N}(\mathbf{q})$ since for fuzzy sets $A, B \in (\mathcal{T}^\vee)^\wedge$ with $A \leq B$ we get

$$\begin{aligned}
(\mathbf{p} - \mathbf{q})(A) &= \mathbf{p}(A) - \mathbf{q}(A) \\
&= \mathbf{p}(A \mathsf{T}_\lambda P) + \mathbf{p}(A \mathsf{T}_\lambda Q) - \mathbf{q}_1(A \mathsf{T}_\lambda P) - \mathbf{q}_2(A \mathsf{T}_\lambda Q) \\
&= (\mathbf{p} - \mathbf{q}_1)(A \mathsf{T}_\lambda P) + (\mathbf{p} - \mathbf{q}_2)(A \mathsf{T}_\lambda Q) \\
&\leq (\mathbf{p} - \mathbf{q}_1)(B \mathsf{T}_\lambda P) + (\mathbf{p} - \mathbf{q}_2)(B \mathsf{T}_\lambda Q) \\
&= \mathbf{p}(B) - \mathbf{q}(B) \\
&= (\mathbf{p} - \mathbf{q})(B),
\end{aligned}$$

where the inequality is a consequence of $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{N}(\mathbf{p})$. To complete the proof of Claim 2, it remains to show that $\mathbf{q} \gg \mathbf{q}_1$ and $\mathbf{q} \gg \mathbf{q}_2$, i.e., $\mathbf{q} - \mathbf{q}_1$ and $\mathbf{q} - \mathbf{q}_2$ are monotone on $(\mathcal{T}^\vee)^\wedge$. For each $A \in (\mathcal{T}^\vee)^\wedge$ we get

$$\begin{aligned}
(\mathbf{q} - \mathbf{q}_1)(A) &= \mathbf{q}_1(A \mathsf{T}_\lambda P) + \mathbf{q}_2(A \mathsf{T}_\lambda Q) - \mathbf{q}_1(A \mathsf{T}_\lambda P) - \mathbf{q}_1(A \mathsf{T}_\lambda Q) \\
&= (\mathbf{q}_2 - \mathbf{q}_1)(A \mathsf{T}_\lambda Q) \\
&\geq 0,
\end{aligned}$$

showing that $\mathbf{q} - \mathbf{q}_1$ is nonnegative. This also implies that the restriction of $(\mathbf{q} - \mathbf{q}_1)^\vee$ to the σ-algebra of all crisp sets of the form $L \cap Q$ with $L \in \mathcal{T}^\vee$ is nonnegative and, therefore, monotone. Applying Theorem 6.1 to the T_λ-measure $\mathbf{q}_2 - \mathbf{q}_1$ which is monotone on the T_λ-tribe of all fuzzy subsets of Q contained in $(\mathcal{T}^\vee)^\wedge$, we get for all $A, B \in (\mathcal{T}^\vee)^\wedge$ with $A \leq B$ and for a suitable $\mathcal{T}^\vee$-measurable function

$k\colon X \longrightarrow [0,1]$:

$$\begin{aligned}
(\mathbf{q}-\mathbf{q}_1)(A) &= (\mathbf{q}_2-\mathbf{q}_1)(A \mathbin{\top_\lambda} Q)\\
&= \int_{\{A \top_\lambda Q>0\}} (k+(1-k)\cdot A)\,\mathrm{d}(\mathbf{q}_2-\mathbf{q}_1)^\vee\\
&= \int_{\{A>0\}\cap Q} (k+(1-k)\cdot A)\,\mathrm{d}(\mathbf{q}_2-\mathbf{q}_1)^\vee\\
&\le \int_{\{A>0\}\cap Q} (k+(1-k)\cdot B)\,\mathrm{d}(\mathbf{q}_2-\mathbf{q}_1)^\vee\\
&\quad + \int_{\{A=0\}\cap\{B>0\}\cap Q} (k+(1-k)\cdot B)\,\mathrm{d}(\mathbf{q}_2-\mathbf{q}_1)^\vee\\
&= \int_{\{B>0\}\cap Q} (k+(1-k)\cdot B)\,\mathrm{d}(\mathbf{q}_2-\mathbf{q}_1)^\vee\\
&= (\mathbf{q}_2-\mathbf{q}_1)(B \mathbin{\top_\lambda} Q)\\
&= (\mathbf{q}-\mathbf{q}_1)(B).
\end{aligned}$$

Monotonicity of $\mathbf{q}-\mathbf{q}_2$ is shown by complete analogy, interchanging the roles of $\mathbf{q}_1$ and $\mathbf{q}_2$ as well as of P and Q, thus completing the proof of Claim 2.
Now let $\tilde{\mathbf{m}}$ be the greatest T_λ-measure (with respect to the dominance relation $\gg$) in $\mathcal{N}(\mathbf{m})$ resulting from Claims 1 and 2. Since $\tilde{\mathbf{m}}$ is defined on $(\mathcal{T}^\vee)^\wedge$, Theorem 6.1 yields the representation

$$\tilde{\mathbf{m}}(A) = \int_{\{A>0\}} [l-(1-l)\cdot A]\,\mathrm{d}(\tilde{\mathbf{m}})^\vee \tag{43}$$

where $(\tilde{\mathbf{m}})^\vee$ is the restriction of $\tilde{\mathbf{m}}$ to $\big((\mathcal{T}^\vee)^\wedge\big)^\vee = \mathcal{T}^\vee$ and $l\colon X \longrightarrow [0,1]$ is a suitable $\mathcal{T}^\vee$-measurable function. Remark 7.2(ii) tells us that the unique decomposition $((\tilde{\mathbf{m}})_{\mathbf{L}}, (\tilde{\mathbf{m}})_\lambda)$ of $\tilde{\mathbf{m}}$ (compare Proposition 7.1) is given by

$$(\tilde{\mathbf{m}})_{\mathbf{L}}(A) = \int_X (1-l)\cdot A\,\mathrm{d}(\tilde{\mathbf{m}})^\vee, \tag{44}$$

$$(\tilde{\mathbf{m}})_\lambda(A) = \int_{\{A>0\}} (1-l)\,\mathrm{d}(\tilde{\mathbf{m}})^\vee. \tag{45}$$

Let $(\mathbf{m}_{\mathbf{L}}, \mathbf{m}_\lambda)$ be the unique decomposition of $\mathbf{m}$ resulting from Proposition 7.1, let $\check{\mathbf{m}}$ be the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$, and let $f\colon X \longrightarrow [0,1]$ be the $\mathcal{T}^\vee$-measurable function such that (34) and (35) hold.

Claim 3. $(\tilde{\mathbf{m}})_{\mathbf{L}} = \mathbf{m}_{\mathbf{L}}$.

From $\tilde{\mathbf{m}} \in \mathcal{N}(\mathbf{m})$ it follows that the function $\mathbf{r}\colon \mathcal{T} \longrightarrow \mathbb{R}$ defined by $\mathbf{r} = \mathbf{m} - \tilde{\mathbf{m}}$ is a monotone T_λ-measure. Using $\mathbf{m} - (\tilde{\mathbf{m}})_{\mathbf{L}} = \mathbf{r} + (\tilde{\mathbf{m}})_\lambda$, the T_λ-measure $\mathbf{m} - (\tilde{\mathbf{m}})_{\mathbf{L}}$ is monotone too. Since $\mathbf{m}_{\mathbf{L}}$ satisfies Proposition 7.1(iv), we get

$$\mathbf{m}_{\mathbf{L}} \ge (\tilde{\mathbf{m}})_{\mathbf{L}}. \tag{46}$$

Conversely, note that $\mathbf{m} - \mathbf{m}_{\mathbf{L}}$ is a monotone T_λ-measure on $\mathcal{T}$, which, together with the fact that $\tilde{\mathbf{m}}$ is the greatest element in $\mathcal{N}(\mathbf{m})$, implies that $\tilde{\mathbf{m}} \gg \mathbf{m}_{\mathbf{L}}$, from which we deduce that $\tilde{\mathbf{m}} - \mathbf{m}_{\mathbf{L}}$ is monotone, showing that $\mathbf{m}_{\mathbf{L}} \in \mathcal{M}(\tilde{\mathbf{m}})$. Since $(\tilde{\mathbf{m}})_{\mathbf{L}}$ is the greatest element of $\mathcal{M}(\tilde{\mathbf{m}})$, we get $(\tilde{\mathbf{m}})_{\mathbf{L}} \ge \mathbf{m}_{\mathbf{L}}$ and, together with (46), $(\tilde{\mathbf{m}})_{\mathbf{L}} = \mathbf{m}_{\mathbf{L}}$.

Claim 4. The monotone T_λ-measure $\mathring{\mathbf{m}}\colon \mathcal{T} \longrightarrow [0,\infty[$ defined by

$$\mathring{\mathbf{m}} = \mathbf{m} - \tilde{\mathbf{m}} \tag{47}$$

is monotonically irreducible.

In the case $\tilde{\mathbf{m}} = \mathbf{m}$ the claim is proved by definition. Suppose that $\tilde{\mathbf{m}} \neq \mathbf{m}$, i.e., $\mathring{\mathbf{m}}$ is different from the zero measure. Then Claim 3 yields

$$\begin{aligned}
\mathring{\mathbf{m}} &= \mathbf{m} - \tilde{\mathbf{m}} \\
&= \mathbf{m_L} + \mathbf{m}_\lambda - (\tilde{\mathbf{m}})_\mathbf{L} - (\tilde{\mathbf{m}})_\lambda \\
&= \mathbf{m}_\lambda - (\tilde{\mathbf{m}})_\lambda \\
&= \mathbf{m}_\lambda - (\mathbf{m}_\lambda)^\sim + (\mathbf{m}_\lambda)^\sim - (\tilde{\mathbf{m}})_\lambda.
\end{aligned}$$

If we are able to show that

$$(\mathbf{m}_\lambda)^\sim = (\tilde{\mathbf{m}})_\lambda, \tag{48}$$

then the proof is complete because of the maximality of $(\mathbf{m}_\lambda)^\sim$ in $\mathcal{N}(\mathbf{m}_\lambda)$. In order to prove (48), note that $(\mathbf{m}_\lambda)_\mathbf{L}$ is identically zero on $(\mathcal{T}^\vee)^\wedge$ because of $\mathbf{m} = \mathbf{m_L} + (\mathbf{m}_\lambda)_\mathbf{L} + (\mathbf{m}_\lambda)_\lambda$ and the maximality of $\mathbf{m_L}$ in $\mathcal{N}(\mathbf{m})$. Similarly, $((\mathbf{m}_\lambda)^\sim)_\mathbf{L}$ is the zero measure on $(\mathcal{T}^\vee)^\wedge$. Observe that

$$\mathbf{m} - (\mathbf{m}_\lambda)^\sim = \mathbf{m_L} + (\mathbf{m}_\lambda - (\mathbf{m}_\lambda)^\sim),$$

the T_λ-measure on the right hand side being monotone on $\mathcal{T}$. Since $\tilde{\mathbf{m}}$ is the greatest element in $\mathcal{N}(\mathbf{m})$, we have $\tilde{\mathbf{m}} \gg (\mathbf{m}_\lambda)^\sim$. From the monotonicity of $\mathring{\mathbf{m}} = \mathbf{m}_\lambda - (\tilde{\mathbf{m}})_\lambda$ and the maximality of $(\mathbf{m}_\lambda)^\sim$ in $\mathcal{N}(\mathbf{m}_\lambda)$, we deduce that $(\mathbf{m}_\lambda)^\sim - (\tilde{\mathbf{m}})_\lambda$ is monotone and, subsequently, $(\mathbf{m}_\lambda)^\sim \geq (\tilde{\mathbf{m}})_\lambda$. Assume that $(\mathbf{m}_\lambda)^\sim \neq (\tilde{\mathbf{m}})_\lambda$, and define the monotone $T_\mathbf{L}$-measure $\overline{\mathbf{m}}\colon (\mathcal{T}^\vee)^\wedge \longrightarrow [0,\infty[$ by $\overline{\mathbf{m}} = (\tilde{\mathbf{m}})_\mathbf{L} + (\mathbf{m}_\lambda)^\sim$. Then we have

$$\begin{aligned}
\overline{\mathbf{m}} - \tilde{\mathbf{m}} &= (\tilde{\mathbf{m}})_\mathbf{L} + (\mathbf{m}_\lambda)^\sim - \tilde{\mathbf{m}} \\
&= (\tilde{\mathbf{m}})_\mathbf{L} + (\tilde{\mathbf{m}})_\lambda - (\tilde{\mathbf{m}})_\lambda + (\mathbf{m}_\lambda)^\sim - \tilde{\mathbf{m}} \\
&= (\mathbf{m}_\lambda)^\sim - (\tilde{\mathbf{m}})_\lambda,
\end{aligned} \tag{49}$$

the latter being monotone, showing that $\overline{\mathbf{m}} \gg \tilde{\mathbf{m}}$. On the other hand, Claim 3 yields

$$\mathbf{m} - \overline{\mathbf{m}} = \mathbf{m}_\lambda - (\mathbf{m}_\lambda)^\sim,$$

which implies that $\mathbf{m} - \overline{\mathbf{m}}$ is monotone too, i.e., $\overline{\mathbf{m}} \in \mathcal{N}(\mathbf{m})$. Therefore, $\overline{\mathbf{m}} = \tilde{\mathbf{m}}$ which, together with (49), contradicts the assumption $(\mathbf{m}_\lambda)^\sim \neq (\tilde{\mathbf{m}})_\lambda$, thus completing the proof of Claim 4.

Now (47) shows that $\mathbf{m} = \mathring{\mathbf{m}} + \tilde{\mathbf{m}}$, where $\mathring{\mathbf{m}}$ is monotonically irreducible and $\tilde{\mathbf{m}}$ is generated by definition. It remains to show that (41) holds. From Claim 1 we know that $(\tilde{\mathbf{m}})^\vee \leq \check{\mathbf{m}}$ and, therefore, $(\tilde{\mathbf{m}})^\vee$ is absolutely continuous with respect to $\check{\mathbf{m}}$. Thus we may assume that the Radon-Nikodym derivative

$$\frac{\mathrm{d}(\tilde{\mathbf{m}})^\vee}{\mathrm{d}\check{\mathbf{m}}}$$

has values in $[0,1]$ only. Taking into account (45) we obtain for each $A \in (\mathcal{T}^\vee)^\wedge$

$$(\tilde{\mathbf{m}})_\lambda(A) = \int_{\{A>0\}} l \,\mathrm{d}(\tilde{\mathbf{m}})^\vee = \int_{\{A>0\}} l \cdot \frac{\mathrm{d}(\tilde{\mathbf{m}})^\vee}{\mathrm{d}\check{\mathbf{m}}} \,\mathrm{d}\check{\mathbf{m}}.$$

Writing

$$g = l \cdot \frac{\mathrm{d}(\tilde{\mathbf{m}})^\vee}{\mathrm{d}\check{\mathbf{m}}}$$

and taking into account Claim 3 and (34), we have for all $A \in (\mathcal{T}^\vee)^\wedge$

$$\begin{aligned}
\tilde{\mathbf{m}}(A) &= (\tilde{\mathbf{m}})_\mathbf{L}(A) + (\tilde{\mathbf{m}})_\lambda(A) \\
&= \mathbf{m_L}(A) + (\tilde{\mathbf{m}})_\lambda(A) \\
&= \int_{\{A>0\}} (1-f)\cdot A \,\mathrm{d}\check{\mathbf{m}} + \int_{\{A>0\}} g \,\mathrm{d}\check{\mathbf{m}} \\
&= \int_{\{A>0\}} [g + (1-f)\cdot A] \,\mathrm{d}\check{\mathbf{m}}.
\end{aligned}$$

Putting $h = 1 - f$ gives (41). Since $\tilde{\mathbf{m}}$ is unique and l, f and the Radon-Nikodym derivative

$$\frac{\mathrm{d}(\tilde{\mathbf{m}})^{\vee}}{\mathrm{d}\check{\mathbf{m}}}$$

are $\check{\mathbf{m}}$-a.e. uniquely determined, so are g and h. □

A combination of the results of Theorems 4.8 and 7.5 leads to the following:

Corollary 7.6 *If $\lambda \in\,]0,\infty]$, $\mathcal{T}$ is a T_λ-tribe and $\mathbf{m}$ is a finite monotone T_λ-measure on $\mathcal{T}$, then there exists a unique finite nonnegative σ-additive measure $\mathbf{p}$ on $\mathcal{T}^{\vee}$, a $\mathbf{p}$-a.e. uniquely determined $\mathcal{T}^{\vee}$-Markov kernel $K\colon X \times \mathcal{B}_1 \longrightarrow \mathbb{R}$ and a unique monotonically irreducible T_λ-measure $\mathring{\mathbf{m}}$ on $\mathcal{T}$ such that for all $A \in \mathcal{T}$*

$$\mathbf{m}(A) = \mathring{\mathbf{m}}(A) + \int_X K(x, [0, A(x)[)\,\mathrm{d}\mathbf{p}(x).$$

From the results in Mesiar [61] and Mesiar & Navara [63] it follows that for $\lambda \in\,]0,\infty]$ each T_λ-measure $\mathbf{m}$, which is defined on a T_λ-tribe on a countable set, is a generated measure, i.e., $\mathring{\mathbf{m}}$ equals the zero measure.

8 Jordan decomposition of bounded $T_{\mathbf{L}}$-measures

It was shown in Theorem 7.5 that monotone Frank triangular norm-based measures can be decomposed into a sum consisting of a generated measure and a monotonically irreducible measure. Moreover, if the tribe on which the Frank triangular norm-based measure is defined is a generated one, then the monotonically irreducible component is necessarily zero (compare Theorem 6.1).

It is an interesting open question whether the monotonically irreducible component of a Frank triangular norm-based monotone measure on a nongenerated tribe is always identically zero. We shall prove that in the particular case of the $T_{\mathbf{L}}$-measures, this happens even if the $T_{\mathbf{L}}$-measure is not monotone.

The fact that $T_{\mathbf{L}}$-measures are Jordan decomposable can be deduced directly from a result in Schmidt [72]. However, we present the proof due to Butnariu [15] because some technical details of this proof will be used later.

It is natural to ask in which sense $T_{\mathbf{L}}$-measures differ from other Frank triangular norm-based measures. In order to answer this question, we recall that T-measures are always T-countably additive functions but, in general, T-countable additivity does not characterize T-measures (see Remark 4.4). However, $T_{\mathbf{L}}$-countably additive functions are necessarily $T_{\mathbf{L}}$-measures. To be precise, we have the following result:

Proposition 8.1 *If $\mathbf{m}$ is a finite $T_{\mathbf{L}}$-countably additive function on a $T_{\mathbf{L}}$-tribe $\mathcal{T}$, then $\mathbf{m}$ is a $T_{\mathbf{L}}$-measure on $\mathcal{T}$.*

Proof: Observe that

$$(A \mathsf{T}_{\mathsf{L}} \complement(A \mathsf{T}_{\mathsf{L}} B)) \mathsf{T}_{\mathsf{L}} B = \emptyset,$$
$$(A \mathsf{T}_{\mathsf{L}} \complement(A \mathsf{T}_{\mathsf{L}} B)) \mathsf{S}_{\mathsf{L}} B = A \mathsf{S}_{\mathsf{L}} B.$$

Therefore, using the $T_{\mathbf{L}}$-additivity of $\mathbf{m}$ we get $\mathbf{m}(A \mathsf{S}_{\mathsf{L}} B) = \mathbf{m}(A \mathsf{T}_{\mathsf{L}} \complement(A \mathsf{T}_{\mathsf{L}} B)) + \mathbf{m}(B)$. But $A = (A \mathsf{T}_{\mathsf{L}} \complement(A \mathsf{T}_{\mathsf{L}} B)) \mathsf{S}_{\mathsf{L}} (A \mathsf{T}_{\mathsf{L}} B)$, and the fuzzy sets $A \mathsf{T}_{\mathsf{L}} \complement(A \mathsf{T}_{\mathsf{L}} B)$ and $A \mathsf{T}_{\mathsf{L}} B$ are $T_{\mathbf{L}}$-disjoint. Hence,

$$\mathbf{m}(A) = \mathbf{m}(A \mathsf{T}_{\mathsf{L}} \complement(A \mathsf{T}_{\mathsf{L}} B)) + \mathbf{m}(A \mathsf{T}_{\mathsf{L}} B),$$

and this implies

$$\mathbf{m}(A \mathsf{S}_{\mathsf{L}} B) = \mathbf{m}(A) - \mathbf{m}(A \mathsf{T}_{\mathsf{L}} B) + \mathbf{m}(B),$$

showing that $\mathbf{m}$ is a $T_{\mathbf{L}}$-valuation on $\mathcal{T}$. Finally, we show that $T_{\mathbf{L}}$-countably additive functions are left continuous. Let $(A_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence in $\mathcal{T}$ which converges to A. Define $A_0 = \emptyset$ and, for each $n \in \mathbb{N}$, $B_n = A_n \mathsf{T}_{\mathsf{L}} \complement A_{n-1}$. Then for all $n \in \mathbb{N}$

$$(\mathop{\mathbf{S}_{\mathbf{L}}}_{i=1}^{n} B_i) \mathsf{T}_{\mathsf{L}} B_{n+1} = A_n \mathsf{T}_{\mathsf{L}}(A_{n+1} \mathsf{T}_{\mathsf{L}} \complement A_n) = \emptyset,$$

showing that the sequence $(B_n)_{n\in\mathbb{N}}$ is $T_{\mathbf{L}}$-disjoint. Since for each $n \in \mathbb{N}$ we have

$$A_n = \mathop{\mathbf{S}_{\mathbf{L}}}_{k=1}^{n} B_k,$$

we obtain

$$A = \mathop{\mathbf{S}_{\mathbf{L}}}_{n=1}^{\infty} B_n,$$

and, consequently,

$$\lim_{n\to\infty} \mathbf{m}(A_n) = \lim_{n\to\infty} \mathbf{m}\Big(\mathop{\mathbf{S}_{\mathbf{L}}}_{k=1}^{n} B_k\Big) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{m}(B_k) = \sum_{n=1}^{\infty} \mathbf{m}(B_n) = \mathbf{m}\Big(\mathop{\mathbf{S}_{\mathbf{L}}}_{n=1}^{\infty} B_n\Big) = \mathbf{m}(A).$$

Hence, $\mathbf{m}$ is a $T_{\mathbf{L}}$-measure. □

Proposition 8.1 allows us to approach $T_{\mathbf{L}}$-measures on $T_{\mathbf{L}}$-tribes in a way similar to the treatment of (σ-additive) measures on σ-algebras of subsets of X.

Definition 8.2 Let $\mathcal{T}$ be a $T_{\mathbf{L}}$-tribe, $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty, \infty]$ a function such that $\mathbf{m}(\emptyset) = 0$, and $A \in \mathcal{T}$.

(i) The function $\mathbf{m}^+\colon \mathcal{T} \longrightarrow [0, \infty]$ defined by

$$\mathbf{m}^+(A) = \sup\{\mathbf{m}(B) \mid B \in \mathcal{T}(A)\} \tag{50}$$

is called *upper variation* of $\mathbf{m}$, where $\mathcal{T}(A)$ denotes the class of all fuzzy subsets of A contained in $\mathcal{T}$, i.e., $\mathcal{T}(A) = \{B \mid B \in \mathcal{T}, B \leq A\}$.

(ii) The function $\mathbf{m}^-\colon \mathcal{T} \longrightarrow [0, \infty]$ defined by

$$\mathbf{m}^-(A) = \sup\{-\mathbf{m}(B) \mid B \in \mathcal{T}(A)\} \tag{51}$$

is called *lower variation* of $\mathbf{m}$.

(iii) The function $\mathbf{m}^*\colon \mathcal{T} \longrightarrow [0, \infty]$ defined by $\mathbf{m}^* = \mathbf{m}^+ + \mathbf{m}^-$ is called *total variation* of $\mathbf{m}$.

Obviously, these functions are all monotone, and they satisfy

$$\begin{aligned} \mathbf{m}^+ &= (-\mathbf{m})^-, \\ \mathbf{m}^- &= (-\mathbf{m})^+, \\ \mathbf{m}^+(\emptyset) = \mathbf{m}^-(\emptyset) &= \mathbf{m}^*(\emptyset) = 0. \end{aligned} \tag{52}$$

To simplify notation, from now on we shall use the following shortcut: for fuzzy sets $A, B \in \mathcal{T}$ we shall write

$$A \ominus B = A \mathsf{T}_{\mathsf{L}} \complement B.$$

Obviously, each $T_{\mathbf{L}}$-tribe $\mathcal{T}$ is closed under this operation.

Proposition 8.3 *(i) If $\mathbf{m}$ is finitely $T_{\mathbf{L}}$-additive, then $\mathbf{m}^+, \mathbf{m}^-$ and $\mathbf{m}^*$ are finitely $T_{\mathbf{L}}$-additive too.*

(ii) If $\mathbf{m}$ *is a* $T_{\mathbf{L}}$*-measure, then* $\mathbf{m}^+, \mathbf{m}^-$ *and* $\mathbf{m}^*$ *are monotone* $T_{\mathbf{L}}$*-measures.*

Proof: When proving (i), it is sufficient to show that $\mathbf{m}^+$ is $T_{\mathbf{L}}$-additive (compare (52)). Let E and F be $T_{\mathbf{L}}$-disjoint fuzzy sets in $\mathcal{T}$. For each $B \in \mathcal{T}(E \,\mathsf{S}_{\mathsf{L}}\, F)$ the fuzzy sets $C = B \,\mathsf{T}_{\mathsf{L}}\, E$ and $D = B \ominus C$ are elements of $\mathcal{T}$, and they form a $T_{\mathbf{L}}$-partition of B. Since $D \in \mathcal{T}(E)$ and $C \in \mathcal{T}(F)$, we have

$$\mathbf{m}(B) = \mathbf{m}(C) + \mathbf{m}(D) \leq \mathbf{m}^+(F) + \mathbf{m}^+(E)$$

which implies

$$\mathbf{m}^+(E \,\mathsf{S}_{\mathsf{L}}\, F) \leq \mathbf{m}^+(E) + \mathbf{m}^+(F).$$

For each $\varepsilon > 0$ there exist fuzzy sets $A \in \mathcal{T}(E)$ and $B \in \mathcal{T}(F)$ such that

$$\mathbf{m}^+(E) \leq \mathbf{m}(A) + \frac{\varepsilon}{2},$$
$$\mathbf{m}^+(F) \leq \mathbf{m}(B) + \frac{\varepsilon}{2}.$$

From $A \,\mathsf{T}_{\mathsf{L}}\, B \leq E \,\mathsf{T}_{\mathsf{L}}\, F$ we deduce

$$\mathbf{m}^+(E) + \mathbf{m}^+(F) \leq \mathbf{m}(A) + \mathbf{m}(B) + \varepsilon = \mathbf{m}(A \,\mathsf{S}_{\mathsf{L}}\, B) + \varepsilon \leq \mathbf{m}^+(E \,\mathsf{S}_{\mathsf{L}}\, F) + \varepsilon.$$

Taking the limit $\varepsilon \to 0$ we obtain $\mathbf{m}^+(E) + \mathbf{m}^+(F) \leq \mathbf{m}^+(E \,\mathsf{S}_{\mathsf{L}}\, F)$, and this completes the proof of (i).

In order to show the validity of (ii), assume that $\mathbf{m}$ is a $T_{\mathbf{L}}$-measure. According to Proposition 8.1 and equations (52), it is sufficient to prove that $\mathbf{m}^+$ is $T_{\mathbf{L}}$-countably additive. Let $(A_k)_{k \in \mathbb{N}}$ be a $T_{\mathbf{L}}$-disjoint sequence of fuzzy sets in $\mathcal{T}$. Then the sequence $(B_n)_{n \in \mathbb{N}}$ defined by

$$B_n = \mathop{\mathbf{S}_{\mathbf{L}}}_{k=1}^{n} A_k$$

is nondecreasing and, since $\mathbf{m}^+$ is $T_{\mathbf{L}}$-additive, it follows that for each $n \in \mathbb{N}$

$$\mathbf{m}^+(B_n) = \sum_{k=1}^{n} \mathbf{m}^+(A_k).$$

Thus we have

$$\lim_{n \to \infty} \mathbf{m}^+(B_n) = \sum_{k=1}^{\infty} \mathbf{m}^+(A_k).$$

Since for each $n \in \mathbb{N}$

$$B_n \leq \mathop{\mathbf{S}_{\mathbf{L}}}_{k=1}^{\infty} A_k$$

and since $\mathbf{m}^+$ is monotone, we obtain

$$\mathbf{m}^+\Big(\mathop{\mathbf{S}_{\mathbf{L}}}_{k=1}^{\infty} A_k\Big) \geq \sum_{k=1}^{\infty} \mathbf{m}^+(A_k). \tag{53}$$

For each

$$A \in \mathcal{T}\Big(\mathop{\mathbf{S}_{\mathbf{L}}}_{k=1}^{\infty} A_k\Big)$$

we have

$$A = A \,\mathsf{T}_{\mathsf{M}} \left(\mathop{\mathbf{S}_{\mathbf{L}}}_{k=1}^{\infty} A_k\right) = A \,\mathsf{T}_{\mathsf{M}} \left(\mathop{\mathbf{S}_{\mathbf{M}}}_{n=1}^{\infty} B_n\right) = \mathop{\mathbf{S}_{\mathbf{M}}}_{n=1}^{\infty} (A \,\mathsf{T}_{\mathsf{M}}\, B_n) = \lim_{n \to \infty} (A \,\mathsf{T}_{\mathsf{M}}\, B_n).$$

The sequence $\left(A \mathsf{T}_{\mathsf{M}} B_n\right)_{n\in\mathbb{N}}$ is nondecreasing and $\mathbf{m}$ is left-continuous since it is a $T_{\mathbf{L}}$-measure. Hence we have

$$\mathbf{m}(A) = \mathbf{m}(\lim_{n\to\infty}(A \mathsf{T}_{\mathsf{M}} B_n)) = \lim_{n\to\infty}\mathbf{m}(A \mathsf{T}_{\mathsf{M}} B_n) \leq \lim_{n\to\infty}\mathbf{m}^+(B_n)$$

because of $A \mathsf{T}_{\mathsf{M}} B_n \in \mathcal{T}(B_n)$, which implies

$$\mathbf{m}(A) \leq \sum_{k=1}^{\infty}\mathbf{m}^+(A_k).$$

Therefore, we obtain equality even in (53), and the proposition is proved. □

Proposition 8.4 *(i) A finitely $T_{\mathbf{L}}$-additive function $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty,\infty]$ is bounded if and only if $\mathbf{m}^*$ is bounded.*

(ii) If the finitely $T_{\mathbf{L}}$-additive function $\mathbf{m}\colon \mathcal{T} \longrightarrow [-\infty,\infty]$ is bounded, then we have

$$\mathbf{m} = \mathbf{m}^+ - \mathbf{m}^-, \tag{54}$$

i.e., the pair $(\mathbf{m}^+,\mathbf{m}^-)$ is a Jordan decomposition of $\mathbf{m}$.

Proof: We first show (i). If $\mathbf{m}^*$ is bounded, then for each $A \in \mathcal{T}$

$$|\mathbf{m}(A)| \leq \sup\{\mathbf{m}^*(B) \mid B \in \mathcal{T}\} < \infty,$$

i.e., $\mathbf{m}$ is bounded. Conversely, if $\mathbf{m}$ is bounded, then

$$\max\left(\mathbf{m}^+(A),\mathbf{m}^-(A)\right) \leq \sup\left\{\,|\mathbf{m}(B)| \;\middle|\; B \in \mathcal{T}(A)\right\} \leq \sup\left\{\,|\mathbf{m}(B)| \;\middle|\; B \in \mathcal{T}\right\} < \infty$$

for all $A \in \mathcal{T}$. Hence, $\mathbf{m}^+$ and $\mathbf{m}^-$ are both bounded, and, consequently, so is $\mathbf{m}^*$.

In order to prove (ii), assume that $\mathbf{m}$ is bounded. Then $\mathbf{m}^+,\mathbf{m}^-$ and $\mathbf{m}^*$ are bounded too, and for $A \in \mathcal{T}$ and $M \in \mathcal{T}(A)$ we have $\mathbf{m}(M) = \mathbf{m}(A) - \mathbf{m}(A \ominus M)$. Since $A \ominus M \in \mathcal{T}(A)$, it follows that $\mathbf{m}^+(A) \geq \mathbf{m}(A \ominus M)$ and $\mathbf{m}^-(A) \geq -\mathbf{m}(A \ominus M)$. Hence,

$$\mathbf{m}(A) - \mathbf{m}^+(A) \leq \mathbf{m}(A) - \mathbf{m}(A \ominus M) \leq \mathbf{m}(A) + \mathbf{m}^-(A)$$

and, consequently,

$$\mathbf{m}(A) - \mathbf{m}^+(A) \leq \mathbf{m}(M) \leq \mathbf{m}(A) + \mathbf{m}^-(A).$$

From (50) and (51) we deduce

$$\begin{aligned}\mathbf{m}^-(A) &\leq \mathbf{m}^+(A) - \mathbf{m}(A),\\ \mathbf{m}^+(A) &\leq \mathbf{m}(A) + \mathbf{m}^-(A),\end{aligned}$$

and this completes the proof. □

A straightforward combination of Proposition 8.4, Theorem 5.2 and Theorem 3.3(ii) yields the following.

Corollary 8.5 *Let $\mathbf{m}$ be a bounded $T_{\mathbf{L}}$-measure on a $T_{\mathbf{L}}$-tribe $\mathcal{T}$. Then we have:*

(i) $\mathbf{m}$ is continuous in the sense of (15).

(ii) For all $A \in \mathcal{T}$,

$$\mathbf{m}(A) = \int_X A(x)\,\mathrm{d}((\mathbf{m}^+)^\vee - (\mathbf{m}^-)^\vee), \tag{55}$$

where $(\mathbf{m}^+)^\vee$ and $(\mathbf{m}^-)^\vee$ are the restrictions of $\mathbf{m}^+$ and $\mathbf{m}^-$ to $\mathcal{T}^\vee$, respectively.

(iii) $(\mathbf{m}^+)^\vee$ and $(\mathbf{m}^-)^\vee$ are the upper and lower variation of $\check{\mathbf{m}}$, the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$, and, therefore,

$$\check{\mathbf{m}} = (\mathbf{m}^+)^\vee - (\mathbf{m}^-)^\vee. \tag{56}$$

9 Jordan decomposition of finite $T_{\mathbf{L}}$-measures

In Section 8 we have shown that bounded $T_{\mathbf{L}}$-measures on $T_{\mathbf{L}}$-tribes have Jordan decompositions and, therefore, are continuous (in the sense of (15)) and can be represented by integrals (see (55)). The aim of this section is to show that finite $T_{\mathbf{L}}$-measures on $T_{\mathbf{L}}$-tribes are necessarily bounded and, consequently, have Jordan decompositions by finite $T_{\mathbf{L}}$-measures, a result which is due to Butnariu [15].

Lemma 9.1 *Let* $\mathbf{m}$ *be a finite* $T_{\mathbf{L}}$*-measure on a* $T_{\mathbf{L}}$*-tribe* $\mathcal{T}$*. The following are equivalent:*

(i) $\mathbf{m}$ *is bounded.*

(ii) $\mathbf{m}^+(X) < \infty$ *and* $\mathbf{m}^-(X) < \infty$.

Proof: The implication (i) $\Rightarrow$ (ii) is trivial. Assume that (ii) holds. The monotonicity of $\mathbf{m}^+$ and $\mathbf{m}^-$ implies that they are both bounded. Thus $\mathbf{m}^*$ and, consequently, $\mathbf{m}$ are bounded (compare Proposition 8.4). □

Theorem 9.2 *If* $\mathbf{m}$ *is a finite* $T_{\mathbf{L}}$*-measure on a* $T_{\mathbf{L}}$*-tribe* $\mathcal{T}$*, then there exist nonnegative finite* $T_{\mathbf{L}}$*-measures* $\mathbf{m}^+$ *and* $\mathbf{m}^-$ *on* $\mathcal{T}$ *such that*

$$\mathbf{m} = \mathbf{m}^+ - \mathbf{m}^-, \tag{57}$$

i.e., $(\mathbf{m}^+, \mathbf{m}^-)$ *is a Jordan decomposition of* $\mathbf{m}$.

Proof: Let $\mathbf{m}$ be a finite $T_{\mathbf{L}}$-measure on $\mathcal{T}$. According to Proposition 8.4 and Lemma 9.1, it suffices to show that $\mathbf{m}^+(X) < \infty$ and $\mathbf{m}^-(X) < \infty$. In order to show $\mathbf{m}^+(X) < \infty$, a fuzzy set $M \in \mathcal{T}$ such that $\mathbf{m}^+(M) = \infty$ will be called unbounded.

Claim: If M is unbounded, then there exists a nonincreasing sequence $(A_n)_{n\in\mathbb{N}}$ of unbounded fuzzy sets in $\mathcal{T}(M)$ such that $\mathbf{m}(A_n) \geq n$ for all $n \in \mathbb{N}$.

Let M be unbounded and assume that there exists an $n_0 \in \mathbb{N}$ such that $\mathbf{m}(A) < n_0$ for all unbounded fuzzy sets $A \in \mathcal{T}(M)$. Since M is unbounded, there exists an $M_1 \in \mathcal{T}(M)$ such that $\mathbf{m}(M_1) \leq n_0$, which means that M_1 is not unbounded, i.e.,

$$\sup\{\mathbf{m}(A) \mid A \in \mathcal{T}(M_1)\} < \infty. \tag{58}$$

Choose an arbitrary $B \in \mathcal{T}(M)$ and denote $B_1 = B \mathsf{T_M} M_1$, $B_2 = B \ominus B_1$. It is clear that $B_1 \mathsf{T_L} B_2 = \emptyset$, $B_1 \mathsf{S_L} B_2 = B$, $B_1 \in \mathcal{T}(M_1)$ and $B_2 \in \mathcal{T}(M \ominus M_1)$. Therefore, $\mathbf{m}(B) = \mathbf{m}(B_1) + \mathbf{m}(B_2)$ and

$$\mathbf{m}^+(M \ominus M_1) \geq \mathbf{m}(B_2) = \mathbf{m}(B) - \mathbf{m}(B_1) \geq \mathbf{m}(B) - \mathbf{m}^+(M_1).$$

Since this is true for all $B \in \mathcal{T}(M)$, since M is unbounded and since (58) holds, we have

$$\mathbf{m}^+(M \ominus M_1) \geq \mathbf{m}^+(M) - \mathbf{m}^-(M_1) = \infty,$$

i.e., $M \ominus M_1$ is unbounded. Then, by assumption, $\mathbf{m}(M \ominus M_1) < n_0$, and there exists a fuzzy set $C_1 \in \mathcal{T}(M \ominus M_1)$ such that $\mathbf{m}(C_1) \geq 1$. Put $M_2 = M_1 \mathsf{S_L} C_1$. Then $M_1 \mathsf{T_L} C_1 \leq M_1 \mathsf{T_L} (M \ominus M_1) = \emptyset$ and

$$\mathbf{m}(M_2) = \mathbf{m}(M_1) + \mathbf{m}(C_1) \geq n_0 + 1. \tag{59}$$

Using the assumption again yields that M_2 is not unbounded. Similar reasoning as above shows that $M \ominus M_2$ is unbounded too. Hence, $\mathbf{m}(M \ominus M_2) < n_0$ and $\mathbf{m}(C_2) \geq 1$ for some $C_2 \in \mathcal{T}(M \ominus M_2)$. Denote $M_3 = M_2 S_{\mathbf{L}} C_2$. Since $M_2 \mathsf{T_L} C_2 \leq M_2 \mathsf{T_L} (M \ominus M_2) = \emptyset$ it follows from (59) that we have $\mathbf{m}(M_3) = \mathbf{m}(M_2) + \mathbf{m}(C_2) \geq n_0 + 2$, i.e., M_3 is not unbounded. In this way we obtain fuzzy sets

$M_1, M_2, M_3, \ldots$ in $\mathcal{T}(M)$, none of which are unbounded, and $C_1, C_2, C_3, \ldots$ such that $C_n \in \mathcal{T}(M \ominus M_n)$ and $\mathbf{m}(C_n) \geq 1$ for all $n \in \mathbb{N}$. Moreover, we can show that for all $n \in \mathbb{N}$

$$\sum_{i=1}^{n} C_i \leq M - M_1. \tag{60}$$

Indeed, if $n = 1$ this is true because of the definition of C_1. If (60) holds for all $k \in \{1, 2, \ldots, n\}$, then we have for each $x \in X$

$$\begin{aligned}
C_{n+1}(x) &\leq \max(0, M(x) - M_{n+1}(x)) \\
&= M(x) - M_{n+1}(x) \\
&= M(x) - (M_n(x) + C_n(x)) \\
&= M(x) - (M_{n-1}(x) + C_{n-1}(x) + C_n(x)) \\
&\;\;\vdots \\
&= M(x) - M_1(x) - \sum_{i=1}^{n} C_i(x).
\end{aligned}$$

Hence, (60) is proved by induction. An immediate consequence of (60) is that the sequence $(C_n)_{n\in\mathbb{N}}$ is $T_{\mathbf{L}}$-disjoint which, together with $\mathbf{m}(C_n) \geq 1$, yields

$$\mathbf{m}\Big(\mathop{\mathrm{S}_{\mathbf{L}}}_{n=1}^{\infty} C_n\Big) = \sum_{n=1}^{\infty} \mathbf{m}(C_n) = \infty.$$

Since $\mathbf{m}$ is finite, this contradicts the assumption that $\mathbf{m}(A) < n_0$ for all unbounded fuzzy sets $A \in \mathcal{T}(M)$. Therefore, there exists an unbounded fuzzy set $A_1 \in \mathcal{T}(M)$ such that $\mathbf{m}(A_1) \geq 1$. Since A_1 is unbounded, there exists an unbounded fuzzy set $A_2 \in \mathcal{T}(A_1)$ such that $\mathbf{m}(A_2) \geq 2$, and so on, showing that the claim holds.

To complete the proof of the theorem, assume that X is unbounded and choose the nonincreasing sequence $(A_n)_{n\in\mathbb{N}}$ of unbounded fuzzy sets in $\mathcal{T}$ which exists according to the claim. For each $n \in \mathbb{N}$ denote

$$B_n = \mathop{\mathrm{S}_{\mathbf{L}}}_{k=1}^{n-1} (A_k \ominus A_{k+1}).$$

Clearly, since $(A_n)_{n\in\mathbb{N}}$ is nonincreasing, the sequence $(A_k \ominus A_{k+1})_{k\in\mathbb{N}}$ is $T_{\mathbf{L}}$-disjoint, and therefore,

$$\mathbf{m}(B_n) = \sum_{k=1}^{n-1} \mathbf{m}(A_k \ominus A_{k+1}) = \mathbf{m}(A_1) - \mathbf{m}(A_n).$$

The sequence $(B_n)_{n\in\mathbb{N}}$ is nondecreasing which leads to

$$\mathbf{m}\big(\lim_{n\to\infty} B_n\big) = \lim_{n\to\infty} \mathbf{m}(B_n) = \mathbf{m}(A_1) - \lim_{n\to\infty} \mathbf{m}(A_n) = -\infty,$$

where the latter equality follows from $\mathbf{m}(A_n) \geq n$ and $\mathbf{m}(A_1) < \infty$. This contradicts the finiteness of $\mathbf{m}$. Therefore, the assumption that X is unbounded is false and, consequently, $\mathbf{m}^+(X) < \infty$.

The inequality $\mathbf{m}^-(X) < \infty$ is shown in complete analogy, $\mathbf{m}$ is bounded by Lemma 57, and the Jordan decomposition exists because of Proposition 8.4. □

Since each bounded $T_{\mathbf{L}}$-measure is automatically finite, in the proof of Theorem 9.2 we have implicitly proven the following result:

Corollary 9.3 *A $T_{\mathbf{L}}$-measure $\mathbf{m}$ on a $T_{\mathbf{L}}$-tribe $\mathcal{T}$ is bounded if and only if it is finite.*

Combining Corollary 8.4 and Theorem 9.2 we realize that finite $T_{\mathbf{L}}$-measures on $T_{\mathbf{L}}$-tribes are exactly (Lebesgue) integrals:

Corollary 9.4 *Let $\mathbf{m}$ be a finite $T_{\mathbf{L}}$-measure on a $T_{\mathbf{L}}$-tribe $\mathcal{T}$. Then we have*

(i) $\mathbf{m}$ is continuous in the sense of (15).

(ii) If $(\mathbf{m}^+)^\vee$ and $(\mathbf{m}^-)^\vee$ are the restrictions of $\mathbf{m}^+$ and $\mathbf{m}^-$ to $\mathcal{T}^\vee$, respectively, then for all $A \in \mathcal{T}$

$$\mathbf{m}(A) = \int_X A(x)\, \mathrm{d}((\mathbf{m}^+)^\vee - (\mathbf{m}^-)^\vee). \tag{61}$$

(iii) $(\mathbf{m}^+)^\vee$ and $(\mathbf{m}^-)^\vee$ are the upper and the lower variation, respectively, of $\check{\mathbf{m}}$, the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$, and

$$\check{\mathbf{m}} = (\mathbf{m}^+)^\vee - (\mathbf{m}^-)^\vee. \tag{62}$$

(iv) If (P, Q) is a Hahn decomposition of X with respect to the σ-additive measure $\check{\mathbf{m}}$ on $\mathcal{T}^\vee$, then for all $A \in \mathcal{T}$

$$\mathbf{m}(Q) = -\mathbf{m}^-(Q) \leq \mathbf{m}(A) \leq \mathbf{m}^+(P) = \mathbf{m}(P). \tag{63}$$

10 Absolute continuity of $T_{\mathbf{L}}$-measures

$T_{\mathbf{L}}$-measures on not necessarily generated $T_{\mathbf{L}}$-tribes will play an essential role in the following exposition. Some of the important properties of such $T_{\mathbf{L}}$-measures are the absolute continuity and the singularity of $T_{\mathbf{L}}$-measures. These will be studied in this section.

Definition 10.1 Let $\mathbf{m}$ and $\mathbf{n}$ be $T_{\mathbf{L}}$-measures on a $T_{\mathbf{L}}$-tribe $\mathcal{T}$. The measure $\mathbf{n}$ is called $\mathbf{m}$-*absolutely continuous* (briefly $\mathbf{n} \ll \mathbf{m}$) if for each $A \in \mathcal{T}$

(AC) $\quad \mathbf{n}(A) = 0 \quad$ whenever $\quad \mathbf{m}^*(A) = 0.$

Absolute continuity is a reflexive and transitive relation. Moreover, if $\mathbf{n}_1 \ll \mathbf{m}$ and $\mathbf{n}_2 \ll \mathbf{m}$, then also $\mathbf{n}_1 \pm \mathbf{n}_2 \ll \mathbf{m}$. There are several characterizations of the absolute continuity which are rather easy to prove (see Lemma 11.2 in Butnariu & Klement [18]).

Lemma 10.2 *Let $\mathbf{m}$ and $\mathbf{n}$ be $T_{\mathbf{L}}$-measures on a $T_{\mathbf{L}}$-tribe $\mathcal{T}$. The following are equivalent:*

(i) $\mathbf{n} \ll \mathbf{m}$.

(ii) For all $A \in \mathcal{T}$: $\mathbf{n}(A) = 0$ whenever $\mathbf{m}^*(A) = 0$.

(iii) For all $A \in \mathcal{T}$: $\mathbf{n}^+(A) = 0 = \mathbf{n}^-(A)$ whenever $\mathbf{m}^*(A) = 0$.

(iv) For all $A \in \mathcal{T}$: $\mathbf{n}^*(A) = 0$ whenever $\mathbf{m}^*(A) = 0$.

(v) $\mathbf{n}^* \ll \mathbf{m}^*$.

(vi) For each $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that for each $A \in \mathcal{T}$

$$\mathbf{n}^*(A) < \varepsilon \quad \text{whenever} \quad \mathbf{m}^*(A) < \delta_\varepsilon.$$

(vii) For each $\varepsilon > 0$, there exists an $\eta_\varepsilon > 0$ such that for each $A \in \mathcal{T}$

$$|\mathbf{n}(A)| < \varepsilon \quad \text{whenever} \quad \mathbf{m}^*(A) < \eta_\varepsilon.$$

(viii) $\check{\mathbf{n}} \ll \check{\mathbf{m}}$.

(ix) $(\check{\mathbf{n}})^* \ll (\check{\mathbf{m}})^*$.

A property which is dual to absolute continuity is that of singularity. Some characterizations of the singularity which are dual to the characterizations of the absolute continuity and which are listed below (see Lemma 11.4 in Butnariu & Klement [18]).

Definition 10.3 Let $\mathbf{m}$ and $\mathbf{n}$ be $T_{\mathbf{L}}$-measures on a $T_{\mathbf{L}}$-tribe $\mathcal{T}$. The $T_{\mathbf{L}}$-measure $\mathbf{n}$ is said to be $\mathbf{m}$-*singular* (briefly $\mathbf{n} \perp \mathbf{m}$) if there exists a fuzzy set $A \in \mathcal{T}$ such that

$$\mathbf{n}^*(A) = \mathbf{m}^*(\complement A) = 0.$$

Lemma 10.4 *Let* $\mathbf{m}$ *and* $\mathbf{n}$ *be* $T_{\mathbf{L}}$*-measures on a* $T_{\mathbf{L}}$*-tribe* $\mathcal{T}$*. The following are equivalent:*

(i) $\mathbf{n} \perp \mathbf{m}$.

(ii) $\mathbf{m} \perp \mathbf{n}$.

(iii) There exists a fuzzy subset A *of* X *such that* $A \in \mathcal{T}$ *and* $\mathbf{m}^*(A) = \mathbf{n}^*(\complement A) = 0$.

(iv) $\mathbf{n}^+ \perp \mathbf{m}$ *and* $\mathbf{n}^- \perp \mathbf{m}$.

(v) $\mathbf{n}^* \perp \mathbf{m}^*$.

(vi) $\check{\mathbf{n}} \perp \check{\mathbf{m}}$.

(vii) $(\check{\mathbf{n}})^* \perp (\check{\mathbf{m}})^*$.

The following properties of the absolute continuity and the singularity of finite $T_{\mathbf{L}}$-measures will be used later.

Proposition 10.5 *(i) If* $\mathbf{n}_1, \mathbf{n}_2$ *and* $\mathbf{m}$ *are finite* $T_{\mathbf{L}}$*-measures on a* $T_{\mathbf{L}}$*-tribe* $\mathcal{T}$ *such that* $\mathbf{n}_1 \perp \mathbf{m}$ *and* $\mathbf{n}_2 \perp \mathbf{m}$*, then* $\mathbf{n}_1 \pm \mathbf{n}_2 \perp \mathbf{m}$.

(ii) If $\mathbf{m}$ *and* $\mathbf{n}$ *are finite* $T_{\mathbf{L}}$*-measures on a* $T_{\mathbf{L}}$*-tribe* $\mathcal{T}$ *such that* $\mathbf{n} \ll \mathbf{m}$ *and* $\mathbf{n} \perp \mathbf{m}$*, then* $\mathbf{n}$ *equals the zero measure.*

Proof: The proof of (i) is trivial. In order to show (ii), let $\mathbf{m}$ and $\mathbf{n}$ be $T_{\mathbf{L}}$-measures on $\mathcal{T}$ such that $\mathbf{n} \ll \mathbf{m}$ and $\mathbf{n} \perp \mathbf{m}$. Since $\mathbf{n} \ll \mathbf{m}$, it follows that for each $k \in \mathbb{N}$ there is a $\delta_{\frac{1}{k}} > 0$ such that $\mathbf{n}^*(F) < \frac{1}{k}$ whenever $F \in \mathcal{T}$ and $\mathbf{m}^*(F) < \delta_{\frac{1}{k}}$. Since $\mathbf{n} \perp \mathbf{m}$, there is a set $A \in \mathcal{T}$ such that $\mathbf{n}^*(\complement A) = \mathbf{m}^*(A) = 0$. Clearly, $\mathbf{n}^*(A) = 0$ since $\mathbf{m}^*(A) < \delta_{\frac{1}{k}}$ for all $k \in \mathbb{N}$. If $E \in \mathcal{T}$, then

$$\mathbf{n}^*(E) = \mathbf{n}^*(E \mathsf{T}_{\mathsf{L}} A) + \mathbf{n}^*(E \mathsf{T}_{\mathsf{L}} \complement A) = 0$$

because of $E \mathsf{T}_{\mathsf{L}} A \in \mathcal{T}(A), E \mathsf{T}_{\mathsf{L}} \complement A \in \mathcal{T}(\complement A)$ and the monotonicity of $\mathbf{n}^*$. □

11 Vector $T_{\mathbf{L}}$-measures with Darboux property

In Section 9 we proved that finite $T_{\mathbf{L}}$-measures on $T_{\mathbf{L}}$-tribes have bounded range. The question is now whether or under which conditions the range of a finite $T_{\mathbf{L}}$-measure on a $T_{\mathbf{L}}$-tribe (generated or not) is exactly an interval. This problem is studied here in a more general context, i.e., we ask whether or under which conditions a finite vector valued $T_{\mathbf{L}}$-measure on a $T_{\mathbf{L}}$-tribe has a convex and compact range. For measures defined on a σ-algebra of crisp subsets of X (which are particular $T_{\mathbf{L}}$-measures) this problem was approached in Poprugenko [68], and it was solved in Liapounoff [53]. In Halmos [34,35], Blackwell [10], Chernoff [20], and Lindenstrauss [54], Liapounoff's result was reproved and/or extended. In view of Corollary 9.4, in Dvoretzki, Wald & Wolfowitz [30] it was shown that the range of a vector $T_{\mathbf{L}}$-measure on a generated $T_{\mathbf{L}}$-tribe is necessarily convex and compact. We intend to generalize this result by showing that nonatomic $T_{\mathbf{L}}$-measures on $T_{\mathbf{L}}$-tribes (generated or not) have convex and compact range. To this end, we first study the class of $T_{\mathbf{L}}$-measures with Darboux property.

In this section $\mathcal{T}$ denotes a $T_{\mathbf{L}}$-tribe on X. In what follows, all $T_{\mathbf{L}}$-measures are assumed to be defined on $\mathcal{T}$. An *n-vector $T_{\mathbf{L}}$-measure on* $\mathcal{T}$ is a function $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ from $\mathcal{T}$ to $\mathbb{R}^n$ such that each $\mathbf{m}_i$ is a finite $T_{\mathbf{L}}$-measure. If $\mathbf{m}$ is an n-vector $T_{\mathbf{L}}$-measure, we denote the *range* $\mathrm{Ran}(\mathbf{m})$ of $\mathbf{m}$ by

$$\mathrm{Ran}(\mathbf{m}) = \{\mathbf{m}(W) \mid W \in \mathcal{T}\}.$$

Clearly, $0 \in \mathrm{Ran}(\mathbf{m})$ and $\mathrm{Ran}(\mathbf{m})$ is bounded in $\mathbb{R}^n$ (compare Corollary 9.3).

Definition 11.1 An n-vector $T_{\mathbf{L}}$-measure $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ has the *Darboux property* if for each pair (E, F) of fuzzy sets in $\mathcal{T}$ with $E \leq F$ there exists a family $(W_r)_{r \in [0,1]}$ in $\mathcal{T}$ such that for all $r \in [0, 1]$

$$E \leq W_r \leq F, \tag{64}$$

$$\mathbf{m}(W_r) = (1 - r) \cdot \mathbf{m}(E) + r \cdot \mathbf{m}(F). \tag{65}$$

This is just an extension of the notion introduced in Poprugenko [68]. A typical example of a measure having the Darboux property is the Lebesgue measure on the Borel subsets of $[0, 1]$. By Corollary 9.4, finite $T_{\mathbf{L}}$-measures on generated tribes have the Darboux property.

Proposition 11.2 *Let $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ be an n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$. Then $\mathbf{m}$ has the Darboux property if and only if $\mathbf{m}$ is semiconvex in the sense that for each $F \in \mathcal{T}$ there exists a fuzzy set $W \in \mathcal{T}(F)$ such that*

$$\mathbf{m}(W) = \frac{1}{2} \cdot \mathbf{m}(F).$$

Proof: Assume that $\mathbf{m}$ has the Darboux property. Then for each $F \in \mathcal{T}$ and for $E = \emptyset$ there exists a family $(W_r)_{r \in [0,1]} \subseteq \mathcal{T}$ such that (64) and (65) hold. In particular, $W = W_{\frac{1}{2}} \in \mathcal{T}(F)$ and $\mathbf{m}(W) = \frac{1}{2} \cdot \mathbf{m}(F)$, and the semiconvexity of $\mathbf{m}$ follows. Conversely, let $\mathbf{m}$ be semiconvex, consider $E, F \in \mathcal{T}$ such that $E \leq F$ and denote $W_0 = V_0 = E$ and $W_1 = F$. Then there exists a fuzzy set $U_1 \in \mathcal{T}(W_1 \ominus V_0)$ such that $\mathbf{m}(U_1) = \frac{1}{2} \cdot \mathbf{m}(W_1 \ominus V_0)$. Denoting $V_1 = V_0 \, \mathsf{S_L} \, U_1$ and using the $T_{\mathbf{L}}$-disjointness of V_0 and U_1, we obtain

$$\mathbf{m}(V_1) = \mathbf{m}(V_0) + \mathbf{m}(U_1) = \frac{1}{2} \cdot (\mathbf{m}(W_1) + \mathbf{m}(W_0)).$$

By the semiconvexity again, there is a fuzzy set $U_2 \in \mathcal{T}(W_1 \ominus V_1)$ such that $\mathbf{m}(U_2) = \frac{1}{2} \cdot \mathbf{m}(W_1 \ominus V_1)$. Denote $V_2 = V_1 \, \mathsf{S_L} \, U_2$, then we get $\mathbf{m}(V_2) = \mathbf{m}(V_1) + \mathbf{m}(U_2) = \frac{3}{4} \cdot \mathbf{m}(W_1) + \frac{1}{4} \cdot \mathbf{m}(W_0)$. Continuing in this way we can generate sequences $(U_n)_{n \in \mathbb{N}}$ and $(V_n)_{n \in \mathbb{N}}$ in $\mathcal{T}$ such that $(U_n)_{n \in \mathbb{N}}$ is $T_{\mathbf{L}}$-disjoint, $(V_n)_{n \in \mathbb{N}}$ is nondecreasing, and for each $n \in \mathbb{N}$ we have

$$\mathbf{m}(V_n) = \frac{1}{2^n} \cdot \mathbf{m}(W_0) + \left(1 - \frac{1}{2^n}\right) \cdot \mathbf{m}(W_1),$$

$$\mathbf{m}(U_n) = \frac{1}{2} \cdot \mathbf{m}(W_1 \ominus V_{n-1}) = \frac{1}{2} \cdot (\mathbf{m}(W_1) - \mathbf{m}(V_{n-1})).$$

Choose an arbitrary $r \in]0, 1[$ and let

$$r = \sum_{i=1}^{\infty} \frac{\rho_i}{2^i}$$

be its (unique) nonterminal binary representation (i.e., the binary representation with $\rho_i \in \{0,1\}$ and $\rho_i \neq 0$ for infinitely many $i \in \mathbb{N}$). For each $k \in \mathbb{N}$ define

$$r_k = \sum_{i=1}^{k} \frac{\rho_i}{2^i},$$

$$Z_{r_k} = W_0 \mathsf{S}_{\mathsf{L}} (\mathop{\mathbf{S_L}}_{i=1}^{k} \rho_i \cdot U_i).$$

Clearly, $Z_{r_k} \in \mathcal{T}(W_1)$ and

$$\mathbf{m}(Z_{r_k}) = \mathbf{m}(W_0) + \sum_{i=1}^{k} \rho_i \cdot \mathbf{m}(U_i) = r_k \cdot \mathbf{m}(W_1) + (1 - r_k) \cdot \mathbf{m}(W_0).$$

The sequence $\left(Z_{r_k}\right)_{k\in\mathbb{N}}$ is nondecreasing. Hence, the limit $W_r = \lim_{k\to\infty} Z_{r_k}$ exists, belongs to $\mathcal{T}(W_1)$ and satisfies

$$\mathbf{m}(W_r) = \lim_{k\to\infty} \mathbf{m}(Z_{r_k}) = r \cdot \mathbf{m}(W_1) + (1 - r) \cdot \mathbf{m}(W_0).$$

Obviously for each $r \in [0,1]$ we have $W_0 \leq W_r \leq W_1$, and thus the proposition is proved. □

Proposition 11.3 *An n-vector $T_{\mathbf{L}}$-measure $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ on $\mathcal{T}$ has the Darboux property if and only if for each pair (E, F) of fuzzy sets in $\mathcal{T}$ with $E \leq F$ there exists a family of fuzzy sets $(W_r)_{r\in[0,1]} \subseteq \mathcal{T}$ such that (64), (65) and*

$$W_r \leq W_s \quad \textit{whenever} \quad 0 \leq r < s \leq 1 \tag{66}$$

are satisfied.

Proof: Sufficiency trivially holds. Assume that $\mathbf{m}$ has the Darboux property and start with $W_0 = E$ and $W_1 = F$. Then there exists $Z_{\frac{1}{2}} \in \mathcal{T}(W_1 \ominus W_0)$ such that $\mathbf{m}(Z_{\frac{1}{2}}) = \frac{1}{2} \cdot \mathbf{m}(W_1 \ominus W_0)$ (compare Proposition 11.2). For $W_{\frac{1}{2}} = W_0 \mathsf{S}_{\mathsf{L}} Z_{\frac{1}{2}}$ we have

$$\mathbf{m}(W_{\frac{1}{2}}) = \frac{1}{2} \cdot \mathbf{m}(W_1) + \frac{1}{2} \cdot \mathbf{m}(W_0).$$

Using again the semiconvexity of $\mathbf{m}$, we obtain a fuzzy set $Z_{\frac{1}{4}} \in \mathcal{T}(W_{\frac{1}{2}} \ominus W_0)$ and a fuzzy set $Z_{\frac{3}{4}} \in \mathcal{T}(W_1 \ominus W_{\frac{1}{2}})$ such that

$$\mathbf{m}(Z_{\frac{1}{4}}) = \frac{1}{2} \cdot \mathbf{m}(W_{\frac{1}{2}} \ominus W),$$
$$\mathbf{m}(W_{\frac{3}{4}}) = \frac{1}{2} \cdot \mathbf{m}(W_1 \ominus W_{\frac{1}{2}}).$$

Denote $W_{\frac{1}{4}} = W_0 \mathsf{S}_{\mathsf{L}} Z_{\frac{1}{4}}$ and $W_{\frac{3}{4}} = W_{\frac{1}{2}} \mathsf{S}_{\mathsf{L}} Z_{\frac{3}{4}}$. It is clear that $W_0 \leq W_{\frac{1}{4}} \leq W_{\frac{1}{2}} \leq W_{\frac{3}{4}} \leq W_1$ and

$$\mathbf{m}(W_{\frac{1}{4}}) = \frac{1}{4} \cdot \mathbf{m}(W_1) + \frac{3}{4} \cdot \mathbf{m}(W_0),$$
$$\mathbf{m}(W_{\frac{3}{4}}) = \frac{3}{4} \cdot \mathbf{m}(W_1) + \frac{1}{4} \cdot \mathbf{m}(W_0).$$

In this way we obtain for each $n \in \mathbb{N}$ a finite family $\left(W_{\frac{k}{2^n}}\right)_{k\in\{1,2,\ldots,2^n-1\}}$ of fuzzy sets in $\mathcal{T}$ such that

$$W_0 \leq W_{\frac{1}{2^n}} \leq W_{\frac{2}{2^n}} \leq \cdots \leq W_{\frac{2^n-1}{2^n}} \leq W_1$$

and

$$\mathbf{m}(W_{\frac{k}{2^n}}) = \frac{k}{2^n} \cdot \mathbf{m}(W_1) + \Big(1 - \frac{k}{2^n}\Big) \cdot \mathbf{m}(W_0)$$

for all $k \in \{0, 1, \ldots, 2^n\}$. Each $r \in]0,1[$ can be written as $r = \lim_{n\to\infty} r_k$ with r_k as above. The fuzzy set $W_r = \lim_{k\to\infty} W_{r_k}$ is contained in $\mathcal{T}$ and satisfies (65). A simple computation shows that the family $(W_r)_{r\in[0,1]}$ also fulfills (64) and (66). □

Corollary 11.4 *An n-vector $T_{\mathbf{L}}$-measure $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$ on $\mathcal{T}$ has the Darboux property if and only if for each $E \in \mathcal{T}$ there exists a family of fuzzy sets $(W_r)_{r\in[0,1]}$ in $\mathcal{T}$ such that the condition (66) holds and for each $r \in [0,1]$*

$$\mathbf{m}(W_r) = r \cdot \mathbf{m}(E) \tag{67}$$

Remark 11.5 Halmos [35] called n-vector measures $\mathbf{m}$ (defined on a σ-algebra $\mathcal{A}$ of crisp subsets of X) *convex measures* if the following property holds:

For each $E \in \mathcal{A}$ the set $\mathcal{K}(\mathbf{m}, E)$ of all $(\mathcal{A} \cap E)$-measurable functions $g\colon E \longrightarrow [0,1[$, such that $\mathbf{m}(\{x \in E \mid g(x) < r\}) = r \cdot \mathbf{m}(E)$ for all $r \in [0,1]$, is convex. Here $\mathcal{A} \cap E$ denotes the restriction of the σ-algebra $\mathcal{A}$ to E, i.e.,

$$\mathcal{A} \cap E = \{A \cap E \mid A \in \mathcal{A}\}.$$

Combining Lemma 4 from [35] and Corollary 11.4, it follows that each convex measure has the Darboux property. Since the convexity of an n-vector measure $\mathbf{m}$ implies its semiconvexity (compare [35, Lemma 2]), an n-vector measure $\mathbf{m}$ (defined on a σ-algebra of crisp subsets of X) is convex if and only if it has the Darboux property.

Now we are able to state the main result of this section, namely, a convexity theorem of Liapounoff type for vector $T_{\mathbf{L}}$-measures with Darboux property:

Theorem 11.6 *If $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$ is an n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$ with Darboux property, then its range* $\mathrm{Ran}(\mathbf{m})$ *is convex.*

Proof: It suffices to prove that for each pair (E, F) of fuzzy sets in $\mathcal{T}$ and for all $r \in [0,1]$ there exists a fuzzy set $G \in \mathcal{T}$ such that

$$\mathbf{m}(G) = r \cdot \mathbf{m}(E) + (1-r) \cdot \mathbf{m}(F). \tag{68}$$

First assume that $E \mathbin{\mathsf{T}_{\mathsf{L}}} F = \emptyset$. Then there exist $E_r \in \mathcal{T}(E)$ and $F_{1-r} \in \mathcal{T}(F)$ with $\mathbf{m}(E_r) = r \cdot \mathbf{m}(E)$ and $\mathbf{m}(F_{1-r}) = (1-r)\cdot\mathbf{m}(F)$ (compare Corollary 11.4). Denote $G = E_r \mathbin{\mathsf{S}_{\mathsf{L}}} F_{1-r}$. Since $E_r \mathbin{\mathsf{T}_{\mathsf{L}}} F_{1-r} = \emptyset$, we have proven (68) in this case.

Consider the case that E and F are not $T_{\mathbf{L}}$-disjoint. Then, however, $A = E \ominus (E \mathbin{\mathsf{T}_{\mathsf{L}}} F), B = F \ominus (E \mathbin{\mathsf{T}_{\mathsf{L}}} F)$ and $C = E \mathbin{\mathsf{T}_{\mathsf{L}}} F$ are $T_{\mathbf{L}}$-disjoint. Hence, there exist $A_r \in \mathcal{T}(A)$ and $B_{1-r} \in \mathcal{T}(B)$ such that $A_r \mathbin{\mathsf{T}_{\mathsf{L}}} B_{1-r} = \emptyset$ and

$$\mathbf{m}(A_r \mathbin{\mathsf{S}_{\mathsf{L}}} B_{1-r}) = r \cdot \mathbf{m}(A) + (1-r) \cdot \mathbf{m}(B).$$

Denote $G = (A_r \mathbin{\mathsf{S}_{\mathsf{L}}} B_{1-r}) \mathbin{\mathsf{S}_{\mathsf{L}}} C$. Then

$$\mathbf{m}(G) = r \cdot \mathbf{m}(A) + (1-r) \cdot \mathbf{m}(B) + \mathbf{m}(C) \tag{69}$$

since A_r, B_{1-r} and C are $T_{\mathbf{L}}$-disjoint. From $\mathbf{m}(A) = \mathbf{m}(E) - \mathbf{m}(C)$ and $\mathbf{m}(B) = \mathbf{m}(F) - \mathbf{m}(C)$ it follows that (68) holds in this case too (compare (69)). □

In the special case of the finite $T_{\mathbf{L}}$-measures Theorem 11.6 can be restated as follows:

Corollary 11.7 *If* $\mathbf{m}$ *is a finite* $T_{\mathbf{L}}$*-measure on* $\mathcal{T}$ *having the Darboux property and if* (P, Q) *is a Hahn decomposition of* X *with respect to the* σ*-additive measure* $\check{\mathbf{m}}$ *on* $\mathcal{T}^{\vee}$*, then for each* $E \in \mathcal{T}$

$$\{\mathbf{m}(W) \mid W \in \mathcal{T}(E)\} = \big[\mathbf{m}(Q \cdot E), \mathbf{m}(P \cdot E)\big].$$

In particular, we have $\mathrm{Ran}(\mathbf{m}) = \big[\mathbf{m}(Q), \mathbf{m}(P)\big]$.

Proof: For $E \in \mathcal{T}$ and $W \in \mathcal{T}(E)$ we get

$$\mathbf{m}(E \cdot Q) = -\mathbf{m}^{-}(E) \leq -\mathbf{m}^{-}(W) \leq \mathbf{m}(W) \leq \mathbf{m}^{+}(W) \leq \mathbf{m}^{+}(E) = \mathbf{m}(P \cdot E).$$

If $r \in \big[\mathbf{m}(E \cdot Q), \mathbf{m}(E \cdot P)\big]$ then there is some $s \in [0,1]$ such that

$$r = \begin{cases} s \cdot \mathbf{m}(E \cdot P) & \text{if } r \geq 0, \\ s \cdot \mathbf{m}(E \cdot Q) & \text{if } r < 0. \end{cases}$$

Hence, in both cases there exists a fuzzy set $W \in \mathcal{T}(E)$ such that $\mathbf{m}(W) = r$ (compare Corollary 11.4). □

Remark 11.8 Taking into account Corollary 11.7, a finite $T_{\mathbf{L}}$-measure $\mathbf{m}$ on $\mathcal{T}$ has the Darboux property if and only if for each $E \in \mathcal{T}$ the set $\{\mathbf{m}(W) \mid W \in \mathcal{T}(E)\}$ is convex, i.e., if and only if this set is an interval. Generally speaking, if $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$ is an n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$ with $n \geq 2$ and if for each $E \in \mathcal{T}$ the set

$$\mathrm{Ran}_E(\mathbf{m}) = \{\mathbf{m}(W) \mid W \in \mathcal{T}(E)\}$$

is convex, then $\mathbf{m}$ has the Darboux property (compare Proposition 11.2). However, we do not know whether the converse also holds. Clearly, for $E = X$ the set $\mathrm{Ran}_E(\mathbf{m})$ is convex by virtue of Theorem 11.6. But the proof of Theorem 11.6 can not be applied for $E \neq X$ since, in general, we do not have $A \, \mathsf{S}_{\mathsf{L}} \, B \in \mathcal{T}(E)$ for each pair (A, B) in $\mathcal{T}(E)$. Such an implication, however, is substantially involved in the proof.

12 Nonatomic $T_{\mathbf{L}}$-measures

In the preceding section we have shown that the range $\mathrm{Ran}(\mathbf{m})$ of an n-vector $T_{\mathbf{L}}$-measure $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ with Darboux property is convex. Under which conditions for the components $\mathbf{m}_i$ of $\mathbf{m}$ can we deduce the Darboux property for $\mathbf{m}$? Note that if $\mathbf{m}$ has the Darboux property, each of its components $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ necessarily has the Darboux property. In this section we show that the converse is also true. This is one of the results of Butnariu [13,14], and it involves ideas contained in the proof of the Liapounoff theorem given in Halmos [35].

Let $\mathcal{T}$ be a $T_{\mathbf{L}}$-tribe. All $T_{\mathbf{L}}$-measures considered in this section are assumed to be defined on $\mathcal{T}$. If $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ is an n-vector $T_{\mathbf{L}}$-measure, then define the finite $T_{\mathbf{L}}$-measure $\mathbf{m}^t \colon \mathcal{T} \longrightarrow \mathbb{R}$ by

$$\mathbf{m}^t = \mathbf{m}_1^* + \mathbf{m}_2^* + \cdots + \mathbf{m}_n^*.$$

Definition 12.1 Let $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ be an n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$.

(i) A $T_{\mathbf{L}}$-measure $\mathbf{n}$ on $\mathcal{T}$ is called $\mathbf{m}$-*absolutely continuous* if it is $\mathbf{m}^t$-absolutely continuous, i.e., if $\mathbf{n} \ll \mathbf{m}^t$.

(ii) A $T_{\mathbf{L}}$-measure $\mathbf{n}$ on $\mathcal{T}$ is called *atomic* if there exists a fuzzy set $A \in \mathcal{T}$ with $\mathbf{n}(A) \neq \emptyset$ such that for each $B \in \mathcal{T}(A)$ we have either $\mathbf{n}(B) = 0$ or $\mathbf{n}(B) = \mathbf{n}(A)$.

(iii) A $T_{\mathbf{L}}$-measure $\mathbf{n}$ on $\mathcal{T}$ is called *nonatomic* if it is not atomic, i.e., if for each $A \in \mathcal{T}$ with $\mathbf{n}(A) \neq \emptyset$ there exists a $B \in \mathcal{T}(A)$ such that $\mathbf{n}(B) \notin \{0, \mathbf{n}(A)\}$.

(iv) The n-vector $T_{\mathbf{L}}$-measure $\mathbf{m}$ is called *nonatomic* if each of its components $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ is nonatomic.

Theorem 12.2 *Let $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ be an n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$. The following are equivalent:*

(i) $\mathbf{m}$ has the Darboux property.

(ii) For each $i \in \{1, 2, \ldots, n\}$ the component $\mathbf{m}_i$ has the Darboux property.

(iii) $\mathbf{m}$ is nonatomic.

Proof: The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial. It remains to show that (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). The proof of these implications will be done in the following sequence of lemmas. $\square$

Lemma 12.3 *Assume that $E, F \in \mathcal{T}$ with $E \,\mathsf{T}_{\mathsf{L}}\, F = \emptyset$, and let*

$$\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$$

be a nonnegative n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$ with Darboux property and $\mathbf{n}$ a finite $T_{\mathbf{L}}$-measure on $\mathcal{T}$ such that $\mathbf{n} \ll \mathbf{w}$. Then there exists a family of fuzzy sets $(H_r)_{r\in[0,1]}$ in $\mathcal{T}$ such that for each $r \in [0,1]$

$$\mathbf{w}(H_r) = r \cdot \mathbf{w}(E) + (1-r) \cdot \mathbf{w}(F), \tag{70}$$

$$H_0 = F, \qquad H_1 = E. \tag{71}$$

Moreover, for the function $f_{\mathbf{n}} \colon [0,1] \longrightarrow \mathbb{R}$ given by $f_{\mathbf{n}}(r) = \mathbf{n}(H_r)$ we have

$$f_{\mathbf{n}} \text{ is continuous.} \tag{72}$$

Proof: If $\mathbf{w}(E) = 0$ or $\mathbf{w}(F) = 0$, then such a family $(H_r)_{r\in[0,1]}$ exists by Corollary 11.4. Assume that $\mathbf{w}(E) \neq 0$ and $\mathbf{w}(F) \neq 0$. Since $\mathbf{w}$ has the Darboux property, by Proposition 11.3 there are two families $(W_r)_{r\in[0,1]}$ and $(Z_r)_{r\in[0,1]}$ in $\mathcal{T}$ such that for $r, s \in [0,1]$ with $r < s$ we have

$$W_0 = Z_0 = \emptyset, \qquad W_1 = E, \qquad Z_1 = F, \tag{73}$$

$$W_r \leq W_s, \qquad Z_r \leq Z_s, \tag{74}$$

$$\mathbf{w}(W_r) = r \cdot \mathbf{w}(E), \qquad \mathbf{w}(Z_r) = r \cdot \mathbf{w}(F). \tag{75}$$

Since $E \,\mathsf{T}_{\mathsf{L}}\, F = \emptyset$ it follows that $W_r \,\mathsf{T}_{\mathsf{L}}\, T_{1-r} = \emptyset$ for all $r \in [0,1]$. Denote $H_r = W_r \,\mathsf{S}_{\mathsf{L}}\, Z_{1-r}$. Then (70) and (71) are clearly satisfied. Choose $s, r \in [0,1]$ with $s > r$. Then

$$\begin{aligned} |f_{\mathbf{n}}(r) - f_{\mathbf{n}}(s)| &= |\mathbf{n}(Z_{1-r} \ominus Z_{1-s}) - \mathbf{n}(W_s \ominus W_r)| \\ &\leq |\mathbf{n}(W_s \ominus W_r)| + |\mathbf{n}(Z_{1-r} \ominus Z_{1-s})|. \end{aligned} \tag{76}$$

For a given $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that for each $A \in \mathcal{T}$ we have

$$|\mathbf{n}(A)| < \frac{\varepsilon}{2} \quad \text{whenever} \quad \mathbf{w}^t(A) < \delta_\varepsilon \tag{77}$$

because of $\mathbf{n} \ll \mathbf{w}^t$. Since $\mathbf{w}$ is nonnegative, we get

$$\mathbf{w}^t(W_s \ominus W_r) = \mathbf{w}^t(W_s) - \mathbf{w}^t(W_r) = (s-r) \cdot \mathbf{w}^t(E)$$

and

$$\mathbf{w}^t(Z_{1-r} \ominus Z_{1-s}) = \mathbf{w}^t(Z_{1-r}) - \mathbf{w}^t(Z_{1-s}) = (s-r) \cdot \mathbf{w}^t(F)$$

because of (74) and (75). Denote

$$\eta_\varepsilon = \min\Big(1, \frac{\delta_\varepsilon}{\mathbf{w}^t(E)}, \frac{\delta_\varepsilon}{\mathbf{w}^t(F)}\Big).$$

If $0 < s - r < \eta_\varepsilon$ then we deduce (compare (77)) $|\mathbf{n}(W_s \ominus W_r)| < \frac{\varepsilon}{2}$ and $|\mathbf{n}(Z_{1-r} \ominus Z_{1-s})| < \frac{\varepsilon}{2}$. Taking into account (76) it follows that $|f_\mathbf{n}(r) - f_\mathbf{n}(s)| < \varepsilon$ whenever $0 < s - r < \eta(\varepsilon)$. Hence, $f_\mathbf{n}$ is continuous. □

Lemma 12.4 *If $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$ is a nonnegative n-vector $T_\mathbf{L}$-measure on $\mathcal{T}$ which has the Darboux property, then for each finite $T_\mathbf{L}$-measure $\mathbf{n}$ with $\mathbf{n} \ll \mathbf{w}$ the $(n+1)$-vector $T_\mathbf{L}$-measure $\mathbf{w}' = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n, \mathbf{n})$ has the Darboux property.*

Proof: It is sufficient to show that $\mathbf{w}'$ is semiconvex (compare Proposition 11.2). Suppose that $A \in \mathcal{T}$ satisfies $\mathbf{w}(A) \neq 0$. Then there exists an $E \in \mathcal{T}(A)$ with $\mathbf{w}(E) = \frac{1}{2} \cdot \mathbf{w}(A)$. If $\mathbf{n}(E) = \frac{1}{2} \cdot \mathbf{n}(A)$, we are done. Assume that $\mathbf{n}(E) \neq \frac{1}{2} \cdot \mathbf{n}(A)$ and denote $F = A \ominus E$. Clearly, $\frac{1}{2} \cdot \mathbf{n}(A)$ is contained in the interval with the boundaries $\mathbf{n}(E)$ and $\mathbf{n}(F)$. Since E and F satisfy Lemma 12.3, there exists a family $(H_r)_{r\in[0,1]}$ in $\mathcal{T}$ such that (70), (71) and (72) hold. Hence, there is an $r_0 \in [0,1]$ such that

$$\mathbf{n}(H_{r_0}) = f_\mathbf{n}(r_0) = \frac{1}{2} \cdot \mathbf{n}(A)$$

because of $f_\mathbf{n}(0) = \mathbf{n}(F)$, $f_\mathbf{n}(1) = \mathbf{n}(E)$ and the continuity of $f_\mathbf{n}$. For $H = H_{r_0}$ we deduce that $\mathbf{n}(H) = \frac{1}{2} \cdot \mathbf{n}(A)$ and

$$\mathbf{w}(H) = r_0 \cdot \mathbf{w}(E) + (1 - r_0) \cdot \mathbf{w}(F) = \frac{1}{2} \cdot \mathbf{w}(A),$$

i.e., $\mathbf{w}'(H) = \frac{1}{2} \cdot \mathbf{w}(A)$. □

Lemma 12.5 *If $\mathbf{m}$ and $\mathbf{n}$ are finite $T_\mathbf{L}$-measures on $\mathcal{T}$, if $\mathbf{n} \ll \mathbf{m}$ and if $\mathbf{m}$ is nonnegative and has the Darboux property, then $\mathbf{n}$ also has the Darboux property. Moreover, $\mathbf{m} \pm \mathbf{n}$ are finite $T_\mathbf{L}$-measures with Darboux property.*

Proof: Indeed, if in Lemma 12.4 we take $n = 1$ and $\mathbf{w}_1 = \mathbf{m}$, then the 2-vector measure $(\mathbf{m}, \mathbf{n})$ has the Darboux property, and the result follows. □

Lemma 12.6 *If $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ is a nonnegative n-vector $T_\mathbf{L}$-measure on $\mathcal{T}$ with $\mathbf{m}_i \ll \mathbf{m}_{i-1}$ for all $i \in \{2, 3, \ldots, n\}$ and if $\mathbf{m}_1$ has the Darboux property, then $\mathbf{m}$ itself has the Darboux property.*

Proof: For $n = 1$ the result is trivial. Assume that the result holds for each $k < n$. In order to show its validity for $k = n$ consider the $(n-1)$-vector $T_\mathbf{L}$-measure $\mathbf{w} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_{n-1})$ which has the Darboux property. Since

$$\mathbf{m}_n \ll \mathbf{m}_{n-1} \ll \mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_{n-1} = \mathbf{w}^t,$$

it follows that $\mathbf{m}_n \ll \mathbf{w}$. Hence, $(\mathbf{w}, \mathbf{m}_n) = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ has the Darboux property by Lemma 12.4. □

Lemma 12.7 *If $\mathbf{n}$ is a finite nonatomic nonnegative $T_\mathbf{L}$-measure on $\mathcal{T}$, then for each $A \in \mathcal{T}$ with $\mathbf{n}(A) \neq 0$ there exists a sequence $(A_k)_{k\in\mathbb{N}_0}$ in $\mathcal{T}$ such that for all $k \in \mathbb{N}$*

(i) $A = A_0 \geq A_1 \geq A_2 \geq \ldots \geq A_k$,

(ii) $0 < \mathbf{n}(A_k) \leq \frac{1}{2} \cdot \mathbf{n}(A_{k-1})$,

(iii) $\lim_{k\to\infty} A_k = \emptyset$.

Proof: Put $B_0 = A$ and assume that we have already obtained fuzzy sets $B_0 \geq B_1 \geq B_2 \geq \ldots \geq B_k$ in $\mathcal{T}$ such that $0 < \mathbf{n}(B_i) \leq \frac{1}{2^i} \cdot \mathbf{n}(A)$ and $\mathbf{n}(B_i) \leq \frac{1}{2} \cdot \mathbf{n}(B_{i-1})$ for $i \in \{1, 2, \ldots, k\}$. Then there exists $B \in \mathcal{T}(B_k)$ with $0 < \mathbf{n}(B) < \mathbf{n}(B_k)$ because of the nonatomicity of $\mathbf{n}$. Define B_{k+1} by

$$B_{k+1} = \begin{cases} B & \text{if } \mathbf{n}(B) \leq \frac{1}{2} \cdot \mathbf{n}(B_k), \\ B_k \ominus B & \text{otherwise.} \end{cases}$$

Clearly

$$0 < \mathbf{n}(B_{k+1}) \leq \frac{1}{2} \cdot \mathbf{n}(B_k)$$

and $B_{k+1} \leq B_k$. Moreover, we have

$$\mathbf{n}(B_{k+1}) \leq \frac{1}{2^{k+1}} \cdot \mathbf{n}(A).$$

The sequence $(B_k)_{k\in\mathbb{N}} \subseteq \mathcal{T}$ obtained in this way is nonincreasing and its limit $W = \lim_{k\to\infty} B_k$ belongs to $\mathcal{T}$ (compare Theorem 3.3(i)). Since $\mathbf{n}(B_k) \leq \frac{1}{2^k} \cdot \mathbf{n}(A)$ for all $k \in \mathbb{N}$, it follows that $\mathbf{n}(W) = 0$ because of the continuity of $\mathbf{n}$. If we define $A_k = B_k \ominus W$ for $k \in \mathbb{N}$ and $A_0 = A$, then the sequence $(A_k)_{k\in\mathbb{N}_0}$ satisfies the properties (i)–(iii). □

Lemma 12.8 *If* $\mathbf{n}$ *is a nonnegative nonatomic finite* $T_{\mathbf{L}}$*-measure on* $\mathcal{T}$*, then* $\mathbf{n}$ *has the Darboux property.*

Proof: We prove that $\mathbf{n}$ is semiconvex. For each $A \in \mathcal{T}$ with $\mathbf{n}(A) \neq 0$, there exists a sequence $(A_k)_{k\in\mathbb{N}_0} \subseteq \mathcal{T}$ such that (i)–(iii) in Lemma 12.7 are satisfied. Denote $W_k = A_{k-1} \ominus A_k$ for each $k \in \mathbb{N}$. From (i) it follows that $(W_k)_{k\in\mathbb{N}}$ is a $T_{\mathbf{L}}$-disjoint sequence in $\mathcal{T}(A)$. For $k \in \mathbb{N}$ and $r \in \{0, 1\}$ define the fuzzy set

$$W_{k,r} = \begin{cases} W_k & \text{if } r = 1, \\ \emptyset & \text{if } r = 0. \end{cases}$$

If $t \in [0, 1]$ and

$$t = \sum_{k=1}^{\infty} \frac{t_k}{2^k}$$

its unique nonterminal binary representation, denote

$$t * W = \mathop{\mathbf{S_L}}_{k=1}^{\infty} W_{k,t_k}.$$

Clearly, $t * W \in \mathcal{T}(A)$ and

$$\mathbf{n}(t * W) = \sum_{k=1}^{\infty} t_k \cdot \mathbf{n}(W_k).$$

We claim that the function $g\colon [0, 1] \longrightarrow \mathbb{R}$ defined by $g(t) = \mathbf{n}(t * W)$ is continuous. Indeed, if $0 \leq s < t \leq 1$, then there exists a smallest $j \in \mathbb{N}$ such that $t_j = 1$ and $s_j = 0$. Hence,

$$g(t) - g(s) = \mathbf{n}(W_j) + \sum_{k=j+1}^{\infty} (t_k - s_k) \cdot \mathbf{n}(W_k) \geq \mathbf{n}(W_j) - \sum_{k=j+1}^{\infty} \mathbf{n}(W_k).$$

Since the latter sum equals $\mathbf{n}(A_j)$, it follows that $g(t) - g(s) \geq \mathbf{n}(A_{j-1}) - 2 \cdot \mathbf{n}(A_j) \geq 0$ because of Lemma 12.7(ii). Now observe that

$$g(t) - g(s) \leq \sum_{k=1}^{\infty} (t_k - s_k) \cdot \mathbf{n}(A_{k-1}) \leq 2 \cdot (t - s) \cdot \mathbf{n}(A),$$

and the continuity of g follows. The function g satisfies $g(0) = 0$ and

$$g(1) = \mathbf{n}\Big(\mathop{\mathrm{S}_{\mathbf{L}}}_{k=1}^{\infty} W_k\Big) = \mathbf{n}(A).$$

Hence, there exists a $t^\circ \in [0,1]$ such that $g(t^\circ) = \frac{1}{2} \cdot g(1)$, i.e., $\mathbf{n}(B) = \frac{1}{2} \cdot \mathbf{n}(A)$ and $B \in \mathcal{T}(A)$ for $B = t^\circ * W$, and $\mathbf{n}$ is semiconvex. $\square$

Lemma 12.9 *If $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_n$ are nonnegative finite $T_{\mathbf{L}}$-measures on $\mathcal{T}$ having the Darboux property, then $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2 + \ldots + \mathbf{n}_n$ has the Darboux property.*

Proof: We prove that $\mathbf{n}$ is nonatomic. Assume that $A \in \mathcal{T}$ and $\mathbf{n}(A) \neq 0$. Then $\mathbf{n}_i(A) \neq 0$ for at least one index i. Since $\mathbf{n}_i$ has the Darboux property, there exists a fuzzy set $B \in \mathcal{T}(A)$ such that $\mathbf{n}_i(B) = \frac{1}{2} \cdot \mathbf{n}_i(A) < \mathbf{n}_i(A)$. Hence, $0 < \mathbf{n}(B) < \mathbf{n}(A)$ since all $\mathbf{n}_i$ are monotone (see also Proposition 5.1). $\square$

Lemma 12.10 *If $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2 \ldots, \mathbf{m}_n)$ is a nonnegative n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$ and if each of the $T_{\mathbf{L}}$-measures $\mathbf{m}_1$, $\mathbf{m}_2$,..., $\mathbf{m}_n$ has the Darboux property, then $\mathbf{m}$ itself has the Darboux property.*

Proof: For $i \in \{1, 2, \ldots, n\}$ denote $\mathbf{m}_i^0 = \sum_{k=i}^{n} \mathbf{m}_k$. Each $\mathbf{m}_i^0$ has the Darboux property by Lemma 12.9. Observe that $\mathbf{m}_i^0 \ll \mathbf{m}_{i-1}^0$ whenever $i \in \{2, 3, \ldots, n\}$. Thus $\mathbf{m}^0 = (\mathbf{m}_1^0, \mathbf{m}_2^0, \ldots, \mathbf{m}_n^0)$ has the Darboux property by Lemma 12.4. Let $f = (f_1, \ldots, f_n)$ be the linear operator $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$f_i(y_1, y_2, \ldots, y_n) = \sum_{k=i}^{n} y_k.$$

It is clear that f is one-to-one and onto (its associated determinant is different from zero). Therefore, f^{-1} exists and is linear. From $\mathbf{m}^0 = f \circ \mathbf{m}$ it follows that $\mathbf{m} = f^{-1} \circ \mathbf{m}^0$ and, subsequently, that $\mathbf{m}$ is semiconvex. $\square$

Now we are ready to show that, in Theorem 12.2, property (ii) implies indeed property (i).

Lemma 12.11 *If $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ is an n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$ and if each component $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ has the Darboux property, then $\mathbf{m}$ itself has the Darboux property.*

Proof: For each index $i \in \{1, 2, \ldots, n\}$ there exists a Hahn decomposition (P_i, Q_i) of X with respect to $\check{\mathbf{m}}_i$ (the restriction of $\mathbf{m}_i$ to $\mathcal{T}^{\vee}$). Denote $E_i^0 = P_i$ and $E_i^1 = Q_i$. For $a = (a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ denote $E(a) = \bigcap_{i=1}^{n} E_i^{a_i}$. Clearly, $E(a) \in \mathcal{T}$. Define the finite $T_{\mathbf{L}}$-measures $\mathbf{m}_1^a, \mathbf{m}_2^a, \ldots, \mathbf{m}_n^a$ by

$$\mathbf{m}_i^a(A) = (-1)^{a_i} \cdot \mathbf{m}_i(A \mathbin{\mathsf{T}_{\mathsf{L}}} E(a)).$$

Each $\mathbf{m}_i^a$ is nonnegative since

$$A \mathbin{\mathsf{T}_{\mathsf{L}}} E(a) \leq \begin{cases} Q_i & \text{if } a_i = 1, \\ P_i & \text{if } a_i = 0. \end{cases}$$

We can even show that each $\mathbf{m}_i^a$ has the Darboux property: if $A \in \mathcal{T}$ and $\mathbf{m}_i^a(A) \neq 0$, then there exists a $B \in \mathcal{T}(A)$ such that

$$\mathbf{m}_i^a(B) = (-1)^{a_i} \cdot \mathbf{m}_i(B \mathsf{T}_{\mathsf{L}} E(a)) = \frac{1}{2} \cdot (-1)^{a_i} \cdot \mathbf{m}_i(A \mathsf{T}_{\mathsf{L}} E(a)) = \frac{1}{2} \cdot \mathbf{m}_i^a(A)$$

since $\mathbf{m}_i$ is semiconvex and $E(a)$ is a subset of X. Hence, $\mathbf{m}^a = (\mathbf{m}_1^a, \mathbf{m}_2^a, \ldots, \mathbf{m}_n^a)$ is a nonnegative n-vector $T_{\mathbf{L}}$-measure with Darboux property (compare Lemma 12.10). Since $\{E(a) \mid a \in \{0,1\}^n\}$ is a partition of X, we have

$$\mathbf{m} = \sum_{a \in \{0,1\}^n} (-1)^a \cdot \mathbf{m}^a, \tag{78}$$

where the n-vector $T_{\mathbf{L}}$-measure $(-1)^a \cdot \mathbf{m}^a$ is given by

$$(-1)^a \cdot \mathbf{m}^a = ((-1)^{a_1} \cdot \mathbf{m}_1^a, (-1)^{a_2} \cdot \mathbf{m}_2^a, \ldots, (-1)^{a_n} \cdot \mathbf{m}_n^a).$$

By the Darboux property of each $\mathbf{m}^a$, for each $A \in \mathcal{T}$ and for each $a \in \{0,1\}^n$, there exists an $A(a) \in \mathcal{T}(A)$ such that $\mathbf{m}^a(A(a)) = \frac{1}{2} \cdot \mathbf{m}^a(A)$. Denote $B(a) = A(a) \mathsf{T}_{\mathsf{L}} E(a)$ and

$$B = \sum_{a \in \{0,1\}^n} B(a).$$

Then because of (78) and the $T_{\mathbf{L}}$-disjointness of the family $(B(a))_{a \in \{0,1\}^n}$ we have

$$\mathbf{m}(B) = \sum_{a \in \{0,1\}^n} \mathbf{m}(B(a)) = \sum_{a \in \{0,1\}^n} (-1)^a \cdot \mathbf{m}^a(A(a)) = \frac{1}{2} \cdot \sum_{a \in \{0,1\}^n} (-1)^a \cdot \mathbf{m}^a(A) = \frac{1}{2} \cdot \mathbf{m}(A).$$

This shows that $\mathbf{m}$ is semiconvex. □

The following lemma verifies implication (iii) ⇒ (ii) of Theorem 12.2, and this will complete the proof of this theorem.

Lemma 12.12 *If $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ is a nonatomic n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$ then each component $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ has the Darboux property.*

Proof: If $\mathbf{m}$ is nonatomic, each of its components $\mathbf{m}_i$ is finite and nonatomic. If each $\mathbf{m}_i$ is nonnegative, then the result follows from Lemma 12.8. Assume that $\mathbf{m}_i$ has also some negative values. If we can show that $\mathbf{m}_i^+$ and $\mathbf{m}_i^-$ are also nonatomic, then they have the Darboux property (compare Lemma 12.8) and, consequently, the 2-vector $T_{\mathbf{L}}$-measure $(\mathbf{m}_i^+, \mathbf{m}_i^-)$ (compare Lemma 12.10) and the $T_{\mathbf{L}}$-measure $\mathbf{m}_i = \mathbf{m}_i^+ - \mathbf{m}_i^-$ also have the Darboux property due to $\mathbf{m}_i = f(\mathbf{m}_i^+, \mathbf{m}_i^-)$ and to the fact that the linear function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(y_1, y_2) = y_1 - y_2$ preserves semiconvexity. If $A \in \mathcal{T}$ with $\mathbf{m}_i^+(A) \neq 0$, then

$$\mathbf{m}_i(P_i \mathsf{T}_{\mathsf{L}} A) = \mathbf{m}_i^+(A) \neq 0$$

for each Hahn decomposition (P_i, Q_i) of X with respect to $\check{\mathbf{m}}_i$ (compare Corollary 9.4). Since $\mathbf{m}_i$ is nonatomic, there exists a fuzzy set $B \in \mathcal{T}(P_i \mathsf{T}_{\mathsf{L}} A)$ such that

$$0 \neq \mathbf{m}_i(B) \neq \mathbf{m}_i(P_i \mathsf{T}_{\mathsf{L}} A) = \mathbf{m}_i^+(A).$$

As a consequence, $\mathbf{m}_i^+(B) = \mathbf{m}_i(P_i T_{\mathbf{L}} B) = \mathbf{m}_i(B) \notin \{0, \mathbf{m}_i^+(A)\}$, i.e., $\mathbf{m}_i^+$ is nonatomic. The nonatomicity of $\mathbf{m}_i^-$ can be seen analogously. □

Theorem 12.2 together with Corollary 9.3 allows Theorem 11.6 to be restated as follows:

Corollary 12.13 *If $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$ is a finite nonatomic n-vector $T_{\mathbf{L}}$-measure on $\mathcal{T}$, then its range* $\mathrm{Ran}(\mathbf{m})$ *is convex and bounded. In the case of $n = 1$, moreover,*

$$\mathrm{Ran}(\mathbf{m}) = \big[\mathbf{m}(Q), \mathbf{m}(P)\big],$$

(P, Q) being a Hahn decomposition of X with respect to the σ-additive measure $\check{\mathbf{m}}$.

13 A Liapounoff type theorem for $T_{\mathbf{L}}$-measures

In Section12 we have shown that the range $\mathrm{Ran}(\mathbf{m})$ of a nonatomic $T_{\mathbf{L}}$-measure $\mathbf{m}$ on a $T_{\mathbf{L}}$-tribe is convex and bounded (compare Corollary 12.13). In this section we show that $\mathrm{Ran}(\mathbf{m})$ is also compact. The idea of this proof is extracted from the proof of the Liapounoff theorem in Halmos [35]. Throughout this section $\mathcal{T}$ will denote a $T_{\mathbf{L}}$-tribe, and all $T_{\mathbf{L}}$-measures are assumed to be defined on $\mathcal{T}$.

Theorem 13.1 *If* $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$ *is a nonatomic* n*-vector* $T_{\mathbf{L}}$*-measure on* $\mathcal{T}$*, then its range* $\mathrm{Ran}(\mathbf{m})$ *is convex and compact.*

Proof: Convexity and boundedness of $\mathrm{Ran}(\mathbf{m})$ follow from Corollary 12.13. Without loss of generality, we may assume that the affine dimension of $\mathrm{Ran}(\mathbf{m})$ equals n. Otherwise, all the following arguments can be carried over from $\mathbb{R}^n$ to the linear hull of $\mathrm{Ran}(\mathbf{m})$. It remains to show that $\mathrm{Ran}(\mathbf{m})$ is closed. To this end, it is sufficient to show that

$$H \cap \mathrm{cl}(\mathrm{Ran}(\mathbf{m})) \subseteq \mathrm{Ran}(\mathbf{m}), \tag{79}$$

where cl stands for the topological closure, for each supporting hyperplane H of $\mathrm{Ran}(\mathbf{m})$ (compare Rockafellar [69, Theorem 11.6.2 and Corollary 11.7.1]). Let H be such a supporting hyperplane. Then there exists an $r \in \mathbb{R}$ and a linear function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $H = \{x \in \mathbb{R}^n \mid f(x) = r\}$ and $f(\mathbf{m}(E)) \geq r$ for all $E \in \mathcal{T}$. The proof is carried out in several steps.

Claim 1. For each crisp set $A \in \mathcal{T}^{\vee}$ the function $\mathbf{m}_A\colon \mathcal{T} \longrightarrow \mathbb{R}$ defined by

$$\mathbf{m}_A(E) = \mathbf{m}(\complement A \cdot E) - \mathbf{m}(A \cdot E) \tag{80}$$

is a nonatomic n-vector $T_{\mathbf{L}}$-measure and $\mathrm{Ran}(\mathbf{m}_A) = \mathrm{Ran}(\mathbf{m}) - \mathbf{m}(A)$.

Indeed, a simple computation shows that $\mathbf{m}_A$ is a n-vector $T_{\mathbf{L}}$-measure. For $E \in \mathcal{T}$ we have

$$\mathbf{m}_A(E) = \mathbf{m}((\complement A \cdot E)\, \mathsf{S}_{\mathsf{L}} (A \cdot \complement E)) - \mathbf{m}(A),$$

i.e., $\mathrm{Ran}(\mathbf{m}_A) \subseteq \mathrm{Ran}(\mathbf{m}) - \mathbf{m}(A)$. Conversely, since $A \in \mathcal{T}^{\vee}$, we get

$$\mathbf{m}(E) - \mathbf{m}(A) = \mathbf{m}_A((\complement A \cdot E)\, \mathsf{S}_{\mathsf{L}} (A \cdot \complement E)),$$

i.e., $\mathrm{Ran}(\mathbf{m}) - \mathbf{m}(A) \subseteq \mathrm{Ran}(\mathbf{m}_A)$.

To show the nonatomicity of $\mathbf{m}_A$, observe that for $E \in \mathcal{T}$ there exist $T_{\mathbf{L}}$-disjoint fuzzy sets $F_1 \in \mathcal{T}(\complement A\, \mathsf{T}_{\mathsf{L}}\, E)$ and $F_2 \in \mathcal{T}(A\, \mathsf{T}_{\mathsf{L}}\, E)$ such that

$$\begin{aligned} \mathbf{m}(F_1) &= \frac{1}{2} \cdot \mathbf{m}(\complement A\, \mathsf{T}_{\mathsf{L}}\, E), \\ \mathbf{m}(F_2) &= \frac{1}{2} \cdot \mathbf{m}(A\, \mathsf{T}_{\mathsf{L}}\, E), \end{aligned}$$

since $\mathbf{m}$ is nonatomic (see Theorem 12.2 and Propostition 11.2). Therefore,

$$\mathbf{m}_A(F_1\, \mathsf{S}_{\mathsf{L}}\, F_2) = \mathbf{m}_A(F_1) + \mathbf{m}_A(F_2) = \mathbf{m}(F_1) - \mathbf{m}(F_2) = \frac{1}{2} \cdot \mathbf{m}_A(E),$$

showing that $\mathbf{m}_A$ is nonatomic.

Claim 2. There exists a set $Q \in \mathcal{T}^{\vee}$ such that $\mathbf{m}(Q) \in H \cap \mathrm{Ran}(\mathbf{m})$.
Define the finite $T_{\mathbf{L}}$-measure $\mathbf{u}\colon \mathcal{T} \longrightarrow \mathbb{R}$ by $\mathbf{u} = f \circ \mathbf{m}$. Observe that

$$r = \inf\{\mathbf{u}(E) \mid E \in T\},$$

and let (P, Q) be a Hahn decomposition of X with respect to the σ-additive measure $\check{\mathbf{u}}$. By Corollary 9.4 (iv), we have $\mathbf{u}(Q) = r$, and this proves Claim 2.

Now we are in the position to show (79). Observe that $f \circ \mathbf{m}_Q(E) \geq 0$ for all $E \in \mathcal{T}$, and define $\mathbf{u}_Q = f \circ \mathbf{m}_Q$. Let L be the linear subspace $L = \{x \in \mathbb{R}^n \mid f(x) = 0\}$, i.e., $L = H - \mathbf{m}(Q)$. By Claim 1, (79) is equivalent to

$$L \cap \mathrm{cl}(\mathrm{Ran}(\mathbf{m}_Q)) \subseteq \mathrm{Ran}(\mathbf{m}_Q). \tag{81}$$

Denote $\mathbf{w} = \mathbf{m}_Q^t$ (see Definition 71). Clearly, each coordinate of $\mathbf{m}_Q$ is $\mathbf{w}$-absolute continuous. We prove (81) by induction upon the number n of components of the vector $T_{\mathbf{L}}$-measure $\mathbf{m}$. If $n = 1$ then (81) follows from Corollary 12.13. Assuming that (81) is true for all $k < n$, we want to prove it for $k = n$. Then, by Corollary 60(ii) and Halmos [37, p. 125], there exists a crisp set $B \in \mathcal{T}^\vee$ such that $\mathbf{u}_Q(B) = 0$ and the n-vector $T_{\mathbf{L}}$-measure $\mathbf{w}_B$ defined by $\mathbf{w}_B(E) = \mathbf{w}(B \cdot E)$ is $\mathbf{u}_Q$-absolutely continuous. Define the n-vector $T_{\mathbf{L}}$-measures $\mathbf{w}_1$ and $\mathbf{w}_2$ by $\mathbf{w}_1(E) = \mathbf{m}_Q(B \cdot E)$ and $\mathbf{w}_2(E) = \mathbf{m}_Q(\complement B \cdot E)$, respectively. Since for each $E \in \mathcal{T}$ we have

$$f \circ \mathbf{w}_1(E) = f \circ \mathbf{m}_Q(B \cdot E) = \mathbf{u}_Q(B \cdot E) = 0,$$

we also get $\mathrm{Ran}(\mathbf{w}_1) \subseteq L$. Hence, the affine dimension of $\mathrm{Ran}(\mathbf{w}_1)$ is less than or equal to $n - 1$, implying that $\mathrm{Ran}(\mathbf{w}_1)$ is compact due to the nonatomicity of $\mathbf{w}_1$ and the inductive assumption. Let x be a point in $L \cap \mathrm{cl}(\mathrm{Ran}(\mathbf{m}_Q))$. Then there exists a sequence $(E_l)_{l \in \mathbb{N}} \subseteq \mathcal{T}$ with $\lim_{l \to \infty} \mathbf{m}_Q(E_l) = x$, and therefore,

$$0 = f(x) = f\big(\lim_{l \to \infty} \mathbf{m}_Q(E_l)\big) = \lim_{l \to \infty} \mathbf{u}_Q(E_l).$$

By the $\mathbf{u}_Q$-absolute continuity of $\mathbf{w}_2$ we get $\lim_{l \to \infty} \mathbf{w}_2(E_l) = 0$. Note that $\mathbf{m}_Q = \mathbf{w}_1 + \mathbf{w}_2$ which implies

$$\lim_{l \to \infty} \mathbf{m}_Q(E_l) = \lim_{l \to \infty} \mathbf{w}_1(E_l) = x,$$

or, in other words, $x \in \mathrm{cl}(\mathrm{Ran}(\mathbf{w}_1)) = \mathrm{Ran}(\mathbf{w}_1) \subseteq \mathrm{Ran}(\mathbf{m}_Q)$. Hence, (81) holds. □

Theorem 13.1 obviously extends the Liapounoff Theorem [53]. Observing that $T_{\mathbf{L}}$-measures on generated tribes are necessarily nonatomic (compare Corollary 9.4 (ii)), Theorem 13.1 contains the theorems in Dvoretzki, Wald & Wolfowitz [30] stating that the range of a vector integral is convex and compact as a special case.

14 A Liapounoff type theorem for T_λ-measures

Most recently, in Barbieri & Weber [5] a Liapounoff Theorem has been proven for general T_λ-measures. In order to state these results, some preparatory steps are necessary [5].

Proposition 14.1 *Let $\mathcal{T}$ be a $T_{\mathbf{M}}$-clan and $\mathbf{m}\colon \mathcal{T} \longrightarrow \mathbb{R}$ be a function satisfying* (TM1) *and* (TM2) *(with respect to $T_{\mathbf{M}}$). Then the function $d_{\mathbf{m}}\colon \mathcal{T} \longrightarrow [0, \infty]$ defined by*

$$d_{\mathbf{m}}(A, B) = \sup\big\{|\mathbf{m}(C) - \mathbf{m}(D)| \;\big|\; C, D \in \mathcal{T}, A \,\mathsf{T}_{\mathsf{M}}\, B \subseteq C \subseteq D \subseteq A \,\mathsf{S}_{\mathsf{M}}\, B\big\}$$

is a pseudo-metric on the $T_{\mathbf{M}}$-clan $\mathcal{T}$.

Definition 14.2 Let $\mathcal{T}$ be a $T_{\mathbf{M}}$-clan and $\mathbf{m}\colon \mathcal{T} \longrightarrow \mathbb{R}$ be a function satisfying (TM1) and (TM2) (with respect to $T_{\mathbf{M}}$). Then $\mathcal{T}$ is said to be $\mathbf{m}$-*chained* if, for all $A, B \in \mathcal{T}$ and for each $\varepsilon > 0$, there exist $C_0, C_1, \ldots, C_n$ in $\mathcal{T}$ with $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n = B$ and $d_{\mathbf{m}}(C_{i-1}, C_i) < \varepsilon$ for all $i \in \{1, 2, \ldots, n\}$.

We only mention that, in general, whenever $\mathcal{T}$ is $\mathbf{m}$-chained then $\mathbf{m}$ is nonatomic, but not conversely. However, a $T_{\mathbf{L}}$-measure $\mathbf{m}$ on a T_λ-tribe $\mathcal{T}$ (and, therefore, its restriction $\check{\mathbf{m}}$ to the σ-algebra of crisp sets $\mathcal{T}^\vee$) is nonatomic if and only if $\mathcal{T}$ is $\mathbf{m}$-chained or, equivalently, if $\mathcal{T}^\vee$ is $\check{\mathbf{m}}$-chained. Also note that, in the case of a measure $\mathbf{m}$ on a σ-algebra $\mathcal{A}$ of crisp subsets of X, the pseudo-metric space $(\mathcal{A}, d_{\mathbf{m}})$ is always complete. The following result is also due to [5].

Theorem 14.3 *Let $\mathcal{T}$ be a T_λ-tribe and $\mathbf{m}\colon \mathcal{T} \longrightarrow \mathbb{R}^n$ a nonatomic n-vector T_λ-measure on $\mathcal{T}$ for some $\lambda \in]0,\infty[$. Then we have:*

(i) If $\mathcal{T}$ is $\mathbf{m}$-chained, then $\operatorname{Ran}(\mathbf{m})$ *is convex.*

(ii) If the space $(\mathcal{T}, d_{\mathbf{m}})$ is complete, then $\operatorname{Ran}(\mathbf{m})$ *is compact.*

Theorem 14.3 does no longer hold if we drop the additional assumptions in (i) and (ii), as the following counterexamples show [5].

Example 14.4 (i) Consider the (T_λ-)tribe $[0,1]^{\{x_0\}}$ of all fuzzy subsets of the singleton $\{x_0\}$. Then the function $\mathbf{m}\colon [0,1]^{\{x_0\}} \longrightarrow [0,2]$ defined by

$$\mathbf{m}(A) = \begin{cases} 0 & \text{if } A(x_0) = 0, \\ A(x_0) + 1 & \text{otherwise,} \end{cases}$$

is a nonatomic T_λ-measure for each $\lambda \in]0,\infty[$ whose range, however, is not convex.

(ii) If $\boldsymbol{\lambda}\colon \mathcal{B} \longrightarrow [0,1]$ denotes the usual Lebesgue measure on the σ-algebra $\mathcal{B}$ of all Borel subsets of $[0,1]$, then the function $\mathbf{m}\colon \mathcal{B}^\wedge \longrightarrow [0,1]$ given by (compare (3))

$$\mathbf{m}(A) = \boldsymbol{\lambda}(\operatorname{supp}(A)) - \int_{[0,1]} A \, \mathrm{d}\boldsymbol{\lambda}$$

is a T_λ-measure for each $\lambda \in]0,\infty[$ and $\mathcal{B}^\wedge$ is $\mathbf{m}$-chained, but $\operatorname{Ran}(\mathbf{m})$ is not compact.

Most results of this section can be generalized: the values of the measures in question may be not only numbers in the (extended) real line, but in an arbitrary normed commutative group (see Weber [78] and Barbieri & Weber [4,5]).

References

[1] J. Aczél. *Lectures on Functional Equations and their Applications.* Academic Press, New York, 1966.

[2] R. J. Aumann and L. S. Shapley. *Values of Non-Atomic Games.* Princeton University Press, Princeton, 1974.

[3] G. Barbieri and H. Weber. Measures on clans and on MV-algebras. In this volume.

[4] G. Barbieri and H. Weber. A topological approach to the study of fuzzy measures. In Y. Abramovich, E. Avgerinos, and N. C. Yannelis, editors, *Functional Analysis and Economic Theory*, pages 17–46. Springer, Berlin, 1998.

[5] G. Barbieri and H. Weber. A representation theorem and a Lyapunov theorem for T_s-measures: The solution of two problems of Butnariu and Klement. *J. Math. Anal. Appl.*, 244:408–424, 2000.

[6] H. Bauer. *Probability Theory and Elements of Measure Theory.* Academic Press, London, 1981.

[7] L. P. Belluce. Semi-simple algebras of infinite valued logic and bold fuzzy set theory. *Can. J. Math.*, 38:1356–1379, 1986.

[8] L. P. Belluce. Semi-simple and complete MV-algebras. *Algebra Universalis*, 29:1–9, 1992.

[9] G. Birkhoff. *Lattice Theory.* American Mathematical Society, Providence, 1973.

[10] D. Blackwell. The range of certain vector integrals. *Proc. Amer. Math. Soc.*, 2:390–395, 1951.

[11] L. Borkowski, editor. *J. Łukasiewicz: Selected Works.* Studies in Logic and Foundations of Mathematics. North-Holland, Amsterdam, 1970.

[12] D. Butnariu. Additive fuzzy measures and integrals. I. *J. Math. Anal. Appl.*, 93:436–452, 1983.

[13] D. Butnariu. Decomposition and range for additive fuzzy measures. *Fuzzy Sets and Systems*, 10:135–155, 1983.

[14] D. Butnariu. Non-atomic fuzzy measures and games. *Fuzzy Sets and Systems*, 17:39–52, 1985.

[15] D. Butnariu. Fuzzy measurability and integrability. *J. Math. Anal. Appl.*, 117:385–410, 1986.

[16] D. Butnariu. Additive fuzzy measures and integrals. III. *J. Math. Anal. Appl.*, 125:288–303, 1987.

[17] D. Butnariu and E. P. Klement. Triangular norm-based measures and their Markov kernel representation. *J. Math. Anal. Appl.*, 162:111–143, 1991.

[18] D. Butnariu and E. P. Klement. *Triangular Norm-Based Measures and Games with Fuzzy Coalitions.* Kluwer Academic Publishers, Dordrecht, 1993.

[19] C. C. Chang. Algebraic analysis of many valued logics. *Trans. Amer. Math. Soc.*, 88:467–490, 1958.

[20] H. Chernoff. An extension of a result of Liapounoff on the range of a vector measure. *Proc. Amer. Math. Soc.*, 2:720–726, 1951.

[21] A. H. Clifford. Naturally totally ordered commutative semigroups. *Amer. J. Math.*, 76:631–646, 1954.

[22] A. C. Climescu. Sur l'équation fonctionelle de l'associativité. *Bull. École Polytechn. Iassy*, 1:1–16, 1946.

[23] B. De Baets and R. Mesiar. $\mathcal{T}$-partitions. *Fuzzy Sets and Systems*, 97:211–223, 1998.

[24] M. de Glas. Representation of Łukasiewicz' many-valued algebras. The atomic case. *Fuzzy Sets and Systems*, 14:175–185, 1984.

[25] A. Di Nola. Representation and reticulation by quotients of MV algebras. *Ricerche Mat.*, 40:291–297, 1991.

[26] D. Dubois, M. Grabisch, and H. Prade. Gradual rules and the approximation of functions. In *Proceedings 2nd International Conference on Fuzzy Logic and Neural Networks, Iizuka*, pages 629–632, 1992.

[27] D. Dumitrescu. Fuzzy partitions with the connectives T_∞, S_∞. *Fuzzy Sets and Systems*, 47:193–195, 1992.

[28] D. Dumitrescu. Fuzzy measures and the entropy of fuzzy partitions. *J. Math. Anal. Appl.*, 176:359–373, 1993.

[29] A. Dvoretzky, A. Wald, and J. Wolfowitz. Elimination of randomization in certain problems of statistics and game theory. *Proc. Nat. Acad. Sci. U.S.A.*, 36:256–259, 1950.

[30] A. Dvoretzky, A. Wald, and J. Wolfowitz. Relations among certain ranges of vector measures. *Pacific J. Math.*, 1:59–74, 1951.

[31] M. J. Frank. On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. *Aequationes Math.*, 19:194–226, 1979.

[32] S. Gottwald. *A Treatise on Many-Valued Logic.* Research Studies Press, Baldock, 2001.

[33] P. Hájek. *Metamathematics of Fuzzy Logic.* Kluwer Academic Publishers, Dordrecht, 1998.

[34] P. R. Halmos. On the set of values of a finite measure. *Bull. Amer. Math. Soc.*, 53:138–141, 1947.

[35] P. R. Halmos. The range of a vector measure. *Bull. Amer. Math. Soc.*, 54:416–421, 1948.

[36] P. R. Halmos. *Measure Theory.* Van Nostrand Reinhold, New York, 1950.

[37] P. R. Halmos. *Measure Theory.* Springer, Berlin, 1974.

[38] U. Höhle. Fuzzy equalities and indistinguishability. In *Proceedings EUFIT '93, Aachen*, volume 1, pages 358–363, 1993.

[39] U. Höhle. On the fundamentals of fuzzy set theory. *J. Math. Anal. Appl.*, 201:786–826, 1996.

[40] U. Höhle. Many-valued equalities, singletons and fuzzy partitions. *Soft Computing*, 2:134–140, 1998.

[41] U. Höhle and E. P. Klement. Plausibility measures — a general framework for possibility and fuzzy probability measures. In H. J. Skala, S. Termini, and E. Trillas, editors, *Aspects of Vagueness*, pages 31–50. Reidel, Dordrecht, 1983.

[42] U. Höhle and S. Weber. Uncertainty measures, realizations and entropies. In J. Goutaias, R.P.S. Mahler, and H.T. Nguyen, editors, *Random Sets: Theory and Applications*, pages 259–295. Springer, Heidelberg, 1997.

[43] F. Klawonn and R. Kruse. Equality relations as a basis for fuzzy control. *Fuzzy Sets and Systems*, 54:147–156, 1993.

[44] E. P. Klement. Characterization of finite fuzzy measures using Markoff-kernels. *J. Math. Anal. Appl.*, 75:330–339, 1980.

[45] E. P. Klement. Fuzzy σ-algebras and fuzzy measurable functions. *Fuzzy Sets and Systems*, 4:83–93, 1980.

[46] E. P. Klement. Characterization of fuzzy measures constructed by means of triangular norms. *J. Math. Anal. Appl.*, 86:345–358, 1982.

[47] E. P. Klement. Construction of fuzzy σ-algebras using triangular norms. *J. Math. Anal. Appl.*, 85:543–565, 1982.

[48] E. P. Klement, R. Mesiar, and E. Pap. A characterization of the ordering of continuous t-norms. *Fuzzy Sets and Systems*, 86:189–195, 1997.

[49] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms.* Kluwer Academic Publishers, Dordrecht, 2000.

[50] E. P. Klement and M. Navara. A characterization of tribes with respect to the Łukasiewicz t-norm. *Czechoslovak Math. J.*, 47(122):689–700, 1997.

[51] E. P. Klement and M. Navara. A survey on different triangular norm-based fuzzy logics. *Fuzzy Sets and Systems*, 101:241–251, 1999.

[52] R. Kruse, J. Gebhardt, and F. Klawonn. *Foundations of Fuzzy Systems.* J. Wiley & Sons, Chichester, 1994.

[53] A. A. Liapounoff. Sur les fonctions-vecteurs completement additives. *Bull. Acad. Sci. URSS Ser. Math.*, 4:465–478, 1940.

[54] J. Lindenstrauss. A short proof of Liapounoff's convexity theorem. *J. Math. Mech.*, 15:971–973, 1966.

[55] C. M. Ling. Representation of associative functions. *Publ. Math. Debrecen*, 12:189–212, 1965.

[56] J. Łukasiewicz. O logice trówartosciowej. *Ruch Filozoficzny*, 5:170–171, 1920. (English translation contained in [11]).

[57] K. Menger. Statistical metrics. *Proc. Nat. Acad. Sci. U.S.A.*, 8:535–537, 1942.

[58] K. Menger. Ensembles flous et fonctions aleatoires. *C. R. Acad. Sci. Paris Sér. A*, 232:2001–2003, 1951.

[59] K. Menger. Probabilistic geometry. *Proc. Nat. Acad. Sci. U.S.A.*, 37:226–229, 1951.

[60] K. Menger. Probabilistic theories of relations. *Proc. Nat. Acad. Sci. U.S.A.*, 37:178–180, 1951.

[61] R. Mesiar. Fundamental triangular norm based tribes and measures. *J. Math. Anal. Appl.*, 177:633–640, 1993.

[62] R. Mesiar. On the structure of T_s-tribes. *Tatra Mt. Math. Publ.*, 3:167–172, 1993.

[63] R. Mesiar and M. Navara. T_s-tribes and T_s-measures. *J. Math. Anal. Appl.*, 201:91–102, 1996.

[64] R. Mesiar and J. Rybárik. Pan-operations structure. *Fuzzy Sets and Systems*, 74:365–369, 1995.

[65] M. Navara. A characterization of triangular norm based tribes. *Tatra Mt. Math. Publ.*, 3:161–166, 1993.

[66] R. B. Nelsen. *An Introduction to Copulas*, volume 139 of *Lecture Notes in Statistics*. Springer, New York, 1999.

[67] J. (von) Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, 1944.

[68] G. Poprugenko. Sur la propriété de Darboux des fonctions continues d'ensemble. *Fundam. Math.*, 12:254–268, 1928.

[69] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.

[70] A. Rose and J. B. Rosser. Fragments of many-valued statement calculi. *Trans. Amer. Math. Soc.*, 87:1–53, 1958.

[71] E. Ruspini. A new approach to clustering. *Inform. Control*, 15:22–32, 1969.

[72] K. D. Schmidt. A general Jordan decomposition. *Arch. Math.*, 38:556–564, 1982.

[73] B. Schweizer and A. Sklar. Associative functions and abstract semigroups. *Publ. Math. Debrecen*, 10:69–81, 1963.

[74] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. North-Holland, New York, 1983.

[75] A. Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231, 1959.

[76] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, 1974.

[77] H. Thiele and N. Schmechel. On the mutual definability of fuzzy equivalence relations and fuzzy partitions. In *Proceedings of the International Joint Conference of the Fourth IEEE International Conference on Fuzzy Systems and the Second International Fuzzy Engineering Symposium, Yokohama*, volume III, pages 1383–1390. IEEE Press, 1995.

[78] H. Weber. On modular functions. *Funct. et Approx.*, 24:35–52, 1996.

[79] O. Wyler. Clans. *Compositio Math.*, 17:172–189, 1966.

[80] L. A. Zadeh. Fuzzy sets. *Inform. Control*, 8:338–353, 1965.

[81] L. A. Zadeh. Probability measures of fuzzy events. *J. Math. Anal. Appl.*, 23:421–427, 1968.

[82] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.