WEAK CONVERGENCE OF ORBITS OF NONLINEAR OPERATORS IN REFLEXIVE BANACH SPACES

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March 11, 2003

Abstract

Let K be a closed convex subset of a reflexive Banach space X. We consider self-mappings of K which are bounded on bounded subsets of K and satisfy a relaxed form of nonexpansivity with respect to a given convex function f. The family of these operators is endowed with the topology of uniform convergence on bounded subsets of K. We show that "almost all" such operators T share the property that they have a fixed point z_T such that, for any $x \in K$, the orbit $\left\{T^k x\right\}_{k=1}^{\infty}$ converges weakly to z_T. Here the meaning of "almost all" is in the sense of Baire's categories: the collection of all those operators which do not have this property is contained in a countable union of nowhere dense sets.

Keywords: Bregman distance, fixed point, nonexpansivity of an operator with respect to a convex function, orbit, reflexive Banach space, total convexity, uniform convexity, weak convergence.

2000 Mathematics Subject Classification: 47H09, 47H30, 54E35, 54E52, 65K99.

1 INTRODUCTION

Let $(X, \|\cdot\|)$ be a reflexive Banach space, let $f : X \to (-\infty, +\infty]$ be a proper convex function with the domain

$$\mathcal{D} = \{x \in X : f(x) < +\infty\},$$

and define the functions $D_f, D_f^{\#} : \mathcal{D} \times \mathcal{D} \to [0, +\infty]$ by

$$D_f(y,x) = f(y) - f(x) + f^{\circ}(x, x-y) \tag{1}$$

and

$$D_f^{\#}(y,x) = f(y) - f(x) - f^{\circ}(x, y-x), \tag{2}$$

where

$$f^{\circ}(x, y-x) := \lim_{t \searrow 0} t^{-1} \left[f(ty + (1-t)x) - f(x) \right].$$

Suppose that K is a nonempty, closed and convex subset of $\mathcal{D}^{\circ}$, the interior of $\mathcal{D}$, and that the function f is strictly convex on K. An operator $T : K \to K$ is called *relatively nonexpansive with respect to* f (cf. [14]) if there exists a vector $z \in K$ such that

$$D_f(z, Tx) \le D_f(z, x) \tag{3}$$

for all $x \in K$. Note that, in these circumstances, z is a fixed point of T because

$$0 \le D_f(z, Tz) \le D_f(z,z) = 0,$$

and this cannot hold unless $D_f(z, Tz) = 0$ which implies $z = Tz$ (cf. Proposition 1.1.4 in [8]).

A fixed point z of T satisfying (3) is called a *nonexpansivity pole of* T *with respect to* f and the collection of all such fixed points is denoted by $\operatorname{Nexp}_f(T)$. An operator T can be relatively nonexpansive with respect to a function f as described above without being nonexpansive in the usual sense of this term, that is, it may not satisfy the inequality

$$\|Tx - Ty\| \le \|x - y\|, \tag{4}$$

or even the inequality

$$D_f(Ty, Tx) \le D_f(y, x), \tag{5}$$

for all $x, y \in K$ (see, for instance, [8, Section 2.1.7]). In a smooth strictly convex space X, if T is an operator satisfying (4) with a fixed point z, then it is necessarily relatively nonexpansive with respect to $f(x) = \|x - z\|^2$. Also, if T satisfies (5) with respect to some function f as described above, and if T has a fixed point, then T is relatively nonexpansive with respect to f.

A question of practical interest concerning the operators $T : K \to K$ which are relatively nonexpansive with respect to an a priori given function f is whether for any $x \in K$, the *orbit* $\{T^k x\}_{k \in \mathbb{N}}$ converges (weakly or strongly) to a fixed point of T. The relevance of this question is due to the fact that significant problems of applied mathematics (convex feasibility problems, optimization problems, variational inequalities and some classes of integral and differential equations) can be reduced to the computation of fixed points of operators which are relatively nonexpansive with respect to functions f satisfying the conditions required in the present paper (cf. [8, Section 2.4]). For particular families of operators T which are totally nonexpansive (in the sense given to this term in [8]) and, therefore, relatively nonexpansive with respect to a function f having some differentiability properties on K, and whose restriction to K, defined by

$$f_K(x) = \begin{cases} f(x) & \text{if } x \in K, \\ +\infty & \text{if } x \notin K, \end{cases} \tag{6}$$

is totally convex, weak and, sometimes, strong convergence of the orbits $\{T^k x\}_{k\in\mathbb{N}}$ to fixed points of T were established under various conditions by many authors (see, for example, [3], [1], [28], [4], [29], [26], [30], [33], [5], [18], [19], [8], [9], [2], [24], [25], [39], [12] and [11]). In this connection, recall (cf. [8]) that the function f_K defined by (6) is called *totally convex* if for each $x \in K$, the modulus of total convexity of f_K at x, given by

$$\nu_{f_K}(x,t) := \inf\left\{D_f^{\#}(y,x) : y \in K,\ \|y-x\| = t\right\},$$

is positive whenever $t > 0$, where $D_f^{\#}$ is defined by (2).

As far as we know, the first study of weak convergence per se of orbits of operators T which are relatively nonexpansive with respect to a convex function f (in fact, a smooth and uniformly convex function f) occurs in [31]. The purpose of the present paper is to show that for large classes of operators $T : K \to K$, which are relatively nonexpansive with respect to a function f, weak convergence of the orbits $\{T^k x\}_{k\in\mathbb{N}}$ to fixed points of T is a *generic property* in the sense that it is satisfied by all but a "small" set of operators in those classes. We prove this under the assumption that, in addition to f being strictly convex on K, the convergence structure induced on K by the function D_f given by (1) is stronger than that induced by the norm of X. In other words, depending on the context, we use either the assumption

$(A1)$ *For each $x \in K$ and each $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in K$ and $D_f(y,x) \le \delta$, then $||x-y|| \le \epsilon$;*

or its stronger version,

$(A2)$ *For each $x \in K$ and each $\epsilon > 0$, there exists $\delta > 0$ such that if $y, z \in K$, $||y-x|| \le \delta$ and $D_f(z,y) \le \delta$, then $||x-z|| \le \epsilon$.*

Naturally, the question whether functions satisfying $(A1)$ or $(A2)$ exist on Banach spaces comes to mind. Lemma 2.1 below contains an answer to this question. It shows that if the function f is totally convex on K, then it satisfies condition $(A1)$. Taking into account [11, Proposition 2.4], it follows that if X is a locally uniformly convex Banach space (as many spaces happen to be - see [23]), then all the functions $f_p = \|\cdot\|^p$ with $p \in (1,\infty)$ are totally convex on X and, therefore, satisfy $(A1)$ no matter how K is chosen. Lemma 2.1 also implies that functions which are uniformly convex on bounded subsets of K satisfy $(A2)$. This and the results in [10] show that in uniformly convex Banach spaces all functions f_p mentioned above satisfy $(A2)$.

With these facts in mind, we consider the set $\mathcal{M}(K)$ of all operators $T : K \to K$ which are bounded on bounded subsets of K. We endow it with the metrizable complete uniformity with the base

$$\mathcal{E}(N,\varepsilon) = \{(T_1,T_2) \in \mathcal{M}(K) \times \mathcal{M}(K) : ||T_1x - T_2x|| \le \epsilon, \forall x \in K \cap \mathbf{B}(0,N)\},$$

where N, ε are positive real numbers and, for any $u \in X$, $\mathbf{B}(u,N)$ denotes the closed ball of center u and radius N in X. The space $\mathcal{M}(K)$ is provided with the topology induced by this uniformity. We denote by $\mathcal{M}_c(K)$ and $\mathcal{M}_u(K)$ the subsets of $\mathcal{M}(K)$ consisting of those operators in $\mathcal{M}(K)$ which are continuous on K and, respectively, uniformly continuous on bounded subsets of K. Both $\mathcal{M}_c(K)$ and $\mathcal{M}_u(K)$ are closed subsets of $\mathcal{M}(K)$ and we endow each one of them with the corresponding relative topology induced by $\mathcal{M}(K)$. Let $\mathcal{M}^f(K)$ be the collection of all $T \in \mathcal{M}(K)$ which satisfy the following conditions:

(I) *The operator T is relatively nonexpansive with respect to f;*

(II) *For some $z_T \in \operatorname{Nexp}_f(T)$, the function $D_f(z_T,\cdot)$ is convex and uniformly continuous on bounded subsets of K and, for each $\alpha > 0$, the level set*

$$L_f(T;\alpha) := \{y \in K : D_f(z_T,y) \le \alpha\}$$

is bounded.

Denote $\mathcal{M}_c^f(K) = \mathcal{M}^f(K) \cap \mathcal{M}_c(K)$ and let $\overline{\mathcal{M}_c^f(K)}$ be the closure of $\mathcal{M}_c^f(K)$ in $\mathcal{M}_c(K)$. Also, let $\mathcal{M}_1^f(K)$ be the collection of all those operators in $\mathcal{M}(K)$ which satisfy condition (I) above as well as the following requirement:

(III) *For some $z_T \in \operatorname{Nexp}_f(T)$, the function $D_f(z_T,\cdot)$ is convex, lower semicontinuous, bounded on bounded subsets of K and, for each $\alpha > 0$, the level set $L_f(T;\alpha)$ is bounded.*

Denote $\mathcal{M}_u^f(K) = \mathcal{M}_1^f(K) \cap \mathcal{M}_u(K)$ and let $\overline{\mathcal{M}_u^f(K)}$ be the closure of $\mathcal{M}_u^f(K)$ in $\mathcal{M}_u(K)$. Theorem 2.1, proved below, shows that the family of all those $T \in \overline{\mathcal{M}_c^f(K)}$ for which the orbits $\{T^kx\}_{k\in\mathbb{N}}$ are weakly convergent to a fixed point of T, no matter how x is chosen in K, contains a countable intersection of open everywhere dense subsets of $\overline{\mathcal{M}_c^f(K)}$. Moreover, for each operator T in this countable intersection of open everywhere dense subsets of $\overline{\mathcal{M}_c^f(K)}$, all its weakly convergent orbits have the same weak limit. Theorems 3.1 and 3.2 give similar results for the space $\overline{\mathcal{M}_u^f(K)}$, instead of $\overline{\mathcal{M}_c^f(K)}$. They show that there exists a countable intersection of open everywhere dense subsets of $\overline{\mathcal{M}_u^f(K)}$ such that, for each operator T contained in it, all the orbits $\{T^kx\}_{k\in\mathbb{N}}$ with $x \in K$ converge weakly and uniformly on bounded subsets of K to a fixed point of T. The main difference between Theorem 3.1 and Theorem 3.2 is in the assumptions under which they are proved. Theorem 3.1 is established under the hypothesis that the space X is separable and f satisfies $(A1)$. In Theorem 3.2

the separability condition on X is dropped but one strengthens the conditions on f by requiring it to satisfy $(A2)$.

It is interesting to observe that the spaces $\overline{\mathcal{M}_c^f(K)}$ and $\overline{\mathcal{M}_u^f(K)}$ contain useful operators. For instance, if the space X is uniformly convex and uniformly smooth, then for any $p \in (1, \infty)$, the function $f = \|\cdot\|^p$ is Fréchet differentiable, totally convex and, therefore, it satisfies condition $(A1)$ and for any closed convex set $C \subseteq X$ which contains the origin, the Bregman projection operators $T = \Pi_C^f$ is well defined (cf. [8]), and satisfies conditions (I) and (II) for $z_T = 0$. According to [40, Corollary 4.1], Π_C^f is continuous and, thus, belongs to $\mathcal{M}_c^f(K)$. If X is a smooth space and, for some $p \in (1, \infty)$, the function $\|\cdot\|^p$ is totally convex and if $z \in K$, then any nonexpansive (in the sense of (4)) operator $T : K \to K$ having z as a fixed point is relatively nonexpansive with respect to the function $f(x) = \|z - x\|^p$ and, therefore, is contained in $\mathcal{M}_u^f(K)$. If X is a Hilbert space and $f = \|\cdot\|^2$, then for any nonempty closed convex subset C of X, the metric projection operator $T = P_C$ is contained in $\mathcal{M}_u^f(K)$.

Theorem 2.1 can be seen as an extension of a result of De Blasi and Myjak [20, Theorem, p. 185] (in this context see also [21]). In fact, De Blasi and Myjak [20] have shown that, if K is bounded, then the set $\mathcal{N}_0(K)$ consisting of all nonexpansive $T : K \to K$ whose orbits converge strongly to the fixed points of T is a residual set in $\mathcal{M}(K)$. Theorem 2.1 says that, even if K is unbounded, a similar property holds for operators which are relatively nonexpansive with respect to a function f satisfying $(A1)$: for most of them (in the sense of Baire's categories), their orbits converge weakly to fixed points. When X is a Hilbert space, then Theorem 2.1 and its Corollary 2.1 enhance another result in [20] according to which $\mathcal{N}(K)$, the subset of $\mathcal{M}(K)$ consisting of all operators $T : K \to K$ which are strict contractions, is of the first Baire category in $\mathcal{M}(K)$: they show that even if one adds to $\mathcal{N}(K)$ all metric projection operators (which, in general, are not strict contractions), one still gets a set which is of the first Baire category in $\mathcal{M}(K)$.

The results presented in this paper also relate to previous results we have presented in [14]. They improve upon those results by showing that in reflexive spaces one can guarantee generic weak convergence of orbits of relatively nonexpansive operators under less demanding requirements concerning the function f and, implicitly, for a larger class of operators. The methods and techniques we use in the proofs of the main results are those of generic analysis which have already been successfully applied in the theory of dynamical systems ([22], [27], [35]), the calculus of variations and optimization ([41], [36], [42]), as well as in nonlinear analysis ([34], [32], [37], [38], [39]).

2 GENERIC WEAK CONVERGENCE FOR THE ORBITS OF OPERATORS IN $\overline{\mathcal{M}^f(K)}$ AND $\overline{\mathcal{M}_c^f(K)}$

In this section we analyze the generic behavior of the orbits generated by operators in $\overline{\mathcal{M}_c^f(K)}$. We start with a series of technical results. The first of them shows the connection between the total convexity of f_K defined by (6) and conditions $(A1)$ and $(A2)$. This justifies the claims made in the Introduction that the class of functions f satisfying either $(A1)$ or $(A2)$ on some spaces is quite large.

Lemma 2.1: (a) *If the function f_K defined by* (6) *is totally convex, then f satisfies condition $(A1)$.*

(b) *If f is (Gâteaux) differentiable at each point $x \in K$ and $(A1)$ holds, then f_K is totally convex.*

(c) *If f_K is uniformly convex on bounded subsets of K, then f satisfies* $(A2)$.

Proof. (a) Suppose that f_K is totally convex, $\varepsilon > 0$ and $x \in K$. If for all $y \in K$ we have $\|y - x\| < \varepsilon$, then there is nothing to prove. Otherwise, there exists $y_0 \in K$ with $\|y_0 - x\| \geq \varepsilon$ and, thus, according to Proposition 1.1.2 in [8], we deduce that $\nu_{f_K}(x, \varepsilon)$ finite. Since f_K is totally convex, we also have that $\nu_{f_K}(x, \varepsilon) > 0$. Let $\delta := \nu_{f_K}(x, \varepsilon)$. If $y \in K$ and $D_f(y, x) \leq \delta$, then

$$\nu_{f_K}(x, \|y - x\|) \leq D_f^{\#}(y, x) \leq D_f(y, x) \leq \nu_{f_K}(x, \varepsilon). \tag{7}$$

The function $\nu_{f_K}(x, \cdot)$ is strictly increasing on its domain (cf. Proposition 1.2.2 in [8]). Therefore, (7) cannot hold unless $\|y - x\| \leq \varepsilon$. Hence $(A1)$ is satisfied with $\delta := \nu_{f_K}(x, \varepsilon)$.

(b) Suppose that f is differentiable on K and that $(A1)$ holds, but $\nu_{f_K}(x^*, t_0) = 0$ for some $x^* \in K$ and for some $t_0 > 0$. Then there exists a sequence $\{x_k\}_{k=1}^{\infty}$ contained in K such that $\|x_k - x\| = t_0$ for all k and $\lim_{k\to\infty} D_f^{\#}(x_k, x^*) = 0$. By the differentiability of f on K, we have $D_f(\cdot, x^*) = D_f^{\#}(\cdot, x^*)$. Hence $\lim_{k\to\infty} D_f(x_k, x^*) = 0$. This and $(A1)$ imply that $t_0 = \lim_{k\to\infty} \|x_k - x^*\| = 0$ which contradicts the assumption that $t_0 > 0$.

(c) Assume that f_K is uniformly convex on bounded subsets of K. Let x be any point in K and take $\varepsilon > 0$. Denote $K_x = K \cap \mathbf{B}(x, \frac{\varepsilon}{2})$ and observe that $\mathrm{Dom}\,(\nu_{f_K}(K_x, \cdot))$, the domain of the function $\nu_{f_K}(K_x, \cdot) : [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_{f_K}(K_x, t) := \inf\left\{\nu_{f_K}(u, t) : \ u \in K_x\right\},$$

contains a nonempty interval $[0, \tau)$ because K is contained in $\mathrm{Int}\,(\mathrm{Dom}\,(f))$ and K_x is bounded. Set

$$\delta = \min\left\{\frac{\varepsilon}{2}, \nu_{f_K}(K_x, \frac{\varepsilon}{p})\right\},$$

where $p \geq 3$ is a number such that $\frac{\varepsilon}{p} \in \mathrm{Dom}\,(\nu_{f_K}(K_x, \cdot))$. Clearly, $\delta > 0$ because f_K is uniformly convex on bounded subsets of K (see [11]). If $y, z \in K$, $\|y - x\| < \delta$ and $D_f(z, y) < \delta$, then $\|y - x\| < \frac{\varepsilon}{2}$ and

$$\nu_{f_K}(y, \|z - y\|) \leq D_f(z, y) < \delta \leq \nu_{f_K}(K_x, \frac{\varepsilon}{p}).$$

The last inequality cannot hold unless

$$\|z - y\| < \frac{\varepsilon}{p},$$

because the function $\nu_{f_K}(K_x, \cdot)$ is strictly increasing on its domain (cf. [16]). Hence we have

$$\|y - x\| < \frac{\varepsilon}{2} \text{ and } \|z - y\| < \frac{\varepsilon}{p} < \frac{\varepsilon}{2}.$$

Consequently,

$$\|z - x\| \leq \|y - x\| + \|y - z\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and this completes the proof. □

From now on we assume that the function f satisfies condition $(A1)$. The next lemma establishes a special property of the functions satisfying condition $(A1)$ which will be used in the sequel.

Lemma 2.2: *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in K and let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers converging to zero. Assume that $\{z_n\}_{n=1}^{\infty}$ is a sequence contained in K such that for each integer $n \geq 1$, the function $D_f(z_n, \cdot) : K \to R^1$ is convex and lower semicontinuous and*

$$\limsup_{i \to \infty} D_f(z_n, x_i) < \epsilon_n.$$

Then the sequence $\{x_i\}_{i=1}^{\infty}$ converges weakly and the sequence $\{z_n\}_{n=1}^{\infty}$ converges strongly to the same point $x_* \in K$.

Proof. Let $\{x_{j_i}\}_{i=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$ which converges weakly to x_*. Clearly, $x_* \in K$ because K is closed and convex. By hypothesis, for each natural number n we have

$$\begin{aligned} D_f(z_n, x_*) \leq & \liminf_{i \to \infty} D_f(z_n, x_{j_i}) \\ \leq & \limsup_{i \to \infty} D_f(z_n, x_{j_i}) \leq \limsup_{i \to \infty} D_f(z_n, x_i) < \epsilon_n. \end{aligned}$$

Since the sequence $\{\epsilon_n\}_{n=1}^{\infty}$ converges to zero, this inequality and $(A1)$ imply that $\{z_n\}_{n=1}^{\infty}$ converges strongly to x_*. Consequently, any weak subsequential limit of $\{x_n\}_{n=1}^{\infty}$ coincides with the unique strong limit of $\{z_n\}_{n=1}^{\infty}$. □

The next lemma can be proved by straightforward induction.

Lemma 2.3: *Let $n \geq 1$ be an integer, $x \in K$, $B \in \mathcal{M}_c(K)$, and $\epsilon > 0$. Then there exist a neighborhood $\mathcal{U}$ of B in $\mathcal{M}_c(K)$ and a number $\delta > 0$ such that for each $S \in \mathcal{U}$ and for each $y \in K \cap \mathbf{B}(x, \delta)$, we have $||S^n y - B^n x|| \leq \epsilon$.*

Let T be an operator in $\mathcal{M}(K)$ and suppose that $z_T \in \mathrm{Nexp}_f(T)$. As noted in the Introduction, in this case z_T is a fixed point of T. For each $\gamma \in (0,1)$, we define the perturbed mapping $T_\gamma : K \to K$ by

$$T_\gamma x = \gamma z_T + (1-\gamma)Tx. \tag{8}$$

Clearly, $T_\gamma \in \mathcal{M}(K)$ and, if $T \in \mathcal{M}_c(K)$ or, respectively, $T \in \mathcal{M}_u(K)$, then $T_\gamma \in \mathcal{M}_c(K)$ or, respectively, $T_\gamma \in \mathcal{M}_u(K)$. Note that T_γ converges to T as $\gamma \to 0^+$. It is easy to see that z_T is a fixed point of T_γ. Moreover, if $D_f(z_T,\cdot)$ is convex on K, then for each $\gamma \in (0,1)$,

$$D_f(z_T, T_\gamma x) \le (1-\gamma)D_f(z_T, x) \tag{9}$$

for all $x \in K$, i.e., T_γ is also relatively nonexpansive with respect to f and $z_T \in \mathrm{Nexp}_f(T_\gamma)$. In particular, this happens when z_T is a nonexpansivity pole of T with respect to f for which either condition (II) or condition (III) holds. The following result concerning the operators T_γ is a modification of Lemma 8.1 in [13].

Lemma 2.4: *Suppose that $T \in \mathcal{M}^f(K)$ and let $z_T \in \mathrm{Nexp}_f(T)$ be such that condition (II) is satisfied. Let $\epsilon, \gamma \in (0,1)$ and let K_0 be a nonempty bounded subset of K. Then there exist a natural number N and a neighborhood $\mathcal{U}$ of T_γ in $\mathcal{M}(K)$ such that for each $S \in \mathcal{U}$, each $x \in K_0$, and each integer $n \ge N$, we have $D_f(z_T, S^n x) \le \epsilon$.*

Proof. Since T and z_T satisfy condition (II), the function $D_f(z_T,\cdot)$ is uniformly continuous on K_0 and, thus, there exists a real number $c_0 > 1$ such that

$$D_f(z_T, x) \le c_0 \text{ for all } x \in K_0. \tag{10}$$

Also, the set $L_f(T, c_0+2)$ is bounded. Let $c_1 > 0$ be a number such that $L_f(T, c_0+2)$ is included in the closed ball $\mathbf{B}(0, c_1)$. Then we have

$$x \in K \text{ and } D_f(z_T, x) \le c_0 + 2 \Rightarrow ||x|| \le c_1. \tag{11}$$

Set $K_1 = \{x \in K : ||x|| \le c_1\}$ and $K_2 = \{x \in K : ||x|| \le c_1 + 2\}$. Clearly, $K_0 \subset K_1$. By (II) again, the function $D_f(z_T,\cdot)$ is uniformly continuous on K_2. Therefore there exists $\delta_0 \in (0, 8^{-1}\epsilon)$ such that

$$|D_f(z_T, x_1) - D_f(z_T, x_2)| \le 16^{-1}\gamma\epsilon \tag{12}$$

for each $x_1, x_2 \in K_2$ satisfying $||x_1 - x_2|| \le \delta_0$. Choose a natural number N such that

$$16^{-1}N\gamma\epsilon > c_0 + 2 \tag{13}$$

and define

$$\mathcal{U} = \{S \in \mathcal{M}(K) : ||Sx - T_\gamma x|| \le \delta_0,\ \forall x \in K_1\}. \tag{14}$$

Suppose that $S \in \mathcal{U}$ and $x \in K_0$. We are going to show that

$$D_f(z_T, S^n x) \le \epsilon \text{ for all integers } n \ge N. \tag{15}$$

To this end, we first show by induction that, for all integers $n \geq 0$,

$$D_f(z_T, S^n x) \leq c_0 \tag{16}$$

and

$$D_f(z_T, S^{n+1}x) \leq (1-\gamma)D_f(z_T, S^n x) + 16^{-1}\gamma\epsilon. \tag{17}$$

By (10), the inequality (16) is valid for $n = 0$ because S^0 is the identity operator. Assume that (16) is true for some integer $n \geq 0$. Then, we deduce from (11) that $S^n x \in K_1$ and $||S^n x|| \leq c_1$. This and (14) imply that

$$||S^{n+1}x - T_\gamma S^n x|| \leq \delta_0. \tag{18}$$

By (16) and (9) we also have

$$D_f(z_T, T_\gamma(S^n x)) \leq (1-\gamma)D_f(z_T, S^n x) \leq (1-\gamma)c_0, \tag{19}$$

which, together with (11), implies that $||T_\gamma(S^n x)|| \leq c_1$. Thus, (18) implies that

$$T_\gamma(S^n x) \in K_1 \text{ and } S^{n+1}x \in K_2.$$

Combining this, (18) and the definition of δ_0 (see (12)), we obtain that

$$\left|D_f(z_T, S^{n+1}x) - D_f(z_T, T_\gamma(S^n x))\right| \leq 16^{-1}\gamma\epsilon.$$

Consequently, we have

$$D_f(z_T, S^{n+1}x) \leq 16^{-1}\gamma\epsilon + (1-\gamma)D_f(z_T, S^n x) \leq 16^{-1}\gamma\epsilon + (1-\gamma)c_0 \leq c_0,$$

because of (19). Hence, (17) holds and $D_f(z_T, S^{n+1}x) \leq c_0$. Therefore (16) and (17) are valid for all integers $n \geq 0$.

Now, let $n \geq 0$ be an integer. If $D_f(z_T, S^n x) \leq \epsilon$, then, according to (17), we obtain

$$D_f(z_T, S^{n+1}x) \leq (1-\gamma)\epsilon + 16^{-1}\gamma\epsilon \leq \epsilon.$$

Therefore, to prove (15) it is sufficient to show that $D_f(z_T, S^n x) \leq \epsilon$ for some integer $n \in [0, N]$. Assume the contrary. Then $D_f(z_T, S^n x) > \epsilon$ for all integers $n \in [0, N]$. Taking (17) into account, we see that for all $n = 0, \ldots, N$,

$$D_f(z_T, S^n x) - D_f(z_T, S^{n+1}x) \geq \gamma D_f(z_T, S^n x) - 16^{-1}\gamma\epsilon \geq 2^{-1}\gamma\epsilon.$$

When combined with (13), these inequalities yield

$$D_f(z_T, x) \geq D_f(z_T, x) - D_f(z_T, S^{N+1}x) \geq 2^{-1}\gamma\epsilon N > c_0 + 2,$$

which contradicts (10). Thus (15) holds and the proof is complete. $\square$

At this stage we are in a position to establish our generic convergence result for orbits of operators in $\overline{\mathcal{M}_c^f(K)}$. To this end, let $\overline{\mathcal{M}^f(K)}$ be the closure in $\mathcal{M}(K)$ of the set of all operators $T : K \to K$ satisfying (I) and (II). Clearly, $\overline{\mathcal{M}_c^f(K)} \subset \overline{\mathcal{M}^f(K)}$.

Theorem 2.1: (a) *There exists a set $\mathcal{F} \subset \overline{\mathcal{M}^f(K)}$ which is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}^f(K)}$ such that for each $B \in \mathcal{F}$, there is $z_* \in K$ with the property that for each $x \in K$, the orbit $\{B^n x\}_{n\in\mathbb{N}}$ converges weakly to z_*.*

(b) *There exists a set $\mathcal{F}_c \subset \mathcal{F} \cap \overline{\mathcal{M}_c^f(K)}$ which is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}_c^f(K)}$ such that for each $B \in \mathcal{F}_c$, there is a fixed point $z_* \in K$ of B with the property that for each $x \in K$, the orbit $\{B^n x\}_{n\in\mathbb{N}}$ converges weakly to z_*.*

Proof. Fix $\theta \in K$. For each natural number i, set

$$K_i = \{x \in K : \ ||x - \theta|| \le i\}. \tag{20}$$

With each $T \in \mathcal{M}^f(K)$, we associate a point $z_T \in \mathrm{Nexp}_f\,(T)$ for which condition (II) holds and define the operators $T_\gamma : K \to K$, $\gamma \in (0,1)$, by (8). According to Lemma 2.4, for each $T \in \mathcal{M}^f(K)$, each $\gamma \in (0,1)$ and each integer $i \ge 1$, there exist a natural number $N(T,\gamma,i)$ and an open neighborhood $\mathcal{U}(T,\gamma,i)$ of T_γ in $\mathcal{M}(K)$ such that the following properties hold:

$(P1)$ *For each $S \in U(T,\gamma,i)$, each $x \in K_i$ and each integer $n \ge N(T,\gamma,i)$, $D_f(z_T, S^n x) \le 2^{-i}$;*

$(P2)$ *For each $S \in U(T,\gamma,i)$, we have $||Sx - T_\gamma x|| \le 2^{-i}$ for all $x \in K_i$.*

Define

$$\mathcal{F} = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,q) : T \in \mathcal{M}^f(K),\ \gamma \in (0,1)\}] \cap \overline{\mathcal{M}^f(K)},$$

and

$$\mathcal{F}_c = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,q) : T \in \mathcal{M}_c^f(K),\ \gamma \in (0,1)\}] \cap \overline{\mathcal{M}_c^f(K)}.$$

Clearly, the set $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}^f(K)}$, the set $\mathcal{F}_c$ is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}_c^f(K)}$, and $\mathcal{F}_c \subset \mathcal{F}$.

Let $B \in \mathcal{F}$. For each integer $q \ge 1$, there exist $T_q \in \mathcal{M}^f(K)$ and $\gamma_q \in (0,1)$ such that

$$B \in \mathcal{U}(T_q, \gamma_q, q), \quad \forall q \ge 1. \tag{21}$$

Let $x \in K$ and denote by z_{T_q} the nonexpansivity pole of the operator T_q for which condition (II) is satisfied with T_q instead of T (such nonexpansivity poles exist because $T_q \in \mathcal{M}^f(K)$). There exists an integer $q_x \ge 1$ such that $x \in K_q$ for all integers $q \ge q_x$, where the sets K_q are given by (20). Note that according to $(P1)$ and (21), we have that whenever $q \ge q_x$,

$$D_f(z_{T_q}, B^i x) \le 2^{-q}, \ \forall i \ge N(T_q, \gamma_q, q). \tag{22}$$

So, all but finitely many vectors $B^i x$ are contained in the level set $L(T_{q_x}, 2^{-q_x})$ which is bounded by condition (II). Hence the sequence $\{B^i x\}_{i=1}^{\infty}$ is bounded too. Also, by (22) it results that if $q \ge q_x$, then

$$\limsup_{i\to\infty} D_f(z_{T_q}, B^i x) \le 2^{-q}.$$

When combined with Lemma 2.2, this implies that the sequence $\{z_{T_q}\}_{q=1}^{\infty}$ converges strongly to some vector $z_* \in K$ which is also the weak limit of the sequence $\{B^i x\}_{i\in\mathbb{N}}$. Thus (a) is proved.

Now suppose that $B \in \mathcal{F}_c$. A reasoning similar to the one presented above shows that, for each positive integer q, there exist $T_q \in \mathcal{M}_c^f(K)$ and $\gamma_q \in (0,1)$ such that (21) holds. Let z_{T_q} be a nonexpansivity pole of T_q such that condition (II) is satisfied with T_q instead of T. For each $\gamma \in (0,1)$, define

$$T_{q,\gamma} = \gamma z_{T_q} + (1-\gamma)T_q. \tag{23}$$

Since z_{T_q} is a fixed point of T_{q,γ_q}, we have

$$\begin{aligned} Bz_* - z_* &= (Bz_* - Bz_{T_q}) + (Bz_{T_q} - z_{T_q}) + (z_{T_q} - z_*) \\ &= (z_{T_q} - z_*) + (Bz_* - Bz_{T_q}) + (Bz_{T_q} - T_{q,\gamma_q} z_{T_q}), \end{aligned}$$

and the last expression converges strongly to zero as $q \to \infty$ because B is continuous and $(P2)$ ensures that

$$\lim_{q\to\infty} ||Bz_{T_q} - T_{q,\gamma_q} z_{T_q}|| = 0.$$

Thus $Bz_* = z_*$ and the proof is complete. $\square$

When X is a Hilbert space and $f = \|\cdot\|^2$, condition $(A1)$ is satisfied and the space $\mathcal{M}_c^f(X)$ consists of all those continuous operators $T : X \to X$ which are bounded on bounded sets and for some $z_T \in X$ satisfy

$$\|z_T - Tx\| \leq \|z_T - x\|, \quad \forall x \in X. \tag{24}$$

In particular, $\mathcal{M}_c^f(X)$ contains all nonexpansive operators having fixed points in X and, in particular, all metric projections on nonempty closed convex subsets of X, that is, all operators $P_C : X \to X$ defined by

$$P_C(x) = \arg\min \{\|y - x\| : y \in C\},$$

where C is a nonempty closed convex subset of X. Let $T \in \mathcal{M}_c^f(X)$ and denote by $\mathrm{Fix}(T)$ the set of fixed points of T. As noted above, $z_T \in \mathrm{Fix}(T)$. When $\mathrm{Fix}(T)$ is not a singleton, then the orbits of T starting at two different points in $\mathrm{Fix}(T)$ are stationary and converge to their respective different initial points. Hence, if $\mathrm{Fix}(T)$ is not a singleton, then T does not belong to the subset $\mathcal{F}_c$ of $\overline{\mathcal{M}_c^f(X)}$ the existence of which is guaranteed by Theorem 2.1, because all the orbits of the operators in $\mathcal{F}_c$ converge weakly to the same limit. This leads to the following result which shows that the set of nonexpansive operators with more than one fixed point in a Hilbert space is "small" in $\overline{\mathcal{M}_c^f(X)}$ (in the Baire category sense).

Corollary 2.1: *If X is a Hilbert space and $f = \|\cdot\|^2$, then the set of all nonexpansive operators on X which have more than one fixed point is of the first Baire category in $\overline{\mathcal{M}_c^f(X)}$. In particular, the set of all metric projections* *on closed convex subsets of X which are not singletons is of the first Baire category in $\overline{\mathcal{M}_c^f(X)}$.*

A similar result can be deduced from Theorem 2.1 in the case where X is not necessarily a Hilbert space, but still has the property that for some $p \in (1, +\infty)$, the function $f = \|\cdot\|^p$ is totally convex and Fréchet differentiable on X. In this situation, the function f satisfies $(A1)$ by Lemma 2.1. For any closed convex nonempty subset $C \subset X$, the Bregman projection operator $\Pi_C^f : X \to X$, defined by

$$\Pi_C^f(x) = \arg\min \{D_f(y, x) : y \in C\},$$

is well defined and, for any $z \in C$, satisfies the following *strong nonexpansivity condition with respect to f* :

$$D_f(z, \Pi_C^f(x)) + D_f(\Pi_C^f(x), x) \leq D_f(z, x), \quad \forall x \in X.$$

Consequently, $C \subseteq \mathrm{Nexp}_f\left(\Pi_C^f\right)$. Moreover, condition (II) is satisfied whenever $0 \in C$ because $D_f(0, \cdot)$ is convex. According to a result from [40], this implies that $\Pi_C^f \in \mathcal{M}_c^f(X)$. Observe that if C is not a singleton, then not all orbits of Π_C^f converge weakly to the same limit point. In view of Theorem 2.1(b), this implies that if C is not a singleton, then $\Pi_C^f \notin \mathcal{F}_c$. Hence we obtain the following result which applies, in particular, to any locally uniformly convex and uniformly smooth Banach space.

Corollary 2.2: *If for some $p \in (1, \infty)$ the function $f = \|\cdot\|^p$ is totally convex and Fréchet differentiable on X, then the set of Bregman projections Π_C^f on closed convex subsets C of X which are not singletons and contain 0 is of the first Baire category in $\overline{\mathcal{M}_c^f(X)}$.*

3 GENERIC WEAK CONVERGENCE FOR THE ORBITS OF OPERATORS IN $\overline{\mathcal{M}_u^f(K)}$

In this section we establish generic convergence properties for the orbits of the operators in $\overline{\mathcal{M}_u^f(K)}$ under each of the following assumptions: (i) X is separable and f satisfies $(A1)$ or, alternatively, (ii) f satisfies $(A2)$ [which is a stronger assumption than $(A1)$]. So, let f be a function which satisfies condition $(A1)$. We first prove several technical results concerning the operators in $\overline{\mathcal{M}_u^f(K)}$. They involve the operators T_γ associated with $T \in \overline{\mathcal{M}_u^f(K)}$ by (8).

Lemma 3.1: *Suppose that $T \in \mathcal{M}_u^f(K)$ and let $z_T \in \mathrm{Nexp}_f(T)$ be such that condition (III) is satisfied. Let $\epsilon, \gamma \in (0,1)$ and let K_0 be a nonempty bounded subset of K. Then there exist a natural number N and a neighborhood $\mathcal{U}$ of T_γ in $\mathcal{M}_u(K)$ such that for each $S \in \mathcal{U}$, each $x \in K_0$, and each integer $n \geq N$, there exist vectors $y_1^{(n)} \in K$ and $y_2^{(n)} \in X$ such that*

$$S^n x = y_1^{(n)} + y_2^{(n)}, \; D_f(z_T, y_1^{(n)}) \leq \epsilon, \text{ and } ||y_2^{(n)}|| \leq \epsilon. \tag{25}$$

Proof. Since $T \in \mathcal{M}_u^f(K)$ and $z_T \in \mathrm{Nexp}_f(T)$ satisfy condition (III), we may assume without loss of generality that

$$\{y \in K : \exists x \in K \text{ such that } D_f(z_T, x) \leq 1 \text{ and } ||x - y|| \leq 2\} \subset K_0. \tag{26}$$

Observe that, according to condition (III), there exists a number $c_0 > 0$ such that

$$D_f(z_T, x) \leq c_0, \quad \forall x \in K_0. \tag{27}$$

Choose a natural number $N \geq 4$ such that

$$(1-\gamma)^N(c_0+1) \leq 8^{-1}\epsilon. \tag{28}$$

This, together with (27) and (9), implies that for each $x \in K_0$, and any integer $n \geq N$,

$$D_f(z_T, T_\gamma^n x) \leq (1-\gamma)^n D_f(z_T, x) \leq (1-\gamma)^N c_0 \leq 8^{-1}\epsilon. \tag{29}$$

According to [13, Proposition 5.1], there exists a neighborhood $\mathcal{U}$ of T_γ in $\mathcal{M}_u(K)$ such that for each $x \in K_0$, all integers $n = N, N+1, \ldots, 4N$, and any $S \in \mathcal{U}$,

$$||T_\gamma^n x - S^n x|| \leq 8^{-1}\epsilon. \tag{30}$$

Assume that $S \in \mathcal{U}$ and $x \in K_0$. It follows from (29) and (30) that if $n \in \{N, \ldots, 4N\}$, then (25) is valid for $y_1^{(n)} = T_\gamma^n x \in K$ and $y_2^{(n)} = S^n x - T_\gamma^n x$. Let $q \geq 4N$ be an integer such that $y_1^{(n)} \in K$ and $y_2^{(n)} \in X$ satisfying (25) exist for all $n \in \{N, \ldots, q\}$. We now prove that for $n = q+1$, there also exist $y_1^{(n)} \in K$ and $y_2^{(n)} \in X$ such that (25) is valid. To this end, note that since (25) is true for $n = q - N$, it follows from (26) that $S^{q-N}x \in K_0$. Therefore, using (29) and (30), we obtain that

$$D_f(z_T, T_\gamma^{N+1}(S^{q-N}x)) \leq 8^{-1}\epsilon \quad \text{and} \quad ||T_\gamma^{N+1}(S^{q-N}x) - S^{q+1}x|| \leq 8^{-1}\epsilon.$$

This shows that (25) is true for $n = q+1$ with $y_1^{(n)} = T_\gamma^{N+1}(S^{q-N}x)$ and $y_2^{(n)} = S^{q+1}x - T_\gamma^{N+1}(S^{q-N}x)$. By induction, (25) is true for any integer $n \geq N$ and the proof is complete. $\square$

Denote by X^* the (topological) dual of the space X provided with the usual norm

$$||l||_* = \sup\{|l(y)| : \ y \in X, \ ||y|| \leq 1\}.$$

With these notations we state our next result.

Lemma 3.2: *Suppose that $T \in \mathcal{M}_u^f(K)$ and let $z_T \in \mathrm{Nexp}_f(T)$ be such that condition (III) is satisfied. Let $\epsilon, \gamma \in (0,1)$, $l \in X^*$, and let K_0 be a nonempty bounded subset of K. Then there exist a natural number N and a neighborhood $\mathcal{U}$ of T_γ in $\mathcal{M}_u(K)$ such that for each $S \in \mathcal{U}$, each $x \in K_0$, and each integer $n \geq N$, we have $|l(S^n x - z_T)| \leq \epsilon$.*

Proof. We may assume without loss of generality that $||l||_* = 1$. First we claim that there exists $\delta \in (0,1)$ such that

$$x \in K \text{ and } D_f(z_T, x) \le \delta \Rightarrow |l(x - z_T)| \le 8^{-1}\epsilon. \tag{31}$$

Assume the contrary. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset K$ such that

$$\lim_{m\to\infty} D_f(z_T, x_m) = 0 \text{ and } |l(x_n - z_T)| \ge 8^{-1}\epsilon \tag{32}$$

for any positive integer n. Since $T \in \mathcal{M}_u^f(K)$, condition (III) is satisfied and, consequently, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. We may assume without loss of generality either that

$$l(x_n - z_T) \ge 8^{-1}\epsilon, \tag{33}$$

for all positive integers n, or that

$$l(x_n - z_T) \le -8^{-1}\epsilon. \tag{34}$$

for all positive integers n. Denote by E_n the closure in the norm topology of the convex hull of the set $\{x_i : i \ge n\}$. Since the space X is reflexive, there exists $x_* \in \cap_{n=1}^{\infty} E_n$. The function $D_f(z_T, \cdot) : K \to R^1$ is convex and lower semicontinuous (see condition (III)). Therefore, (32) together with (33) or, alternatively, together with (34), imply that

$$D_f(z_T, x_*) = 0 \quad \text{and} \quad |l(x_* - z_T)| \ge 8^{-1}\epsilon.$$

This contradicts assumption $(A1)$. Thus, the claim is proved.

According to Lemma 3.1, there exist a natural number N and a neighborhood $\mathcal{U}$ of T_γ in $\mathcal{M}_u(K)$ such that for each $S \in \mathcal{U}$, each $x \in K_0$, and each integer $n \ge N$, there exist $y_1^{(n)} \in K$ and $y_2^{(n)} \in X$ with the property

$$S^n x = y_1^{(n)} + y_2^{(n)},\ D_f(z_T, y_1^{(n)}) \le \delta, \text{ and } ||y_2^{(n)}|| \le 8^{-1}\epsilon. \tag{35}$$

Fix $S \in \mathcal{U}$, $x \in K_0$, and let $n \ge N$ be an integer. Then, according to (31) and (35), we get

$$|l(z_T - y_1^n)| \le 8^{-1}\epsilon,\ |l(y_2^n)| \le 8^{-1}\epsilon, \text{ and } |l(z_T - S^n x)| \le 2^{-1}\epsilon$$

because $||l||_* = 1$. □

Now we are in a position to prove a result which shows that in the space $\overline{\mathcal{M}_u^f(K)}$ weak convergence of the orbits is a generic property.

Theorem 3.1: *Suppose that the space X is separable. Then there exists a set $\mathcal{F} \subset \overline{\mathcal{M}_u^f(K)}$ which is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}_u^f(K)}$ such that, for each $B \in \mathcal{F}$, there exists a vector $z_* \in K$ with the property that if $x \in K$, then the orbit $\{B^i x\}_{i=1}^{\infty}$ converges weakly to z_*, uniformly on bounded subsets of K.*

Proof. Since the space X is separable and reflexive, its dual space X^* is also separable. Consequently, the closed unit ball $B^*(0,1)$ of X^* is also separable. Thus there exists a sequence $\{l_i\}_{i=1}^{\infty} \subset B^*(0,1)$ such that its strong closure is $B^*(0,1)$. Let T be an operator in $\mathcal{M}_u^f(K)$ and $\gamma \in (0,1)$. Then, obviously, $T_\gamma \in \mathcal{M}_u^f(K)$. Fix $\theta \in K$ and, for each integer $i \geq 1$, set

$$K_i = \{x \in K : ||x - \theta|| \leq i\}. \tag{36}$$

Let $T \in \mathcal{M}_u^f(K)$, $\gamma \in (0,1)$, and suppose that $n \geq 1$ is an integer. According to Lemmas 3.1 and 3.2, there exist an open neighborhood $\mathcal{U}(T,\gamma,n)$ of T_γ in $\mathcal{M}_u(K)$ and a natural number $N(T,\gamma,n)$ such that the following property holds:

$(P3)$ *For each $S \in U(T,\gamma,n)$, each $x \in K_n$, and each integer $j \geq N(T,\gamma,n)$,*

$$|l_i(S^j x - z_T)| \leq 2^{-n},\ i = 1,\ldots,n,$$

and there exist vectors $y_1^{(j)} \in K$ and $y_2^{(j)} \in X$ such that

$$S^j x = y_1^{(j)} + y_2^{(j)},\ D_f(z_T, y_1^{(j)}) \leq 2^{-n}, \text{ and } ||y_2^{(j)}|| \leq 2^{-n},$$

where z_T denotes the element of $\mathrm{Nexp}_f\,(T)$ for which condition (III) is satisfied.

Take

$$\mathcal{F} = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,q) : T \in \mathcal{M}_u^f(K),\ \gamma \in (0,1)\}] \cap \overline{\mathcal{M}_u^f(K)}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}_u^f(K)}$. Let $B \in \mathcal{F}$. For each integer $q \geq 1$, there exist $T_q \in \mathcal{M}_u^f(K)$ and $\gamma_q \in (0,1)$ such that $B \in \mathcal{U}(T_q,\gamma_q,q)$. Denote by z_{T_q} the element in $\mathrm{Nexp}_f(T_q)$ for which condition (III) is satisfied. Let x be any vector in K. There exists an integer $q_0 \geq 1$ with the property that $x \in K_{q_0}$. Using $(P3)$ and taking into account that for any positive integer q the operator T_q satisfies condition (III), we deduce that the sequence $\{B^i x\}_{i=1}^{\infty}$ is bounded and, therefore, it has weakly convergent subsequences (because X is reflexive). We claim that the weak limit $w - \lim_{j\to\infty} B^j x$ exists in K and that for each integer $q \geq q_0$,

$$|l_i(w - \lim_{j\to\infty} B^j x - z_{T_q})| \leq 2^{-q},\ i = 1,\ldots,q. \tag{37}$$

In order to prove this claim, observe that according to $(P3)$, for each weakly convergent subsequence $\{B^{k_j} x\}_{j=1}^{\infty}$ of $\{B^i x\}_{i=1}^{\infty}$ and for any integer $q \geq q_0$, we have

$$|l_i(w - \lim_{j\to\infty} B^{k_j} x - z_{T_q})| \leq 2^{-q},\ i = 1,\ldots,q.$$

Letting $q \to \infty$, we obtain that the sequences $\{B^{k_j} x\}_{j=1}^{\infty}$ and $\{z_{T_q}\}_{q=1}^{\infty}$ have the same weak limit. Hence, for each $x \in K$, the weak limit of $\{B^i x\}_{i=1}^{\infty}$ exists (all weakly convergent subsequences of this sequence have the same weak limit), the weak limit of $\{z_{T_q}\}_{q=1}^{\infty}$ exists too and the two weak limits coincide. Hence there exists a vector $z_* \in K$ such that

$$w - \lim_{q\to\infty} z_{T_q} = z_* = w - \lim_{j\to\infty} B^j x$$

for all $x \in K$, and

$$|l_i(z_* - z_{T_q})| \le 2^{-q},\ i = 1, \ldots, q.$$

Moreover, if $q \ge 1$ is any integer, $x \in K_q$, and $j \ge N(T_q, \gamma_q, q)$, then

$$|l_i(B^j x - z_*)| \le 2^{-q+1},\ i = 1, \ldots, q. \tag{38}$$

This shows that the weak convergence of the sequences $\{B^i x\}_{i=1}^{\infty}$ is uniform on bounded subsets of K. □

Theorem 3.1 has been proven under the assumption that the space X is separable and f satisfies $(A1)$. Its conclusion still holds in spaces which are not necessarily separable provided that the function f satisfies $(A2)$. In order to show this we need the following result.

Lemma 3.3: *Assume that the function f satisfies condition $(A2)$. Suppose that $\{x_n\}_{n=1}^{\infty} \subset K$ is bounded, $\{\epsilon_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\lim_{n\to\infty} \epsilon_n = 0$, $\{z_n\}_{n=1}^{\infty} \subset K$, and that for each integer $n \ge 1$, the function $D_f(z_n, \cdot) : K \to R^1$ is convex and lower semicontinuous. If for any integer $n > 0$ and for each integer $i \ge 1$, there exist vectors $y_i' \in K$ and $y_i'' \in X$ such that*

$$x_i = y_i' + y_i'', \tag{39}$$

and

$$\limsup_{i\to\infty} D_f(z_n, y_i') < \epsilon_n \quad \text{and} \quad \limsup_{i\to\infty} ||y_i''|| < \epsilon_n, \tag{40}$$

then there exists a vector $x_* \in K$ such that the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly and the sequence $\{z_n\}_{n=1}^{\infty}$ converges strongly to x_*.

Proof. Let x_* be a weak subsequential limit of the bounded sequence $\{x_i\}_{i=1}^{\infty}$ and let $\{x_{i_j}\}_{j=1}^{\infty}$ be a subsequence of $\{x_i\}_{i=1}^{\infty}$ which converges weakly to x_*. Fix an integer $n \ge 1$. It follows from (39) and (40) that there exists a positive integer $j(n)$ such that

$$D_f(z_n, y_{i_j}') \le \epsilon_n \quad \text{and} \quad ||y_{i_j}' - x_{i_j}|| \le \epsilon_n$$

for all $j \ge j(n)$. Taking into account the weak lower semicontinuity of the norm and the convexity and lower semicontinuity of $D_f(z_n, \cdot)$, and extracting a subsequence if necessary, we deduce that there exists a vector $\bar{y}_n \in K$ such that

$$D_f(z_n, \bar{y}_n) \le \epsilon_n \quad \text{and} \quad ||\bar{y}_n - x_*|| \le \epsilon_n.$$

Hence, by $(A2)$, it follows that $\{z_n\}_{n=1}^{\infty}$ converges strongly to x_*. Consequently, any subsequential weak limit of $\{x_n\}_{n=1}^{\infty}$ is necessarily the strong limit of $\{z_n\}_{n=1}^{\infty}$. This implies that $\{x_n\}_{n=1}^{\infty}$ converges weakly to x_*. □

With this result in mind, we proceed to establish generic weak convergence of powers for the operators in $\overline{\mathcal{M}_u^f(K)}$ under the assumption $(A2)$ regarding f.

Theorem 3.2: *If the function f satisfies condition $(A2)$, then there exists a set $F \subset \overline{\mathcal{M}_u^f(K)}$ which is a countable intersection of open everywhere dense* *sets in $\overline{\mathcal{M}_u^f(K)}$ such that, for each $B \in F$, there exists a fixed point $z_* \in K$ of the operator B which has the property that all the orbits $\left\{B^i x\right\}_{i=1}^{\infty}$ with $x \in K$ converge weakly to z_*.*

Proof. Fix $\theta \in K$ and, for each integer $i \geq 1$, set

$$K_i = \{x \in K : \ ||x - \theta|| \leq i\}.$$

Let $T \in \mathcal{M}_u^f(K)$, $\gamma \in (0,1)$, and let $n \geq 1$ be an integer. If $z_T \in \operatorname{Nexp}_f(T)$ is such that condition (III) is satisfied, then, by Lemma 3.1, there exist an open neighborhood $\mathcal{U}(T,\gamma,n)$ of T_γ in $\mathcal{M}_u(K)$ and a natural number $N(T,\gamma,n)$ such that for each $S \in \mathcal{U}(T,\gamma,n)$, each $x \in K_n$, and each integer $j \geq N(T,\gamma,n)$, there exist vectors $y_1^{(j)} \in K$ and $y_2^{(j)} \in X$ with the property that

$$S^j x = y_1^{(j)} + y_2^{(j)}, \ D_f(z_T, y_1^{(j)}) \leq 2^{-n}, \text{ and } ||y_2^{(j)}|| \leq 2^{-n}. \tag{41}$$

Also, for each $S \in \mathcal{U}(T,\gamma,n)$ and each $x \in K_n$, we have

$$\|Sx - T_\gamma x\| \leq 2^{-n}. \tag{42}$$

Define

$$\mathcal{F} = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,q) : T \in \mathcal{M}_u^f(K), \ \gamma \in (0,1)\}] \cap \overline{\mathcal{M}_u^f(K)}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}_u^f(K)}$. Consider an operator $B \in \mathcal{F}$. For each integer $q \geq 1$, there exist $T_q \in \mathcal{M}_u^f(K)$ and $\gamma_q \in (0,1)$ such that $B \in \mathcal{U}(T_q,\gamma_q,q)$. Let $x \in K$. According to (41) and condition (III), the sequence $\{B^i x\}_{i=1}^{\infty}$ is bounded. For any positive integer q, let $z_{T_q} \in \operatorname{Nexp}_f(T_q)$ be such that condition (III) holds with T_q instead of T. Lemma 3.3 (applied with $z_i = z_{T_i}$, $x_i = B^i x$, $\epsilon_i = 2^{-i}$, $i = 1,2,\ldots$), when combined with (41), implies that $\{B^i x\}_{i=1}^{\infty}$ converges weakly and that the sequence $\{z_{T_q}\}_{q=1}^{\infty}$ converges strongly to the same point z_*. Since x is an arbitrary element of K, it follows that all the orbits $\left\{B^i x\right\}_{i=1}^{\infty}$ with $x \in K$ converge weakly to the same limit z_*. Now observe that

$$\begin{aligned}
Bz_* - z_* &= \left(Bz_* - Bz_{T_q}\right) + \left(Bz_{T_q} - z_{T_q}\right) + \left(z_{T_q} - z_*\right) \\
&= \left(Bz_* - Bz_{T_q}\right) + \left(z_{T_q} - z_*\right) + \left(Bz_{T_q} - T_{q,\gamma_q} z_{T_q}\right) + \left(T_{q,\gamma_q} z_{T_q} - z_{T_q}\right) \\
&= \left(Bz_* - Bz_{T_q}\right) + \left(z_{T_q} - z_*\right) + \left(Bz_{T_q} - T_{q,\gamma_q} z_{T_q}\right),
\end{aligned}$$

where T_{q,γ_q} is given by (23). Note that $Bz_* - Bz_{T_q}$ and $z_{T_q} - z_*$ converge strongly to zero as $q \to \infty$ because $\{z_{T_q}\}_{q=1}^{\infty}$ converges strongly to z_* and B is continuous. Since for each positive integer q, we have that $B \in \mathcal{U}(T_q,\gamma_q,q)$ and (42) holds for B instead of S, and T_q and γ_q instead of T and γ, respectively, we also deduce that $Bz_{T_q} - T_{q,\gamma_q} z_{T_q}$ converges strongly to zero and this completes the proof. $\square$

ACKNOWLEDGEMENTS

The work of the first two authors was partially supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities (Grants 293/97 and 592/00). The second author was also partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund.

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